TETRAVALENT EDGE-TRANSITIVE CAYLEY GRAPHS WITH ODD NUMBER OF VERTICES

CAI HENG LI AND ZAI PING LU

Abstract. A characterisation is given of edge-transitive Cayley graphs of valency 4 on odd number of vertices. The characterisation is then applied to solve several problems in the area of edge-transitive graphs: answering a question proposed by Xu (1998) regarding normal Cayley graphs; providing a method for constructing edge-transitive graphs of valency 4 with arbitrarily large vertex-stabiliser; constructing and characterising a new family of half-transitive graphs. Also this study leads to a construction of the first family of arc-transitive graphs of valency 4 which are non-Cayley graphs and have a ‘nice’ isomorphic 2-factorisation.

1. Introduction

A graph $\Gamma$ is a Cayley graph if there exist a group $G$ and a subset $S \subset G$ with $1 \not\in S = S^{-1} := \{g^{-1} \mid g \in S\}$ such that the vertices of $\Gamma$ may be identified with the elements of $G$ in such a way that $x$ is adjacent to $y$ if and only if $yx^{-1} \in S$. The Cayley graph $\Gamma$ is denoted by $\text{Cay}(G, S)$. Throughout this paper, denote by $1$ the vertex of $\text{Cay}(G, S)$ corresponding to the identity of $G$.

It is well-known that a graph $\Gamma$ is a Cayley graph of a group $G$ if and only if the automorphism group $\text{Aut}\Gamma$ contains a subgroup which is isomorphic to $G$ and acts regularly on vertices. In particular, a Cayley graph $\text{Cay}(G, S)$ is vertex-transitive of order $|G|$. However, a Cayley graph is of course not necessarily edge-transitive. In this paper, we investigate Cayley graphs that are edge-transitive.

Small valent Cayley graphs have received attention in the literature. For instance, Cayley graphs of valency 3 or 4 of simple groups are investigated in [5, 6, 28]; Cayley graphs of valency 4 of certain $p$-groups are investigated in [7, 26]. A relation between regular maps and edge-transitive Cayley graphs of valency 4 is studied in [20]. In the main result (Theorem 1.1) of this paper, we characterise edge-transitive Cayley graphs of valency 4 and odd order. To state this result, we need more definitions.

1.1. Let $\Gamma$ be a graph with vertex set $V\Gamma$ and edge set $E\Gamma$. If a subgroup $X \leq \text{Aut}\Gamma$ is transitive on $V\Gamma$ or $E\Gamma$, then the graph $\Gamma$ is said to be $X$-vertex-transitive or $X$-edge-transitive, respectively. A sequence $v_0, v_1, \ldots, v_s$ of vertices of $\Gamma$ is called an $s$-arc if $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s - 1$, and $\{v_i, v_{i+1}\}$ is an edge for $0 \leq i \leq s - 1$. The graph $\Gamma$ is called $(X, s)$-arc-transitive if $X$ is transitive on the $s$-arcs of $\Gamma$; if in addition $X$ is not transitive on the $(s+1)$-arcs, then $\Gamma$ is said to be $(X, s)$-transitive. In particular, a 1-arc is simply called an arc, and an $(X, 1)$-arc-transitive graph is called $X$-arc-transitive.

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A typical method for studying vertex-transitive graphs is taking certain quotients. For an $X$-vertex-transitive graph $\Gamma$ and a normal subgroup $N \triangleleft X$, the normal quotient graph $\Gamma_N$ induced by $N$ is the graph that has vertex set $V\Gamma_N = \{v^N \mid v \in V\Gamma\}$ such that $v_1^N$ and $v_2^N$ are adjacent if and only if $v_1$ is adjacent in $\Gamma$ to some vertex in $v_2^N$. If further the valency of $\Gamma_N$ equals the valency of $\Gamma$, then $\Gamma$ is called a normal cover of $\Gamma_N$.

**Theorem 1.1.** Let $G$ be a finite group of odd order, and let $\Gamma = \text{Cay}(G, S)$ be connected and of valency 4. Assume that $\Gamma$ is $X$-edge-transitive, where $G \leq X \leq \text{Aut}\Gamma$. Then one of the following holds:

1. $G$ is normal in $X$, $X_1 \leq D_8$, and $S = \{a, a^{-1}, a^\tau, (a^\tau)^{-1}\}$, where $\tau \in \text{Aut}(G)$ such that either $o(\tau) = 2$, or $o(\tau) = 4$ and $a^\tau = a^{-1}$;
2. there is a subgroup $M < G$ such that $M \triangleleft X$, and $\Gamma$ is a cover of $\Gamma_M$;
3. $X$ has a unique minimal normal subgroup $N \cong Z_p^k$ with $p$ odd prime and $k \geq 2$ such that
   (i) $G = N \rtimes R \cong Z_p^k \rtimes Z_m$, where $m > 1$ is odd;
   (ii) $X = N \rtimes ((H \rtimes R), O) \cong Z_p^k \rtimes ((Z_2^l \times Z_m), Z_t)$, and $X_1 = H.O$, where $H \cong Z_2^l$ with $2 \leq l \leq k$, and $O \cong Z_t$ with $t = 1$ or 2, satisfying the following statements:
   (a) there exist $x_1, \cdots, x_k \in N$ and $\tau_1, \cdots, \tau_k \in H$ such that $N = \langle x_1, \cdots, x_k \rangle, (x_i, \tau_i) \cong D_{2p}$ and $H = \langle \tau_i \rangle \rtimes C_{H(x_i)}$ for $1 \leq i \leq k$;
   (b) $R$ does not centralise $H$;
   (c) $X/(NH) \cong Z_m$ or $D_{2m}$, and $\Gamma$ is $X$-arc-transitive if and only if $X/(NH) \cong D_{2m}$;
4. $\Gamma$ is $(X, s)$-transitive, and $X, X_1, s$ and $G$ are as in the following table:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$X_1$</th>
<th>$s$</th>
<th>$G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_5, S_5$</td>
<td>$A_4, S_4$</td>
<td>2</td>
<td>$Z_5$</td>
</tr>
<tr>
<td>$\text{PGL}(2, 7)$</td>
<td>$D_{16}$</td>
<td>1</td>
<td>$Z_7 \rtimes Z_3$</td>
</tr>
<tr>
<td>$\text{PSL}(2, 11), \text{PGL}(2, 11)$</td>
<td>$A_4, S_4$</td>
<td>2</td>
<td>$Z_{11} \rtimes Z_5$</td>
</tr>
<tr>
<td>$\text{PSL}(2, 23)$</td>
<td>$S_4$</td>
<td>2</td>
<td>$Z_{23} \rtimes Z_{11}$</td>
</tr>
</tbody>
</table>

**Remarks on Theorem 1.1:**

(a) The Cayley graph $\Gamma$ in part (1), called normal edge-transitive graph, is studied in [21]. If further $X = \text{Aut}\Gamma$, then $\Gamma$ is called a normal Cayley graph, introduced in [27]. For this type of Cayley graph, the action of $X$ on the graph $\Gamma$ is well-understood.

(b) Part (2) is a reduction from the Cayley graph $\Gamma$ to a smaller graph $\Gamma_M$, which is also an edge-transitive Cayley graph of valency 4. An edge-transitive Cayley graph is called basic if it is not a normal cover of a smaller edge-transitive Cayley graph. Theorem 1.1 shows that if $\Gamma$ is not a normal Cayley graph then $\Gamma$ is a cover of a well-characterised graph, that is a basic Cayley graph satisfying part (3) or part (4).

(c) Construction 3.2 shows that for every group $X$ satisfying part (3) with $O = 1$ indeed acts edge-transitively on some Cayley graphs of valency 4.

(d) Part (4) tells us that there are only three 2-arc-transitive basic Cayley graphs of valency 4 and odd order.
The following corollary of Theorem 1.1 gives a solution to Problem 4 of [27], in particular, answering the question stated there in the negative.

**Corollary 1.2.** There are infinitely many connected basic Cayley graphs of valency 4 and odd order which are not normal Cayley graphs.

The proof of Corollary 1.2 follows from Lemma 3.3.

It is well-known that the vertex-stabiliser for an s-arc-transitive graph of valency 4 with $s \geq 2$ has order dividing $2^{3}3^{6}$, see Lemma 2.5. However, in [22, 2], ‘non-trivial’ arc-transitive graphs of valency 4 which have arbitrarily large vertex-stabiliser are constructed. Part (3) of Theorem 1.1 characterises edge-transitive Cayley graphs of valency 4 and odd order with this property.

**Corollary 1.3.** Let $\Gamma$ be a connected Cayley graph of valency 4 and odd order. Assume that $\Gamma$ is $X$-edge-transitive for $X \leq \text{Aut}\Gamma$. Then $|X_1| > 24$ if and only if $\Gamma$ is a cover of a graph satisfying part (3) of Theorem 1.1 with $l \geq 5$.

This characterisation provides a potential method for constructing edge-transitive graphs of valency 4 with arbitrarily large vertex-stabiliser, see Construction 3.2.

A graph $\Gamma$ is called half-transitive if $\text{Aut}\Gamma$ is transitive on the vertices and the edges but not transitive on the arcs of $\Gamma$. Constructing and characterising half-transitive graphs was initiated by Tutte (1965), and is a currently active topic in algebraic graph theory, see [19, 20, 17] for references. Theorem 1.1 provides a method for characterising some classes of half-transitive graphs of valency 4. The following theorem is such an example.

