Brenti's open problem on the real-rootedness of q-Eulerian polynomials of type D

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Abstract. We prove that, for any positive q, the q-Eulerian polynomial of type D has only real zeros. This settles an open problem of Brenti in 1994. For $q = 1$, our result reduces to the real-rootedness of the Eulerian polynomials of type D, which were originally conjectured by Brenti and recently proved by Savage and Visontai.

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1 Introduction

The main objective of this paper is to study the real-rootedness of the q-Eulerian polynomial of type D. Brenti [4] asked whether there is any $q > 0$ such that the q-Eulerian polynomial of type D has only real zeros. By using the theory of $\mathbf{s}$-Eulerian polynomials [10], we present an affirmative answer to Brenti's problem.

Let us first give an overview of Brenti's problem. Assume that the reader is familiar with Coxeter groups and root systems, see [1, 7]. Let W be a finite Coxeter group generated by $s_1, s_2, \ldots, s_n$. The length of each $\sigma \in W$ is defined as the number of generators in one of its reduced expressions, denoted $\ell(\sigma)$. Let

$$\operatorname{Des}\sigma = \{i \,:\, 1 \le i \le n \text{ if } \ell(\sigma s_i) < \ell(\sigma)\}, \quad \operatorname{des}\sigma = |\operatorname{Des}\sigma|.$$

Each element i in $\operatorname{Des}\sigma$ is said to be a descent of σ. The descent polynomial for a finite Coxeter group W is defined by

$$W(x) \;=\; \sum_{\sigma\in W} x^{\operatorname{des}\sigma}.$$

If W is a Coxeter group of type A_n (or $B_n, D_n, \ldots$), by abuse of notation, we shall write the corresponding descent polynomials as $A_n(x)$ (resp. $B_n(x), D_n(x), \ldots$) instead of $W(x)$. It is known that $A_n(x)$ is the classical Eulerian polynomial, which is known to be real-rooted. Brenti [4] studied the problem of whether $W(x)$ has only real zeros for any general finite Coxeter group. By a simple argument, he showed that it is enough to check the real-rootedness of $W(x)$ for irreducible finite Coxeter groups. Brenti [4] showed that $W(x)$ has only real zeros for any irreducible finite Coxeter group except $D_n(x)$. Recently, Savage and Visontai [10] proved that $D_n(x)$ has only real zeros.

Brenti [4] also introduced the q-analogue of $B_n(x)$ and $D_n(x)$. First, regard the Coxeter group B_n as the set of signed permutations of the set $[n]$, i.e., each element $\sigma \in B_n$ is a permutation of $\{-n, \ldots, -1, 1, \ldots, n\}$ satisfying $\sigma(-i) = -\sigma(i)$ for $1 \leq i \leq n$. Then write σ in one-line notation $(\sigma_1, \sigma_2, \ldots, \sigma_n)$, where $\sigma_i = \sigma(i)$. Let $\operatorname{neg}\sigma$ be the negative numbers in $(\sigma_1, \sigma_2, \ldots, \sigma_n)$, and let $\operatorname{des}_B \sigma = |\operatorname{Des}_B \sigma|$, where

$$\operatorname{Des}_B \sigma = \{0 : \text{if } \sigma_1 < 0\} \cup \{i \in [n-1] : \sigma_i > \sigma_{i+1}\}.$$

Brenti introduced the following q-analogue of $B_n(x)$:

$$B_n(x; q) = \sum_{\sigma \in B_n} q^{\operatorname{neg}\sigma} x^{\operatorname{des}_B \sigma}, \tag{1}$$

which reduces to $B_n(x)$ when $q = 1$. Brenti proved that, for any $q \geq 0$, the polynomial $B_n(x; q)$ has only real zeros, and thus established the real-rootedness of $B_n(x)$.