**Theorem 1.4.** Let $G = N \rtimes \langle g \rangle = \mathbb{Z}_k^2 \times \mathbb{Z}_m < \text{AGL}(1,p^k)$, where $k > 1$ is odd, $p$ is an odd prime and $m$ is the largest odd divisor of $p^k - 1$. Assume that $\Gamma$ is a connected edge-transitive Cayley graph of $G$ of valency 4. Then $\text{Aut}\Gamma = G \rtimes \mathbb{Z}_2$, $\Gamma$ is half-transitive, and $\Gamma \cong \Gamma_i = \text{Cay}(G, S_i)$, where $1 \leq i \leq \frac{m - 1}{2}$, $(m, i) = 1$, and

$$S_i = \{ag^i, a^{-1}g^i, (ag^i)^{-1}, (a^{-1}g^i)^{-1}\}, \quad \text{where } a \in N \setminus \{1\}.$$

Moreover, $\Gamma_i \cong \Gamma_j$ if and only if $pi \equiv j$ or $-j \pmod{m}$ for some $r \geq 0$.

The following result is a by-product of analysing PGL(2,7)-arc-transitive graphs of valency 4. (For two graphs $\Gamma$ and $\Sigma$ which have the same vertex set $V$ and disjoint edge sets $E_1$ and $E_2$, respectively, denote by $\Gamma + \Sigma$ the graph with vertex set $V$ and edge set $E_1 \cup E_2$. For a positive integer $n$ and a cycle $C_m$ of size $m$, denote by $nC_m$ the vertex disjoint union of $n$ copies of $C_m$.)

**Proposition 1.5.** Let $p$ be a prime such that $p \equiv -1 \pmod{8}$, and let $T = \text{PSL}(2, p)$ and $X = \text{PGL}(2, p)$. Then there exists an $X$-arc-transitive graph $\Gamma$ of valency 4 such that the following hold:

(i) $\Gamma = \Delta_1 + \Delta_2$, $\Delta_1 \cong \Delta_2 \cong \frac{\text{AGL}(1,p^2) - 1}{48}C_3$, $T \leq \text{Aut}\Delta_1 \cap \text{Aut}\Delta_2$, and both $\Delta_1$ and $\Delta_2$ are $T$-arc-transitive; in particular, $\Gamma$ is not $T$-edge-transitive;

(ii) $\Gamma$ is a Cayley graph if and only if $p = 7$.

Part (i) of this proposition is proved by Lemma 4.3, and part (ii) follows from Theorem 1.1.

**Remark on Proposition 1.5:** The factorisation $\Gamma = \Delta_1 + \Delta_2$ is an isomorphic 2-factorisation of $\Gamma$. The group $X$ is transitive on $\{\Delta_1, \Delta_2\}$ with $T$ being the kernel. Such isomorphic factorisations are called homogeneous factorisations, introduced
and studied in [18, 9]. The factorisation given in Proposition 1.5 are the first
known example of non-Cayley graphs which have a homogeneous 2-factorisation,
refer to [9, Lemma 2.7] for a characterisation of homogeneous 1-factorisations.

This paper is organized as follows. Section 2 collects some preliminary results
which will be used later. Section 3 gives some examples of graphs appeared in
Theorem 1.1. Then Section 4 constructs the graphs stated in Proposition 1.5.
Finally, in Sections 5 and 6, Theorems 1.1 and 1.4 are proved, respectively.

2. Preliminary results

For a core-free subgroup $H$ of $X$ and an element $a \in X \setminus H$, let $[X : H] = \{Hx \mid x \in X\}$, and define the coset graph $\Gamma := \text{Cos}(X, H, H\{a, a^{-1}\}H)$ to be the graph with vertex set $[X : H]$ such that $\{Hx, Hy\}$ is an edge of $\Gamma$ if and only if $yx^{-1} \in H\{a, a^{-1}\}H$. The properties stated in the following lemma are well-known.

Lemma 2.1. For a coset graph $\Gamma = \text{Cos}(X, H, H\{a, a^{-1}\}H)$, we have

(i) $\Gamma$ is $X$-edge-transitive;
(ii) $\Gamma$ is $X$-arc-transitive if and only if $HaH = Ha^{-1}H$, or equivalently, $HaH = HbH$ for some $b \in X \setminus H$ such that $b^2 \in H \cap H^3$;
(iii) $\Gamma$ is connected if and only if $\langle H, a \rangle = X$;
(iv) the valency of $\Gamma$ equals

$$\text{val}(\Gamma) = \begin{cases} |H : H \cap H^a|, & \text{if } HaH = Ha^{-1}H, \\ 2|H : H \cap H^a|, & \text{otherwise}. \end{cases}$$

Lemma 2.2. Let $\Gamma$ be a connected $X$-vertex-transitive graph where $X \leq \text{Aut}\Gamma$, and let $N \vartriangleleft X$ be intransitive on $V\Gamma$. Assume that $\Gamma$ is a cover of $\Gamma_N$. Then $N$ is semiregular on $V\Gamma$, and the kernel of $X$ acting on $V\Gamma_N$ equals $N$.

Proof. Let $K$ be the kernel of $X$ acting on $V\Gamma_N$. Then $N \vartriangleleft K \vartriangleleft X$. Suppose that $K_v \neq 1$, where $v \in V\Gamma$. Then since $\Gamma$ is connected and $K \vartriangleleft X$, it follows that $K_v^{\Gamma(v)} \neq 1$. Thus the number of $K_v$-orbits in $\Gamma(v)$ is less than $|\Gamma(v)|$, and so the valency of $\Gamma_N$ is less than the valency of $\Gamma$, which is a contradiction. Hence $K_v = 1$, and it follows that $N = K$ is semiregular on $V\Gamma$. \hfill \Box

For a Cayley graph $\Gamma = \text{Cay}(G, S)$, let $\text{Aut}(G, S) = \{\alpha \in \text{Aut}(G) \mid S^\alpha = S\}$. For the normal edge-transitive case, we have a simple lemma.

Lemma 2.3. Let $\Gamma = \text{Cay}(G, S)$ be connected of valency 4. Assume that $\text{Aut}\Gamma$ has a subgroup $X$ such that $\Gamma$ is $X$-edge-transitive and $G \vartriangleleft X$. Then $X \leq N_{\text{Aut}\Gamma}(G) = G \rtimes \text{Aut}(G, S)$, and either $X_1 \leq D_8$, or $\Gamma$ is $(X, 2)$-transitive and $|G|$ is even.

Proof. Since $\Gamma$ is connected, $\langle S \rangle = G$, and so $\text{Aut}(G, S)$ acts faithfully on $S$. Hence $\text{Aut}(G, S) \leq S_4$. By [8, Lemma 2.1], we have that $X \leq N_{\text{Aut}\Gamma}(G) = G \rtimes \text{Aut}(G, S)$. Thus $X_1 \leq \text{Aut}(G, S) \leq S_4$. Suppose that 3 divides $|X_1|$. Then $X_1$ is 2-transitive on $S$. Hence $\Gamma$ is $(X, 2)$-transitive, and all elements in $S$ are involutions, see for example [16]. In particular, $|G|$ is even. On the other hand, if 3 does not divide $|X_1|$, then $X_1$ is a 2-group, and hence $X_1 \leq D_8$. \hfill \Box

Lemma 2.4. Let $G$ be a finite group of odd order, and let $\Gamma = \text{Cay}(G, S)$ be connected and of valency 4. Assume that $N \vartriangleleft X \leq \text{Aut}\Gamma$ such that $G \leq X$ and $\Gamma$ is $X$-edge-transitive. Then one of the following statements holds:

...
(i) $N$ has odd order and $N \leq G$;
(ii) $N$ has even order, and either $N$ is transitive on $V\Gamma$, or $GN$ is transitive on $E\Gamma$.

Proof. Let $Y = GN$. Then $Y$ is transitive on $V\Gamma$. Suppose that $N \not\leq G$. Then $Y$ is not regular on $V\Gamma$. It follows that $Y_1$ is a nontrivial $\{2, 3\}$-group. If $Y_1$ has an orbit of size 3 on $\Gamma(1) = S$, then $Y$ has an orbit on $E\Gamma$ which is a 1-factor of $\Gamma$, which is not possible since $|V\Gamma| = |G|$ is odd. It follows that either $Y_1$ is transitive on $S$, or $Y_1$ has an orbit of size 2 on $S$. In particular, $|Y_1|$ is even, so $|N|$ is even. Therefore, either $N$ has odd order and $N \leq G$, as in part (i), or $N$ has even order.

Assume now that $|N|$ is even. If $Y_1$ is transitive on $S$, then $\Gamma$ is $Y$-arc-transitive and hence $Y$-edge-transitive, so part (ii) holds. Thus assume that $Y_1$ has an orbit of size 2 on $S$. Noting that $N < X$, $N_1 \neq 1$ and $\Gamma$ is connected and $X$-vertex-transitive, it is easily shown that $N_1$ is non-trivial on $S$. Since $N_1 \leq Y_1$, $N_1$ has an orbit $\{x, y\}$ of size 2 on $S$. Suppose that $N$ is intransitive on $V\Gamma$. Let $H = 1^N$ be the $N$-orbit containing 1. Then $H \cap S = \emptyset$ as $\Gamma$ is $X$-edge-transitive. Further, $x^N = (1^N)^x = (1^N)x = Hx$ and $y^N = (1^N)y = (1^N)y = Hx$. Suppose that $\Gamma$ forms a subgroup of $G$. In particular, $xy^{-1} \in H$. If $y = x^{-1}$, then $x^2 = xy^{-1} \in H$, and $x \in H$ as $|H|$ is odd, a contradiction. Thus $S = \{x, y, x^{-1}, y^{-1}\}$. Clearly, $\{x, y\}$ is an orbit of $Y_1$ on $S$. It follows that $Y$ is transitive on $E\Gamma$, as in part (ii).

By the result of [14], there is no 4-arc-transitive graph of valency at least 3 on odd number of vertices. Then by the known results about 2-arc-transitive graphs (see for example [25] or [15, Subsection 3.1]), the following result holds.

Lemma 2.5. Let $\Gamma$ be a connected $(X, s)$-transitive graph of valency 4. Then either $s \leq 4$ or $s = 7$, and further, $s$ and the stabiliser $X_v$ are listed as following:

<table>
<thead>
<tr>
<th>$s$</th>
<th>$X_v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2-group</td>
</tr>
<tr>
<td>2</td>
<td>$A_4 \leq X_v \leq S_4$</td>
</tr>
<tr>
<td>3</td>
<td>$A_4 \times Z_3 \leq X_v \leq S_4 \times S_3$</td>
</tr>
<tr>
<td>4</td>
<td>$Z_4^3 \cdot SL(2, 3) \leq X_v \leq Z_4^3 \cdot GL(2, 3)$</td>
</tr>
<tr>
<td>7</td>
<td>$[3^5] \cdot SL(2, 3) \leq X_v \leq [3^5] \cdot GL(2, 3)$</td>
</tr>
</tbody>
</table>

Moreover, if $|V\Gamma|$ is odd, then $s \leq 3$.

Finally, we quote a result about simple groups, which will be used later.

Lemma 2.6. ([12]) Let $T$ be a non-abelian simple group which has a 2'-Hall subgroup. Then $T = PSL(2, p)$, where $p = 2^e - 1$ is a prime. Further, $T = GH$, where $G = \mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$ and $H = D_{p+1} = D_{2^e}$.

3. Existence of graphs satisfying Theorem 1.1

In this section, we construct examples of graphs satisfying Theorem 1.1.

First consider part (1) of Theorem 1.1. We observe that if $\Gamma$ is a connected normal edge-transitive Cayley graph of a group $G$ of valency 4, then $G = \langle a, a^\tau \rangle$, where $\tau \in \text{Aut}(G)$ such that $a^2 = a$ or $a^{-1}$. Conversely, if $G$ is a group that has a presentation $G = \langle a, a^\tau \rangle$, where $\tau \in \text{Aut}(G)$ such that $a^2 = a$ or $a^{-1}$, then $G$ has
a connected normal edge-transitive Cayley graph of valency 4, that is, \( \text{Cay}(G, S) \) where \( S = \{a, a^{-1}, a^\tau, (a^\tau)^{-1}\} \). Thus we have the following conclusion:

**Lemma 3.1.** Let \( G \) be a group of odd order. Then \( G \) has a connected normal edge-transitive Cayley graph of valency 4 if and only if \( G = \langle a, a^\tau \rangle \), where \( \tau \in \text{Aut}(G) \) such that \( a^{\tau^2} = a \) or \( a^{-1} \).

See Construction 6.1 for an example of such construction.

The following construction produces edge-transitive graphs admitting a group \( X \) satisfying part (3) of Theorem 1.1 with \( O = 1 \).

**Construction 3.2.** Let \( X = N \rtimes (H \rtimes R) \cong \mathbb{Z}_p^k \rtimes (\mathbb{Z}_2 \rtimes \mathbb{Z}_m) \), where \( p \) is an odd prime, \( m \) is odd and \( 2 \leq l \leq k \), such that \( N \cong \mathbb{Z}_p^l, H \cong \mathbb{Z}_2 \) and \( R \cong \mathbb{Z}_m \) satisfy

1. \( N \) is the unique minimal normal subgroup of \( X \);
2. there exist \( x \in N \setminus \{1\} \) and \( \tau \in H \) such that \( x^\tau = x^{-1} \) and \( H = \langle \tau \rangle \times C_H(x) \);
3. \( R \) does not centralise \( H \).

Let \( R = \langle \sigma \rangle \cong \mathbb{Z}_m \), and let \( y = x\sigma \). Set

\[
\Gamma(p, k, l, m) = \text{Cos}(X, H, H\{y, y^{-1}\} H).
\]

The next lemma shows that the graphs constructed here are as required.

**Lemma 3.3.** Let \( \Gamma = \Gamma(p, k, l, m) \) be a graph constructed in Construction 3.2, and let \( G = N \rtimes R \cong \mathbb{Z}_p^k \rtimes \mathbb{Z}_m \). Then \( \Gamma \) is a connected \( X \)-edge-transitive Cayley graph of \( G \) of valency 4, and \( G \) is not normal in \( X \).

**Proof.** By the definition, \( H \) is core-free in \( X \), and hence \( X \leq \text{Aut}\Gamma \). Now \( X = GH \) and \( G \cap H = 1 \), and thus \( G \) acts regularly on the vertex set \( [X : H] \). So \( \Gamma \) is a Cayley graph of \( G \), which has odd order \( p^k m \). Obviously, \( G \) is not normal in \( X \).

For \( x \) and \( \sigma \) defined in Construction 3.2, set \( x_i = x^{\sigma^{i-1}} \) for \( i = 1, 2, \ldots, m \), and let \( \alpha = (\sigma^{-1})^\tau \). Then, as \( y = x\sigma, x_2 = \sigma^{-1} x\sigma \) and \( \tau \in H \), we have

\[
\alpha x_2^2 = ((\sigma^{-1})^\tau \sigma)(\sigma^{-1} x\sigma)^2 = (\sigma^{-1})^\tau x^2 \sigma = (x^{-1} \sigma^{-1})^\tau (x\sigma) = (y^{-1} y \in \langle H, y \rangle).
\]

As \( \tau \in H \) and \( \sigma \) normalises \( H \), we have \( \alpha = (\sigma^{-1})^\tau \sigma = \tau(\sigma^\tau) \in H \). Thus \( x_{2i}^2 = \alpha^{-1}(\alpha x_2^2) \in \langle H, y \rangle \), and as \( x_2 \) has odd order, \( x_2 \in \langle H, y \rangle \). Then \( x_3 = x_2 x_{2i}^2 = x_2^3 \sigma = x_2^3 \in \langle H, y \rangle \). Similarly, we have that \( x_i \in \langle H, y \rangle \) for \( i = 2, 3, \ldots, m \). Then calculation shows that \( y^n = x_1 x_2 \cdots x_m \in \langle H, y \rangle \). Thus \( x = x_1 = x^n x_2 \cdots x_m \in \langle H, y \rangle \), and so \( \sigma = x^{-1} y \in \langle H, y \rangle \). Since \( N \) is a minimal normal subgroup of \( X \), we conclude that \( N = \langle x^{h\sigma^i} \mid h \in H, 0 \leq i \leq m - 1 \rangle \), and hence \( N \leq \langle H, y \rangle \). So \( \langle H, y \rangle \cong \langle N, H, \sigma \rangle = X \), and \( \Gamma \) is connected.

Finally, as \( \sigma \) normalises \( H \) and by condition (b) of Construction 3.2, we have that \( H^\sigma \cap H = C_H(x) \) has index 2 in \( H \). Thus \( H^\sigma \cap H = (H^\sigma \cap H^\sigma^{-1})^\sigma = (H^\sigma \cap H)^\sigma = C_H(x)^\sigma \), which has index 2 in \( H \). Since \( X \leq \text{Aut}\Gamma \), \( \Gamma \) is not a cycle. By Lemma 2.1, \( \Gamma \) is connected, \( X \)-edge-transitive and of valency 4.

We end this section with presenting several groups satisfying (a), (b) and (c) of Construction 3.2, so we obtain examples of graphs satisfying Theorem 1.1.(3).

**Example 3.4.** Let \( p \) be an odd prime, and \( m \) an odd integer.

(i) Let \( X = \langle x_1, \tau_1 \rangle \times \langle x_2, \tau_2 \rangle \times \cdots \times \langle x_m, \tau_m \rangle \times \langle \sigma \rangle \cong D_{2p} \times \mathbb{Z}_m = D_{2p} \times \mathbb{Z}_m \), where \( \langle x_i, \tau_i \rangle \cong D_{2p} \) and \( \langle x_i, \tau_i \rangle^\sigma = \langle x_{i+1}, \tau_{i+1} \rangle \) (reading the subscripts...
mod 1). Then \( N = \langle x_1, x_2, \ldots, x_m \rangle \cong \mathbb{Z}^m_p \) is a minimal normal subgroup of \( X \), and \( H = \langle \tau_1, \tau_2, \ldots, \tau_m \rangle \cong \mathbb{Z}^m_2 \) is such that \( H = \langle \tau_i \rangle \times C_H(x_i) \) for \( 1 \leq i \leq m \).

(ii) Let \( Y < X \) with \( X \) as in part (i) such that \( Y = \langle x_1, x_2, \ldots, x_m \rangle \times \langle \tau_1 \tau_2, \tau_2 \tau_3, \ldots, \tau_{m-1} \tau_m \rangle \times \langle \sigma \rangle \cong \mathbb{Z}^m_2 \rtimes \mathbb{Z}^{m-1}_2 \rtimes \mathbb{Z}_m \). Then \( N = \langle x_1, x_2, \ldots, x_m \rangle \) is a minimal normal subgroup of \( Y \), and \( L := \langle \tau_1 \tau_2, \tau_2 \tau_3, \ldots, \tau_{m-1} \tau_m \rangle \cong \mathbb{Z}^{m-1}_2 \) is such that \( L = \langle \tau_i \tau_{i+1} \rangle \times C_L(x_i) \) for \( 1 \leq i \leq m \).

Thus both \( X \) and \( Y \) satisfy the conditions of Construction 3.2.

**Example 3.5.** Let \( N = \langle x_1, \ldots, x_k \rangle = \mathbb{Z}_p^k \), where \( p \) is an odd prime and \( k \geq 3 \).

Let \( l \) be a proper divisor of \( k \). Let \( \sigma \in \text{Aut}(N) \) be such that

\[
x_i^\sigma = \begin{cases}  
  x_{i+1}, & \text{if } 1 \leq i \leq k-1, \\
  x_1x_{i+1}, & \text{if } i = k.
\end{cases}
\]

Let \( \tau \in \text{Aut}(N) \) be such that

\[
x_j^\tau = \begin{cases}  
  x_{j-1}^{-1}, & \text{if } l \mid j - 1, \\
  x_j, & \text{otherwise}.
\end{cases}
\]

Let \( o(\sigma) = m \), \( H = \langle \tau^{\sigma i-1} \mid 1 \leq i \leq m \rangle \) and \( X = N \rtimes \langle \tau, \sigma \rangle \). Then \( N \) is a minimal normal subgroup of \( X \) and \( H = \langle \tau \rangle \times C_H(x_1) \cong \mathbb{Z}^2_2 \). Thus \( X \) satisfies the conditions of Construction 3.2.