Analogous to the case of type B_n, Brenti gave a combinatorial interpretation of $D_n(x)$ as certain generating function over the set of even signed permutations of $[n]$. Given an even signed permutation σ with one-line notation $(\sigma_1, \sigma_2, \ldots, \sigma_n)$, let $\operatorname{neg}_D \sigma$ be the negative numbers in $(\sigma_2, \ldots, \sigma_n)$, and let $\operatorname{des}_D \sigma = |\operatorname{Des}_D \sigma|$, where

$$\operatorname{Des}_D \sigma = \{0 : \text{if } \sigma_1 + \sigma_2 < 0\} \cup \{i \in [n-1] : \sigma_i > \sigma_{i+1}\}.$$

Brenti introduced the following q-analogue of $D_n(x)$,

$$D_n(x; q) = \sum_{\sigma \in D_n} q^{\operatorname{neg}_D \sigma} x^{\operatorname{des}_D \sigma}, \tag{2}$$

which reduces to $A_{n-1}(x)$ when $q = 0$, and reduces to $D_n(x)$ when $q = 1$. Brenti proposed the following problem.

Problem 1.1 ([4]). *Whether there is any $q > 0$ such that $D_n(x; q)$ has only real zeros?*

In this paper, we give an affirmative answer to the above problem. Our proof was inspired by Savage and Visontai's proof of Brenti's conjecture on the real-rootedness of $D_n(x)$. In the next section, we shall give an overview of their proof. In Section 3, we shall give a proof of the real-rootedness of $D_n(x; q)$ for any $q > 0$.

2 The real-rootedness of $D_n(x)$

The aim of this section is to give an overview of Savage and Visontai's proof of the real-rootedness of $D_n(x)$. Their novel proof is based on the theory of **s**-inversion sequences and the theory of compatible polynomials.

Recall that, given a sequence $\mathbf{s} = (s_1, s_2, \ldots)$ of positive integers, an n-dimensional $\mathbf{s}$-inversion sequence is a sequence $\mathbf{e} = (e_1, \ldots, e_n) \in \mathbb{N}^n$ such that $e_i < s_i$ for each $1 \leq i \leq n$. Denote the set of n-dimensional $\mathbf{s}$-inversion sequences by $\mathfrak{I}_n^{(\mathbf{s})}$. For $\mathbf{s} = (2, 4, 6, \ldots)$, Savage and Visontai introduced a statistic asc_D on inversion sequences $\mathbf{e} = (e_1, \ldots, e_n) \in \mathfrak{I}_n^{(\mathbf{s})}$, which is defined as the cardinality of the following set

$$\mathrm{Asc}_D\, \mathbf{e} = \{i \in [n-1] : \frac{e_i}{i} < \frac{e_{i+1}}{i+1}\} \cup \{0 : \text{if } e_1 + e_2/2 \geq 3/2\}. \tag{3}$$

In this way, the polynomial $D_n(x)$ can be interpreted as the generating function of the statistic asc_D over $\mathfrak{I}_n^{(2,4,6,\ldots)}$, precisely,

$$2D_n(x) = \sum_{e \in \mathfrak{I}_n^{(2,4,6,\ldots)}} x^{\mathrm{asc}_D\, e}.$$

Let $T_n(x) = 2D_n(x)$. Clearly, $T_n(x)$ is real-rooted if and only if $D_n(x)$ is real-rooted. To prove the real-rootedness of $T_n(x)$, Savage and Visontai introduced the following refinement of $T_n(x)$:

$$T_{n,i}(x) \;=\; \sum_{\mathbf{e} \in \mathfrak{I}_n^{(2,4,6,\ldots)}} \chi(e_n = i)\, x^{\mathrm{asc}_D\, \mathbf{e}}\,, \tag{4}$$

where $\chi(\varphi)$ is 1 if the statement φ is true and 0 otherwise. Note that

$$T_n(x) = \sum_{i=0}^{2n-1} T_{n,i}(x).$$

They showed that, for any $n \geq 3$ and $0 \leq i \leq 2n-1$, these refined polynomials satisfy the following simple recurrence relation:

$$T_{n,i}(x) = x \sum_{j=0}^{\lceil \frac{n-1}{n} i \rceil - 1} T_{n-1,j}(x) + \sum_{j=\lceil \frac{n-1}{n} i \rceil}^{2n-3} T_{n-1,j}(x), \tag{5}$$

where $\lceil t \rceil$ represents the smallest integer larger than or equal to t.