For instance, taking \( p = 3, k = 9 \) and \( l = 3 \), so \( m = 39 \), and then applying Construction 3.2, we obtain an \( X \)-edge-transitive Cayley graph \( \Gamma(3, 9, 3, 39) \) of valency 4 of the group \( \mathbb{Z}_3^3 \rtimes \mathbb{Z}_{39} \), where \( X = \mathbb{Z}_3^3 \rtimes (\mathbb{Z}_2^3 \rtimes \mathbb{Z}_{39}) \).

4. A FAMILY OF ARC-TRANSITIVE GRAPHS OF VALENCY 4

Here we construct a family of 4-arc-transitive cubic graphs and their line graphs. The smallest line graph is \( \text{PGL}(2, 7) \)-arc-transitive but not \( \text{PSL}(2, 7) \)-edge-transitive, which is one of the graphs stated in Theorem 1.1 (4).

**Construction 4.1.** Let \( p \) be a prime such that \( p \equiv -1 \pmod{8} \), and let \( T = \text{PSL}(2, p) \) and \( X = \text{PGL}(2, p) \). Then \( T \) has exactly two conjugacy classes of maximal subgroups isomorphic to \( S_4 \) which are conjugate in \( X \). Let \( L, R < T \) be such that \( L, R \cong S_4 \), \( L \cap R \cong D_8 \), and \( L, R \) are not conjugate in \( T \) but \( L \Gamma = R \) for some involution \( \tau \in X \setminus T \).

1. Let \( X = \cos(T, L, R) \) be the coset graph defined as: the vertex set \( V X = \{ T : L \} \cup \{ T : R \} \) such that \( Lx \) is adjacent to \( Ry \) if and only if \( yx^{-1} \in LR \).

2. Let \( \Gamma \) be the line graph of \( X \), that is, the vertices of \( \Gamma \) are the edges of \( X \) and two vertices of \( \Gamma \) are adjacent if and only if the corresponding edges of \( X \) have exactly one common vertex.

Then it follows from the definition that \( X \) is bipartite with parts \( [T : L] \) and \( [T : R] \), and \( T \) acts by right multiplication transitively on the edge set \( E X \). Further, we have the following properties.

**Lemma 4.2.** The following statements hold for the graph \( X \) defined above:

(i) \( X \) is connected and of valency 3;
(ii) \( X \) may also be represented as the coset graph \( \cos(X, L, \tau L) \);
(iii) \( X \) is \( (X, 4) \)-arc-transitive;
(iv) $\Sigma$ is $T$-vertex intransitive and locally $(T,4)$-arc-transitive.

Proof. Since $(L,R) = T$, part (i) follows from the definition, see [10, Lemma 2.7]. Part (ii) follows from the definitions of $\text{Cos}(T,L,R)$ and $\text{Cos}(X,L,L\tau L)$.

Next we study the line graph $\Gamma$ in the following lemma.

Lemma 4.3. Let $\Gamma$ be the line graph of $\Sigma$ defined as in Construction 4.1. Let $v$ be the vertex of $\Gamma$ corresponding to the edge $\{L,R\}$ of $\Sigma$. Then we have the following statements:

(i) $\Gamma$ is connected, and has valency 4 and girth 3;
(ii) $\Gamma$ is $X$-arc-transitive, and $X_v \cong D_{16}$;
(iii) $T$ is transitive on $V\Gamma$ and intransitive on $E\Gamma$, and $T_v \cong D_8$;
(iv) $T$ has exactly two orbits $E_1, E_2$ on $E\Gamma$, and letting $\Delta_1 = (V\Gamma, E_1)$ and $\Delta_2 = (V\Gamma, E_2)$, we have $\Delta_1 \cong \Delta_2 \cong \frac{(p^2-1)}{48}C_3$, and $\Gamma = \Delta_1 + \Delta_2$.

Proof. We first look at the neighbors of the vertex $v$ in $\Gamma$. Let $a \in L$ be of order 3, and let $b = a^7 \in R$. Then the 3 neighbors of $L$ in $\Sigma$ are $R, Ra$ and $Ra^{-1}$; and the 3 neighbors of $R$ are $L, Lb$ and $Lb^{-1}$. Write the corresponding vertices of $\Gamma$ as: $u_1 = \{Lb, R\}$, $u_2 = \{Lb^{-1}, R\}$, $w_1 = \{L, Ra\}$ and $w_2 = \{L, Ra^{-1}\}$. Then the neighborhood $\Gamma(v) = \{u_1, u_2, w_1, w_2\}$.

Thus $\Gamma$ is of valency 4. By the definition of a line graph, $u_1$ is adjacent to $w_2$, and $u_2$ is adjacent to $w_1$. Hence the girth of $\Gamma$ is 3. Since $\Sigma$ is connected, $\Gamma$ is connected too, proving part (i).

Now $T_v = L \cap R \cong D_8$ and $X_v = (L \cap R, \tau) \cong D_{16}$. Since $T$ is transitive on $E\Sigma$ and is not transitive on the vertex set $V\Sigma$, there is no element of $T$ maps the arc $(L,R)$ to the arc $(R,L)$. Since $T_v = L \cap R$, there exist $\sigma_1, \sigma_2 \in T_v$ such that $a^{\sigma_1} = a^{-1}$ and $b^{\sigma_2} = b^{-1}$. Thus $w_1^{\sigma_1} = u_2$ and $w_2^{\sigma_2} = u_2$. So $T_v$ has exactly two orbits on $\Gamma(v)$, that is, $\{u_1, u_2\}$ and $\{w_1, w_2\}$. Further, (b) acts transitively on $\{v, u_1, u_2\}$. It follows that $E_1 := \{u_1, u_2\}^T$ is a self-paired orbital of $T$ on $V\Gamma$. Therefore, $\Gamma$ is not $T$-edge-transitive. Further, since $\tau$ interchanges $L$ and $R$ and also interchanges $a$ and $b$, it follows that $\tau \in X_v$ and $\{u_1, u_2\}^\tau = \{w_1, w_2\}$. Thus $\Gamma$ is $X$-arc-transitive.

Let $E_2 = \{w_1, w_2\}^T$, and let $\Delta_1 = (V\Gamma, E_1)$ with $i = 1, 2$. Then $\Gamma = \Delta_1 + \Delta_2$, and $\Delta_i$ consists of cycles of size 3. Thus $|E_1| = |E_2| = |V\Gamma| = \frac{|X|}{|T|} = \frac{(p^2-1)}{48}$, and $\Delta_i$ consists of $\frac{|E_i|}{3}$ cycles of size 3, that is, $\Delta_i \cong \frac{(p^2-1)}{48}C_3$. Finally, $E_1^\tau = E_2$ and so $\tau$ is an isomorphism between $\Delta_1$ and $\Delta_2$. \hfill $\Box$

5. Proof of Theorem 1.1

Let $G$ be a finite group of odd order, and let $\Gamma = \text{Cay}(G, S)$ be connected and of valency 4. Assume that $\Gamma$ is $X$-edge-transitive, where $G \leq X \leq \text{Aut}\Gamma$, and assume further that $G$ is not normal in $X$.

We first treat the case where $\Gamma$ has no non-trivial normal quotient of valency 4 in Subsection 5.1 and 5.2.

Suppose that each non-trivial normal quotient of $\Gamma$ is a cycle. Let $N$ be a minimal normal subgroup of $X$. Then $N = T^k$ for some simple group $T$ and some
integer $k \geq 1$. Since $|VT| = |G|$ is odd, $X$ has no nontrivial normal 2-subgroups. In particular, $N$ is not a 2-group. Further we have the following simple lemma.

**Lemma 5.1.** Either $N$ is soluble, or $C_X(N) = 1$.

**Proof.** Suppose that $N$ is insoluble and $C := C_X(N) \neq 1$. Then $NC = N \times C$ and $C \vartriangleleft X$. Since $|N|$ is not semiregular on $VT$, $C$ is intransitive. By the assumption that any non-trivial normal quotient of $G$ is a cycle, $\Gamma_C$ is a cycle. Let $K$ be the kernel of $X$ acting on $V \Gamma_C$. Then $X/K \leq \text{Aut}\Gamma_C \cong D_{2c}$, where $c = |V \Gamma_C|$. It follows that $N \leq K$. Let $\Delta$ be an arbitrary $C$-orbit on $V \Gamma$. Then $\Delta$ is $N$-invariant. Consider the action of $NC$ on $\Delta$, and let $D$ be the kernel of $NC$ acting on $\Delta$. Then $NC/D = (ND/D) \times (CD/D)$. Since $C$ is transitive on $\Delta$, $CD/D$ is also transitive on $\Delta$. Then $ND/D$ is semiregular on $\Delta$. Noting that $|\Delta|$ is odd and $ND/D \cong N/(N \cap D) \cong T^{k'}$ for some $k' \geq 0$, it follows that $ND/D$ is trivial on $\Delta$, and hence $N \leq D$. Thus $N$ is trivial on every $C$-orbit, and so $N$ is trivial on $V \Gamma$, which is a contradiction. Therefore, either $N$ is soluble, or $C_X(N) = 1$. $\square$

### 5.1. The case where $N$ is transitive

Assume that $N$ is transitive on the vertices of $\Gamma$. Our goal is to prove that $N = A_5$, PSL$(2, 7)$, PSL$(2, 11)$ or PSL$(2, 23)$ by a series of lemmas. The first shows that $N$ is nonabelian simple.

**Lemma 5.2.** The minimal normal subgroup $N$ is a nonabelian simple group, $X$ is almost simple, and $N = \text{soc}(X)$.

**Proof.** Suppose that $N$ is abelian. Since $N$ is transitive, $N$ is regular, and hence $|N| = |G|$ is odd. By Lemma 2.3, we have that $N \leq G$, and so $G = N \vartriangleleft X$, which is a contradiction. Thus $N = T^k$ is nonabelian. Suppose that $k > 1$. Let $L$ be a normal subgroup of $N$ such that $L \cong T^{k-1}$. Since $N_2 \leq X_1$ is a $\{2, 3\}$-group, it follows that $L$ is intransitive on $V \Gamma$; further, since $|V \Gamma|$ is odd and $|T|$ is even, $L$ is not semiregular. It follows from Lemma 2.2 that $\Gamma_L$ is a cycle. Then $\text{Aut}\Gamma_L$ is a dihedral group. Thus $N$ lies in the kernel of $X$ acting on $V \Gamma L$, and so $N$ is intransitive on $V \Gamma$, which is a contradiction. Thus $k = 1$, and $N = T$ is nonabelian simple. By Lemma 5.1, $C_X(N) = 1$, and hence $N$ is the unique minimal normal subgroup of $X$. Thus $X$ is almost simple, and $N = \text{soc}(X)$. $\square$

The 2-arc-transitive case is determined by the following lemma.

**Lemma 5.3.** Assume $\Gamma$ is $(X, 2)$-arc-transitive. Then one of the following holds:

(i) $X = A_5$ or $S_5$, and $X_1 = A_4$ or $S_4$, respectively, and $G = Z_5$;

(ii) $X = \text{PSL}(2, 11)$ or $\text{PGL}(2, 11)$, and $X_1 = A_4$ or $S_4$, respectively, and $G = Z_{11} \rtimes Z_5$;

(iii) $X = \text{PSL}(2, 23)$, $X_1 = S_4$, and $G = Z_{23} \rtimes Z_{11}$.

**Proof.** Note that $X = GX_1$ and $G \cap X_1 = 1$. By Lemma 2.5, $|X_1|$ is a divisor of $2^43^2 = 144$, and hence a Sylow 2-subgroup of $X$ is isomorphic to a subgroup of $D_8 \times Z_2$. Further, $|N : (G \cap N)| = |G : G|$ divides $|X : G| = |X_1|$. Let $M$ be a maximal subgroup of $N$ containing $G \cap N$. Then $|N : M|$ has size dividing 144, and $N$ is a primitive permutation group on $[N : M]$. Inspecting the list of primitive permutation groups of small degree given in [3, Appendix B], we conclude that $N$ is one of the following groups:

$A_5, A_6, \text{PSL}(2, 7), \text{PSL}(2, 8), \text{PSL}(2, 11), M_{11}, \text{PSL}(2, 17), \text{PSL}(2, 23), \text{PSL}(2, 47), \text{PSL}(2, 71)$ and $\text{PSL}(3, 3)$.
It is known that the groups $M_{11}$, $\text{PSL}(2, 17)$, $\text{PSL}(2, 47)$ and $\text{PSL}(3, 3)$ have a Sylow 2-subgroup isomorphic to $Q_8 \rtimes \mathbb{Z}_2$, $D_{16}$, $D_{16}$ and $\mathbb{Z}_2 \rtimes Q_8$, respectively. Thus $N$ is none of these groups. Suppose that $N = A_5$ or $\text{PSL}(2, 8)$. Then $X = A_5$, $S_6$, $\text{PSL}(2, 8)$ or $\text{PSL}(2, 8) \rtimes \mathbb{Z}_3$. However, $X$ has no factorisation $X = GX_1$ such that $G \cap X_1 = 1$, and $X_1$ is a $(2, 3)$-group, which is a contradiction. Suppose that $N = \text{PSL}(2, 71)$. Then $X = \text{PSL}(2, 71)$ or $\text{PGL}(2, 71)$, and $X_1 = D_{72}$ or $D_{144}$, respectively, and $G = \mathbb{Z}_{71} \rtimes \mathbb{Z}_{35}$. Thus $X_1$ is a maximal subgroup of $X$, and $X$ acts primitively on the vertex set $V = [X : X_1]$. This is not possible, see [24] or [17]. Let $N = \text{PSL}(2, 7)$, then $G = \mathbb{Z}_7$ and $N_4 = S_4$. Then, however, $N$ is $2$-transitive on $V$.

Therefore, $N = A_5$, $\text{PSL}(2, 11)$ or $\text{PSL}(2, 23)$. Now either $X$ is primitive on $V$ or $X = N = \text{PSL}(2, 11)$ and $G = \mathbb{Z}_{11} \rtimes \mathbb{Z}_5$. Then, by [23] and [11], we obtain the conclusion stated in the lemma.

The next lemma determines $X$ for the case where $\Gamma$ is not $(X, 2)$-arc-transitive.

**Lemma 5.4.** Suppose that $\Gamma$ is not $(X, 2)$-arc-transitive. Then $X = \text{PGL}(2, 7)$, $X_1 = D_{16}$ and $G = \mathbb{Z}_7 \rtimes \mathbb{Z}_3$.

**Proof.** Since $\Gamma$ is not $(X, 2)$-arc-transitive, $X_1$ is a $2$-group. Since $X = GX_1$ and $G \cap X_1 = 1$, $G$ is a $2'$-Hall subgroup of $X$. Then $G \cap N$ is a $2'$-Hall subgroup of $N$. By Lemma 5.2, $N$ is nonabelian simple. By Lemma 2.6, $N = \text{PSL}(2, p)$, $G \cap N = \mathbb{Z}_p \times \mathbb{Z}_{e-1}$, and $N_1 = D_{e-1}$, where $p = 2^e - 1$ is a prime. If $e > 3$, then $N_4$ is a maximal subgroup of $N$. Thus $N$ is a primitive permutation group on $V$ and has a self-paired suborbit of length 4, which is not possible, see [24] or [17]. Thus $e = 3$, $N = \text{PSL}(2, 7)$, $G = \mathbb{Z}_7 \rtimes \mathbb{Z}_3$, and $N_1 = D_8$. So $X = \text{PSL}(2, 7)$ or $\text{PGL}(2, 7)$.

Suppose that $X = \text{PSL}(2, 7)$. Now write $\Gamma$ as coset graph $\text{Cos}(X, H, H \{x, x^{-1}\} H)$, where $H = X_1 = D_8$, and $x \in X$ is such that $(H, x) = X$. Let $P = H \cap H^x$. Then $|H : P| = 2$ or 4.

Assume that $|H : P| = 4$. Then $\Gamma$ is $X$-arc-transitive and $P = \mathbb{Z}_2$. By Lemma 2.1, we may assume that $x^2 \in P = H \cap H^x$ and $x$ normalises $P$. If $P \lhd H$, then $P \lhd (H, x) = X = \text{PSL}(2, 7)$, which is a contradiction. Thus $P$ is not normal in $H$, and so $\mathbb{Z}_4^2 \cong N_H(P) \lhd H$. Since $N_X(P) \cong D_8$, we have $N_X(P) \neq H$. So $N_H(P) \lhd (H, N_X(P)) = X$, which is a contradiction. Thus $|H : P| = 2$, and hence $P \lhd L := (H, H^x)$. We conclude that $L \cong S_4$. Then $H$ and $H^x$ are two Sylow $2$-subgroups of $L$, and hence $H^x = H^y$ for some $y \in L$. Thus $H^{xy^{-1}} = H$, that is, $xy^{-1} \in N_X(H) = H$, hence $x \in H_y \subseteq L$. Then $(x, H) \lhd L \neq X$, which is a contradiction. Thus $X \neq \text{PSL}(2, 7)$, and so $X = \text{PGL}(2, 7)$.

5.2. The case where $N$ is intransitive. Assume now that the minimal normal subgroup $N \lhd X$ is intransitive on $V\Gamma$. We are going to prove that part (3) of Theorem 1.1 occurs.

**Lemma 5.5.** The minimal normal subgroup $N$ is soluble, and $N \not\leq G$.

**Proof.** Suppose that $N$ is insoluble. Then $N = T^k$ and $N \not\leq G$, where $T$ is nonabelian simple and $k \geq 1$. Let $Y = NG$. Then by Lemma 2.4 $Y$ is transitive on both of $V\Gamma$ and $E\Gamma$. Let $L \leq N$ be a non-trivial normal subgroup of $Y$. Then $L$ is intransitive, and since $|V\Gamma|$ is odd, $L$ is not semi-regular on $V\Gamma$. Thus the valency of the quotient graph $I_L$ is less than 4. Since $|V\Gamma|$ is odd, $I_L$ is a cycle of size $m \geq 3$. Let $K$ be the kernel of $Y$ acting on the $L$-orbits in $V\Gamma$. Then
Let transitively on \(N\) for some elements hold: a nontrivial 2-group. In particular, \(K_1\) is a 2-group. Since \(K = NK_1 \leq Y\) and \(|Y : N|\) is odd, we know that \(K = N\). Thus \(N\) itself is the kernel of \(X\) acting on \(V\Gamma_N\). It follows that \(Y/N\) is the cyclic regular subgroup of \(\text{Aut}^N\) acting on \(V\Gamma_N\). Thus \(Y = NG = N(a) \cong N.Z_m\) for some \(a \in G \setminus N\).

Since \(X_1\) is a nontrivial 2-group, it is easily shown that \(G\cap N\) is a 2'-Hall subgroup of \(N\), and \(N = (G \cap N)N_1\). Then \(G \cap T = G \cap N \cap T\) is a 2'-Hall subgroup of \(T\). By Lemma 2.6, \(T = PSL(2,p)\) for a prime \(p = 2^k - 1\). In particular, \(\text{Out}(T) \cong Z_2\).

By Lemma 5.1, \(C_X(N) = 1\), and hence \(C_Y(N) = 1\). Then \(N\) is the only minimal normal subgroup of \(Y\) and of \(X\). So the element \(a \in Y \leq \text{Aut}(N) = \text{Aut}(T)/S_k\).

Write \(N = T_1 \times \cdots \times T_k\), where \(T_i \cong T\). Then \(\text{Aut}(N) = (\text{Aut}(T_1) \times \text{Aut}(T_2) \times \cdots \times \text{Aut}(T_k)) \times S_k\), and \(a = b\pi\) where \(b \in \text{Aut}(T_1) \times \text{Aut}(T_2) \times \cdots \times \text{Aut}(T_k)\) and \(\pi \in S_k\).

Since \(N\) is a minimal normal subgroup of \(Y\), we have that \(\langle a \rangle\) acts by conjugation transitively on \(\{T_1, T_2, \ldots, T_k\}\), and hence the permutation \(\pi\) is a k-cycle of \(S_k\). Relabeling if necessary, we may assume \(\pi = (12 \ldots k) \in S_k\). Then \(T_1 = T_1\) and \(T_i = T_{i+1}\), where \(i = 1, \ldots, k-1\). Further, \(a^k = b^{a_1} \cdots b^{a_k} \in \text{Aut}(T_1) \times \text{Aut}(T_2) \times \cdots \times \text{Aut}(T_k) = N \rtimes Z_k\). Since \(a^k\) is of odd order, it follows that \(a^k \in N\). Thus \(Y/N \cong Z_k\), and hence \(m = k\). Set \(a^k = t_1t_2 \cdots t_k\), where \(t_i \in T_i\). Since a centralises \(a^k\), we have \(t_1t_2 \cdots t_k = a^k = (a^k)^{a_1} t_1^{a_1} \cdots t_k^{a_k}\). Since \(t_k^{a_i} \in T_k\), it follows that \(t_1 = t_i^a\) and \(t_i^a = t_{i+1}^a\), where \(i = 1, \ldots, k-1\).

Let \(g = t_1^{-1}a\). Then \(T_i = T_i^{g^{-1}} = T_i^{g^{-i}}\) and \(g^i = a^it_i^{-1} \cdots a^{t_{i-1}}\) (reading the subscripts modular \(k\)), where \(2 \leq i \leq k\). In particular, \(g^k = a^k t_k^{-1} t_{k-1}^{-1} \cdots t_1^{-1} = 1\), and so the order of \(g\) is a divisor of \(k\). Noting that \(Y/N \cong Z_k\) and \(N(g) = \langle N, g \rangle = \langle N, t_1^{-1}\rangle = \langle N, a \rangle = Y\), it follows that \(Y = N \leq \langle g \rangle\).

Let \(H_1 = \langle T_1 \rangle\) and \(H_i := H_i^{g^{i-1}}\) for \(1 \leq i \leq k\), and let \(H = H_1 \times \cdots \times H_k\). Then \(H_i \cong D_{2^i}\) is a Sylow 2-subgroup of \(T_i\). \(H\) is a Sylow 2-subgroup of \(N\), and \(H^g = H\). Since \(\Gamma_N\) is a k-cycle and \(Y/N \cong Z_k\), it follows that \(\Gamma\) is not \(Y\)-arc-transitive. Since \(\Gamma\) is \(Y\)-edge-transitive, we may write \(\Gamma\) as a coset graph \(\Gamma = \text{Cos}(Y, H, \{g^ix, (g^ix)^{-1}\}H)\), where \(1 \leq j < k\) and \(x = x_1 \cdots x_k \in N\) for \(x_1 \in T_1\) such that \(|H : (H \cap H^{x_j})| = 2\) and \((H, g^ix) = Y\). Now \(H^{g^ix} = H^{x_1} \times H^{x_2} \times \cdots \times H^{x_k}\) and \(H \cap H^{g^ix} = (H_1 \cap H^{x_1}) \times \cdots \times (H_k \cap H^{x_k})\). Thus we may assume that \((H : (H_1 \cap H^{x_1})) = 2\) and \(H_1 \cap H^{x_1} = H_i\). Then \(H_1 \cap H^{x_i} = H_i\) for \(i = 2, \cdots, k\). Since \(N_2(H_i) = H_i\), we know that \(x_i \in H_i\) for \(i \geq 2\). If \(k > 3\), then \(H_1\) is maximal in \(T_1\), and hence \(H_1 \cap H^{x_1} \leq \langle H_1, H_1^{x_1}\rangle = T_1\), which is a contradiction.

Thus \(N\) is soluble. Then by Lemma 2.4, we have \(N < G\), completing the proof. \(\square\)

We notice that, since \(N\) is intransitive on \(V\Gamma\), the \(N\)-orbits in \(V\Gamma\) form an \(X\)-invariant partition \(V\Gamma_N\). The next lemma determines the structure of \(X\).

**Lemma 5.6.** Let \(K\) be the kernel of \(X\) acting on \(V\Gamma_N\). Then the following statements hold:

\begin{itemize}
  \item \(K\) is a minimal normal subgroup of \(G\).
  \item \(K\) is a \(1\)-group.
  \item \(K\) is a Sylow 2-subgroup of \(G\).
\end{itemize}
(i) $X/K \cong \mathbb{Z}_m$ or $D_{2m}$ for an odd integer $m > 1$, $K_1 \neq 1$, and $\Gamma$ is $X$-arc-transitive if and only if $X/K \cong D_{2m}$;

(ii) $G = N \times R$, $X = N \times ((K_1 \times R)/O)$ and $R$ does not centralise $K_1$, where $R \cong \mathbb{Z}_m$, and $O = 1$ or $\mathbb{Z}_2$;

(iii) $N \cong \mathbb{Z}_p^k$ for an odd prime $p$, and $K_1 \cong \mathbb{Z}_p^l$, where $2 \leq l \leq k$;

(iv) there exist $x_1, \ldots, x_k \in N$ and $\tau_1, \ldots, \tau_k \in K_1$ such that $N = \langle x_1, \ldots, x_k \rangle$,

$\langle x_1, \tau_1 \rangle \cong D_{2p}$ and $K_1 = \langle \tau_1 \rangle \times C_{K_1}(x_i)$ for $1 \leq i \leq k$.

(v) $N$ is the unique minimal normal subgroup of $X$;

Proof. By Lemma 5.5, $N < G$ is soluble, hence $N \cong \mathbb{Z}_p^k$ for an odd prime $p$ and an integer $k > 1$. In particular, $N$ is semi-regular on $V \Gamma$. Since $\Gamma_N$ is a cycle of size $m$ say, $X/K \leq \text{Aut} \Gamma_N = D_{2m}$. Thus $K = N \times K_1$, $K_1$ is a 2-group, and $X/K \cong \mathbb{Z}_m$ or $D_{2m}$. It follows that $G/N \cong G/K \cong \mathbb{Z}_m$. If $K_1 = 1$, then $K = N$, and hence $G < X$, which contradicts that $G$ is not normal in $X$. Thus $K_1 \neq 1$. Further, $\Gamma$ is $X$-arc-transitive if and only if $X/K \cong D_{2m}$, so we have part (i).

Set $U = N_X(K_1)$. Then $U \neq X$ since $K_1$ is not normal in $X$. Noting that $(|N|, |K_1|) = 1$, it follows that $N_{X\cap N}(K/N) = N_{X\cap N}(NK_1/N) = N_{X}(K_1)/N = \text{Aut}(U\cap N)$. Since $K/N$ is normal in $X/N$, it follows that $X = U_N$. Since $N < X$, $N \cap U < N$. Further $N \cap U < N$ is abelian. Then $N \cap U < N$ is normal. If $N \neq U$, then $K = NK_1 = N \times K_1$, and hence $K_1 < X$, a contradiction. Thus $N \cap U < N$. Further, since $N$ is a minimal normal subgroup of $X$, we know that $N \cap U = 1$, and hence $K \cap U = NK_1 \cap U = (N \cap U)K_1 = K_1$. Now $X/K = U_N/K = U\cap K/K \cong U/(K \cap U) = U/K_1$, and so $U = (K_1 \times R)/O$, where $\text{Aut} \cong \mathbb{Z}_m$, and $O = 1$ or $\mathbb{Z}_2$. Then $G = N \times R$, and $X_1 = K_1\langle x_i \rangle$. Further, since $G$ is not normal in $X$, we conclude that $R$ does not centralise $K_1$, as in part (ii).

Let $Y = KR = N \times (K_1 \times R)$. Then $Y$ has index at most 2 in $X$, and $\Gamma$ is $Y$-edge-transitive by Lemma 2.4, but it is not $Y$-arc-transitive. Thus $\Gamma = \text{Cos}(Y, K_1, K_1\{y, y^{-1}\}K_1)$, where $y \in Y$ is such that $(K_1, y) = Y$ and $K_1 \cap K_1^y$ has index 2 in $K_1$. We may choose $y \in N \times R = G$ such that $Y = \langle \sigma \rangle$ and $y = \sigma x$ where $x \in N$. Then $K_1 \cap K_1^y = K_1 \cap K_1^x$ has index 2 in $K_1$.

We claim that $K_1 \cap K_1^x = C_{K_1}(x)$. Let $\sigma \in K_1 \cap K_1^x$. Then $\sigma^{-1} \sigma x^{-1} \in K_1$. Since $x \in N$ and $N \not\triangleleft N K_1$, we have $\sigma^{-1} \sigma x^{-1} = \langle \sigma^{-1} \sigma \rangle x^{-1} \in N$. Thus $\sigma^{-1} \sigma x^{-1} \in N \cap K_1 = 1$, and so $\sigma^{-1} = \sigma$. Then $\sigma$ centralises $x$. It follows that $K_1 \cap K_1^x \leq C_{K_1}(x)$. Clearly, $C_{K_1}(x) \leq K_1 \cap K_1^x$. Thus $C_{K_1}(x) = K_1 \cap K_1^x$ as claimed.

Since $N$ is a minimal normal subgroup of $X$ and $X = NU$, we have that $N = \langle x \rangle \times \langle x^2 \rangle \times \cdots \times \langle x^k \rangle$ where $\sigma_i \in U$. Then $C_{K_1}(x^{\sigma_i}) = C_{K_1}(x^{\sigma_i}) = \langle K_1 \cap K_1^x \rangle$. The intersection $\cap_{i=1}^k C_{K_1}(x^{\sigma_i}) \leq C_{K_1}(N) = N$, and hence $\cap_{i=1}^k C_{K_1}(x^{\sigma_i}) = 1$. Since each $C_{K_1}(x^{\sigma_i})$ is a maximal subgroup of $K_1$, the Frattini subgroup $\Phi(K_1) \leq \cap_{i=1}^k C_{K_1}(x^{\sigma_i}) = 1$. Hence $K_1$ is an elementary abelian 2-group, say $K_1 \cong \mathbb{Z}_2^l$ for some $l \geq 1$. Noting that $\cap_{i=1}^k C_{K_1}(x^{\sigma_i}) = 1$, it follows that $l \leq k$. Suppose that $l = 1$. Then $K_1 \cong \mathbb{Z}_2$ and hence $[Y : G] = 2$. Then $G \leq Y$, and hence $G \leq Y$. So $G < X$, which contradicts the assumption that $G$ is not normal in $X$. Thus $l > 1$, as in part (iii).

Since $|K_1 : C_{K_1}(x)| = 2$, there is $\tau_1 \in K_1$ such that $K_1 = \langle \tau_1 \rangle \times C_{K_1}(x)$. Let $x_1 = x^{-1} \tau_1$. Then $x_1 \neq 1$, $x_1^{\tau_1} = x_1^{-1}$ and $C_{K_1}(x) = C_{K_1}(x_1)$. Since $N$ is a minimal normal subgroup of $X = NU$, there are $\mu_1, \mu_2, \ldots, \mu_k \in U$ such that $N = \langle x_1^{\mu_1} \rangle \times \cdots \times \langle x_1^{\mu_k} \rangle$. Let $x_i = x_1^{\mu_i}$ and
Let \( \tau_i = x_i^\mu \), where \( i = 1, 2, \ldots, k \). Then \( \mathbb{Z}_2^{k-1} \cong (\mathbb{C}K_1(x_1))^{\mu_i} = \mathbb{C}K_1(x_i) \), and \( K_1 = K_1^{\mu_i} = (\tau_i) \times \mathbb{C}K_1(x_i) \). Further, \( x_i^{\tau_i} = x_i^{\tau_i^\mu} = (x_i^{-1})^\mu = x_i^{-1} \), and hence \( \langle x_i, \tau_i \rangle \cong D_{2p} \), as in part (iv).

Now \( N \cong \mathbb{Z}_k^p \) for an odd prime \( p \) and an integer \( k > 1 \). Suppose that \( X \) has a minimal normal subgroup \( L \neq N \). Then \( N \cap L = 1 \), and \( LK/K \triangleleft X/K \cong \mathbb{Z}_m \) or \( D_{2m} \). It follows that either \( L \leq K \), or \( L \) is cyclic and hence \( |L| \) is an odd prime. If \( L \leq K \), then \( L \) is a 2-group, it is not possible. Hence \( L \) is cyclic. It follows that \( L \) is intransitive and semiregular on \( V \). Then \( \Gamma_L \) is a cycle, and hence \( N \) is isomorphic a subgroup of \( \text{Aut}\Gamma_L \). It follows that \( N \) is cyclic, which is a contradiction. Thus \( N \) is the unique minimal normal subgroup of \( X \), as in part (v). \( \square \)

5.3. **Proof of Theorem 1.1.** If \( G \triangleleft X \), then by Lemma 2.3, we have \( X_1 \leq D_3 \). Thus by Lemma 3.1, \( S = \{a, a^{-1}, a^\tau, (a^\tau)^{-1}\} \) for some involution \( \tau \in \text{Aut}(G) \), as in Theorem 1.1 (1).

We assume that \( G \) is not normal in \( X \) in the following. Let \( M \triangleleft X \) be maximal subject to that \( \Gamma \) is a normal cover of \( \Gamma_M \). By Lemma 2.2, \( M \) is semiregular on \( V\Gamma \) and equals the kernel of \( X \) acting on \( V\Gamma_M \). Thus, setting \( Y = X/M \) and \( \Sigma = \Gamma_M \), \( \Sigma \) is \( Y \)-edge-transitive. Since \( |M| \) is odd, by Lemma 2.3, we have \( M \leq G \). Therefore, \( \Sigma \) is a \( Y \)-edge-transitive Cayley graph of \( G/M \), as in Theorem 1.1 (2).

We note that for the normal subgroup defined in the previous paragraph, we have that \( G \leqslant X \) if and only if \( G/M \triangleleft X/M \). Thus, to complete the proof of Theorem 1.1, we only need to deal with the case where \( M = 1 \), that is, \( \Gamma \) has no non-trivial normal quotients of valency 4. Let \( N \) be a minimal normal subgroup of \( X \). If \( N \) is intransitive on \( V\Gamma \), then by Lemmas 5.5 and 5.6, part (3) of Theorem 1.1 occurs. If \( N \) is transitive on \( V\Gamma \), then by Lemmas 5.2-5.3, Theorem 1.1 (4) occurs. \( \square \)

6. **Proof of Theorem 1.4**

Let \( p \) be an odd prime, and let \( k > 1 \) be an odd integer. Let \( m \) be the largest odd divisor of \( p^k - 1 \), and let

\[
G = N \rtimes \langle g \rangle = \mathbb{Z}_p^k \times \mathbb{Z}_m < \text{AGL}(1, p^k).
\]

It is easily shown that \( \langle g \rangle \) acts by conjugation transitively on the set of subgroups of \( N \) of order \( p \). We first construct a family of Cayley graphs of valency 4 of the group \( G \).

**Construction 6.1.** Let \( i \) be such that \( 1 \leq i \leq m - 1 \), and let \( a \in N \setminus \{1\} \). Let

\[
S_i = \{ag^i, a^{-1}g^i, (ag^i)^{-1}, (a^{-1}g^i)^{-1}\},
\]

\( \Gamma_i = \text{Cay}(G, S_i) \).

The following lemma gives some basic properties about \( G \) and \( \Gamma_i \).

**Lemma 6.2.** Let \( G \) be the group and let \( \Gamma_i \) be the graphs defined above. Then we have the following statements:

(i) \( \text{Aut}(G) = \text{AGL}(1, p^k) \cong \mathbb{Z}_p^k \rtimes \text{GL}(1, p^k) \);

(ii) \( \Gamma_i \) is edge-transitive, and \( \Gamma_i \) is connected if and only if \( i \) is coprime to \( m \);

(iii) \( \Gamma_i \cong \Gamma_{m^{-1}} \), and if \( np^i \equiv j \pmod{m} \), then \( \Gamma_i \cong \Gamma_j \).

*Proof.* See [4, Proposition 12.10] for part (i).

Since \( \text{Aut}(G) = \text{AGL}(1, p^k) \) and \( G < \text{AGL}(1, p^k) \), there is an automorphism \( \tau \in \text{Aut}(G) \) such that \( a^\tau = a^{-1} \) and \( g^\tau = g \). Thus \( S_i^\tau = S_i \) and \( (ag^i)^\tau = a^{-1}g^i \).
and \((ag')^{-1}r = (a^{-1}g')^{-1}\). It follows that \(\Gamma_i\) is edge-transitive. It is easily shown that \((ag, a^{-1}g') = G\) if and only if \((m, i) = 1\). Hence \(\Gamma_i\) is connected if and only if \(i\) is coprime to \(m\).

Since \(g\) normalises \(N\), there exists \(a' \in N\) such that \((ag')^{-1} = a'g^{-i}\) and \((a^{-1}g')^{-1} = (a')^{-1}g^{-i}\). Thus \(S_i = \{a'g^{-i}, (a')^{-1}g^{-i}, (a'g^{-i})^{-1}, ((a')^{-1}g^{-i})^{-1}\}\). Since \(\text{GL}(1, p^k)\) acts transitively on \(N \setminus \{1\}\), there exists an element \(\rho \in \text{Aut}(G)\) such that \((a')^{\rho} = a\) and \(g^{\rho} = g\). Thus \(S_i^\rho = \{ag^{m-i}, a^{-1}g^{m-i}, (ag^{m-i})^{-1}, (a^{-1}g^{m-i})^{-1}\} = S_j\). So \(\Gamma_i \cong \Gamma_{m-i}\).

Suppose that \(p^ri \equiv j \pmod{m}\) for some \(r \geq 0\). Noting that \(\Gamma_{m-j} \cong \Gamma_j\), we may assume that \(p^ri \equiv j \pmod{m}\). Since \(g \in \text{GL}(1, p^k)\), there exists \(\theta \in \Gamma\text{L}(1, p^k)\) such that \(\theta\) normalises \(N\) and \(g^{\theta} = g^\rho\). Thus \(S_i^\theta = \{a'g^{r-i}, a^{-1}g^{r-i}, (a'g^{r-i})^{-1}, (a^{-1}g^{r-i})^{-1}\}\), where \(a' = a^{\theta} \in N\). Since \(\text{GL}(1, p^k)\) is transitive on \(N \setminus \{1\}\) and fixes \(g\), there exists \(c \in \text{GL}(1, p^k)\) such that \((S_i^\theta)^c = S_j\), and so \(\Gamma_i \cong \Gamma_j\).

In the rest of this section, we aim to prove that every connected edge-transitive Cayley graph of \(G\) of valency 4 is isomorphic to some \(\Gamma_i\), so completing the proof of Theorem 1.4.

Let \(\Gamma = \text{Cay}(G, S)\) be connected, edge-transitive and of valency 4. We will complete the proof of Theorem 1.4 by a series of steps, beginning with determining the automorphism group \(\text{Aut}\Gamma\).

**Step 1.** \(G\) is normal in \(\text{Aut}\Gamma\), and \(\text{Aut}\Gamma = G \times \text{Aut}(G, S)\).

Suppose that \(G\) is not normal in \(\text{Aut}\Gamma\). Since \(N\) is the unique minimal normal subgroup of \(G\), it follows from Theorem 1.1 that either part (3) of Theorem 1.1 occurs with \(X = \text{Aut}\Gamma\), or \(\Gamma_X\) is a Cayley graph of \(G/N\) isomorphic to one of the graphs in part (4) of Theorem 1.1. Assume that the later case holds. Then \(G/N \cong \mathbb{Z}_a \times \mathbb{Z}_b \times \mathbb{Z}_c \times \mathbb{Z}_d, \mathbb{Z}_e \times \mathbb{Z}_f\). Therefore, as \(G/N \cong \mathbb{Z}_m\), we have that \(G/N \cong \mathbb{Z}_m \cong \mathbb{Z}_5\). By definition, \(m = 5\) is the largest odd divisor of \(p^k - 1\), which is not possible since \(p\) is an odd prime and \(k > 1\) is odd. Thus the former case occurs, and \(\text{Aut}\Gamma = N \rtimes \langle (H \times \langle g \rangle), O \rangle \cong \mathbb{Z}_p^k \times \langle (\mathbb{Z}_2^l \times \mathbb{Z}_m), \mathbb{Z}_n \rangle\), satisfying the properties in part (3) of Theorem 1.1. In particular, \(2 \leq l \leq k\), and \(C_H(N) = 1\).

By Theorem 1.1 (3), there exist \(\tau_0 \in H \setminus \{1\}\) and \(z_0 \in N\) such that \(H = \langle \tau_0 \rangle \times C_H(z_0)\). It follows that for each \(\sigma \in H\), we have \(z_0^\sigma = z_0\) or \(z_0^{-1}\). Since \(g\) normalises \(H\) and \(\langle g \rangle\) acts transitively on the set of subgroups of order \(p\), it follows that for each \(x \in N\) and each \(\sigma \in H\), we have \(x^\sigma = x\) or \(x^{-1}\).

Suppose that there exist \(x_1, x_2 \in N \setminus \{1\}\) such that \(x_1^\sigma = x_1\) and \(x_2^\sigma = x_2^{-1}\). Then \((x_1x_2)^\sigma = x_1x_2^{-1}\), which equals neither \(x_1x_2\) nor \((x_1x_2)^{-1}\), a contradiction. Thus, as \(\sigma\) does not centralise \(N\), we have \(x^\sigma = x^{-1}\) for all \(x \in N\). Since \(H \cong \mathbb{Z}_2^l\) with \(l \geq 2\), there exists \(\tau \in H \setminus \{\sigma\}\). Then similarly, \(\tau\) inverts all elements of \(N\), that is, \(x^\tau = x^{-1}\) for all elements \(x \in N\). However, now \(x^\sigma \tau = x\) for all \(x \in N\), and hence \(\sigma \tau \in C_H(N) = 1\), which is a contradiction.

Therefore, \(G\) is normal in \(\text{Aut}\Gamma\), and by Lemma 2.3, we have that \(\text{Aut}\Gamma = G \times \text{Aut}(G, S)\).

**Step 2.** \(\text{Aut}\Gamma = G \times \langle (\sigma) = \mathbb{Z}_p^k \times \langle (\sigma) \times \langle f \rangle \rangle \cong N \times \mathbb{Z}_m \cong G \times \mathbb{Z}_2,\) and \(S = \{a^f, a^{-1}f, (a^{-1}f)^{-1}, (a^f)^{-1}\}\) where \(a \in N\) and \(f \in G\) has order \(m\) such that \(a^2 = a^{-1}\); in particular, \(\Gamma\) is not arc-transitive.
By Lemma 6.2, we have $\text{Aut}(G) \cong \text{AGL}(1, p^k) \cong N \rtimes (Z_{q^r-1} \times Z_q)$. Since $k$ is odd, $\text{Aut}(G)$ has a cyclic Sylow 2-subgroup, and thus all involutions of $\text{Aut}(G)$ are conjugate. It is easily shown that every involution of $\text{Aut}(G)$ inverts all elements of $N$. Since $\Gamma$ is edge-transitive and $\text{Aut} \Gamma = G \rtimes \text{Aut}(G, S)$, $\text{Aut}(G, S)$ has even order. On the other hand, since $G$ is of odd order, by Lemma 2.3, we have that $\text{Aut}(G, S)$ is isomorphic to a subgroup of $S_N$. Further, since a Sylow 2-subgroup of $\text{Aut}(G)$ is cyclic, we have that $\text{Aut}(G, S) = \langle \sigma \rangle \cong \mathbb{Z}_2$ or $\mathbb{Z}_4$. It follows that $\sigma$ fixes an element of $G$ of order $m$, say $f \in G$ such that $o(f) = m$ and $f^m = f$. Then $G = N \rtimes \langle f \rangle$, and $X = \text{Aut} \Gamma = G \rtimes \langle f, \sigma \rangle$.

Since $\Gamma$ is connected, $\langle \sigma \rangle = G$ and $\text{Aut}(G, S)$ is faithful on $S$. Hence we may write $S = \{x, y, x^{-1}, y^{-1}\}$ such that either $o(\sigma) = 2$ and $(x, y)^\sigma = (y, x)$, or $o(\sigma) = 4$ and $(x, y)^\sigma = (y, x^{-1})$, refer to Lemma 3.1. Now $x = af^i$, where $a \in N$ and $i$ is an integer. Suppose that $o(\sigma) = 4$. Then $y = x^\sigma = (af^i)^\sigma = a^\sigma f^i$, and $a f^{-i} = f^{-i} a^{-1} = (af^i)^{-1} = x^{-1} = x^\sigma = a^\sigma f = a^{-1} f^i$. It follows that $f^{2i} = 1$, and since $f$ has odd order, $f^i = 1$. Thus $x = a$ and $y = x^\sigma = a^\sigma$, belonging to $N$, and so $\langle S \rangle \leq N < G$, which is a contradiction. Thus $\sigma$ is an involution, and so $(x, y)^\sigma = (y, x)$, $x = af^i$, and $y = x^\sigma = a^\sigma f^i = a^{-1} f^i$. In particular, $\Gamma$ is not arc-transitive, and $S = \{af^i, a^{-1} f^i, (af^i)^{-1}, (a^{-1} f^i)^{-1}\}$.

Step 3. $\Gamma \cong \Gamma_j$ for some $j$ such that $1 \leq j \leq \frac{m-1}{2}$ and $(j, m) = 1$.

By Step 2, we may assume that $\text{Aut} \Gamma = N \rtimes \langle f, \sigma \rangle \leq \text{AGL}(1, p^k)$. Since $g \in G$ has order $m$, it follows from Hall’s theorem that there exists $b \in N$ such that $g^b \in \langle f, \sigma \rangle$. So $f^{b^{-1}} = g^r$ for some integer $r$. Let $\tau = \sigma^{b^{-1}}$. Then $(g, \tau) \cong \langle f, \sigma \rangle \cong \mathbb{Z}_{2m}$, and $G = N \rtimes \langle g \rangle$ and $\text{Aut} \Gamma = N \rtimes \langle g, \tau \rangle$. Further, $T := S^{b^{-1}} = (ag^r, a^{-1} g^r, (ag^r)^{-1}, (a^{-1} g^r)^{-1})$. Let $j \equiv ir \pmod m$ and $1 \leq j \leq m - 1$. Then $T = (ag^j, a^{-1} g^j, (ag^j)^{-1}, (a^{-1} g^j)^{-1})$, and $(j, m) = 1$ as $\Gamma \cong \text{Cay}(G, T)$ is connected. By Lemma 6.2 (iii), $\Gamma_i \cong \Gamma_{m-j}$, and so the statement in Step 3 is true.

Step 4. Let $\Gamma_i$ and $\Gamma_j$ be as in Construction 6.1 with $(i, m) = (j, m) = 1$. Then $\Gamma_i \cong \Gamma_j$ if and only if $p^i \equiv j \pmod m$ for some $r \geq 0$.

By Lemma 6.2, we only need to prove that if $\Gamma_i \cong \Gamma_j$ then $p^i \equiv j \pmod m$ for some $r \geq 0$. Thus suppose that $\Gamma_i \cong \Gamma_j$. By Step 2, we have $\text{Aut} \Gamma_i \cong \text{Aut} \Gamma_j \cong G \rtimes \mathbb{Z}_2$. It follows that $\Gamma_i$ and $\Gamma_j$ are so-called CI-graphs, see [13, Theorem 6.1]. Thus $S_i = S_j = \Gamma$ for some $\gamma \in \text{Aut}(G)$. Since $N$ is a characteristic subgroup of $G$, this $\gamma$ induces an automorphism of $G/N = \langle \gamma \rangle$ such that $\gamma^i = S_j \in S_i$, where $S_i = \{\gamma^i, \gamma^{-i}\}$ and $S_j = \{\gamma^j, \gamma^{-j}\}$ are the images of $S_i$ and $S_j$ under $G \to G/N$, respectively. Thus $\langle \gamma^i \rangle = \langle \gamma^j \rangle$ or $\langle \gamma^i \rangle = \langle \gamma^{-j} \rangle$. Since $\text{Aut}(G) = \text{AGL}(1, p^k)$, it follows that for each element $\rho \in \text{Aut}(G)$, we have $g^\rho = cg^{\rho^c}$ for some $c \in N$ and some integer $r$ with $0 \leq r \leq k-1$. Thus $\langle \gamma^i \rangle = \langle \gamma^{r+1} \rangle$, and hence $p^i \equiv j \pmod m$.

This completes the proof of Theorem 1.4. \hfill $\square$

References


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