The theory of compatible polynomials was developed by Chudnovsky and Seymour [5]. Suppose that $f_1(x), \ldots, f_m(x)$ are polynomials with real coefficients. These polynomials are said to be compatible if, for any nonnegative numbers $c_1, \ldots, c_m$, the polynomial

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_m f_m(x)$$

has only real zeros, and they are said to be pairwise compatible if, for all $1 \leq i < j \leq m$, the polynomials $f_i(x)$ and $f_j(x)$ are compatible. These concepts are defined by Chudnovsky and Seymour [5] in their study of the real-rootedness of independence polynomials of claw-free graphs. The following remarkable lemma shows that how the two concepts are related.

Lemma 2.1 ([5, 2.2])**.** *The polynomials $f_1(x), \ldots, f_m(x)$ with positive leading coefficients are pairwise compatible if and only if they are compatible.*

Savage and Visontai inductively proved that the polynomials $T_{n,i}(x)$ satisfying the recurrence relation (5) are compatible, and hereby obtained the real-rootedness of $T_n(x)$. As a result, Savage and Visontai proved the following result, a long-standing conjecture of Brenti.

Theorem 2.2 ([10, Theorem 3.15])**.** *For any finite Coxeter group W, the descent polynomial $W(x)$ has only real zeros. In particular, the polynomial $D_n(x)$ has only real zeros.*

To prove the compatibility of $T_{n,i}(x)$, Savage and Visontai studied the following transformation between two sequences of polynomials. Given a sequence of polynomials $(f_1(x), \ldots, f_m(x))$ with real coefficients, define another sequence of polynomials $(g_1(x), \ldots, g_{m'}(x))$ by the equations

$$g_k(x) = \sum_{\ell=1}^{t_k-1} x f_\ell(x) + \sum_{\ell=t_k}^{m} f_\ell(x), \quad \text{for } 1 \le k \le m', \tag{6}$$

where $1 \le t_1 \le \ldots \le t_{m'} \le m+1$. Savage and Visontai obtained the following useful result.

Theorem 2.3 ([10, Theorem 2.3])**.** *Given a sequence of real polynomials $f_1(x), \ldots, f_m(x)$ with positive leading coefficients, let $g_1(x), \ldots, g_{m'}(x)$ be defined as in (6). If, for all $1 \le i < j \le m$,*

(1) $f_i(x)$ and $f_j(x)$ are compatible, and

(2) $xf_i(x)$ and $f_j(x)$ are compatible,

then, for all $1 \le i < j \le m'$,

(1') $g_i(x)$ and $g_j(x)$ are compatible, and

(2') $xg_i(x)$ and $g_j(x)$ are compatible.

As pointed out by Savage and Visontai, the description of the above theorem can be simplified by using the notion of interlacing if the polynomials $f_1(x), \ldots, f_m(x)$ have only non-negative coefficients. Given two real-rooted polynomials $f(x)$ and $g(x)$ with positive leading coefficients, let $\{u_i\}$ be the set of zeros of $f(x)$ and $\{v_j\}$ the set of zeros of $g(x)$. We say that $g(x)$ interlaces $f(x)$, denoted $g(x) \preceq f(x)$, if either $\deg f(x) = \deg g(x) = n$ and

$$v_n \le u_n \le v_{n-1} \le \cdots \le v_2 \le u_2 \le v_1 \le u_1, \tag{7}$$

or $\deg f(x) = \deg g(x) + 1 = n$ and

$$u_n \leq v_{n-1} \leq \cdots \leq v_2 \leq u_2 \leq v_1 \leq u_1. \tag{8}$$

If all inequalities in (7) or (8) are strict, then we say that $g(x)$ strictly interlaces $f(x)$, denoted $g(x) \prec f(x)$. Interlacing polynomials have the following properties.

Parallel to the concept of pairwise compatibility, we say that a sequence of real polynomials $(f_1(x), \ldots, f_m(x))$ with positive leading coefficients is mutually interlacing if $f_i(x) \preceq f_j(x)$ for all $1 \leq i < j \leq m$. As far as we know, this definition was first introduced by Fisk [6]. It should be mentioned that the notations of interlacing and mutual interlacing adopted in this paper are a little different from Fisk's. Interlacing of two polynomials is closely related to compatibility in the sense of the following, due to Wagner [11].

Theorem 2.4 ([11, Lemma 3.4])**.** *Suppose that $f(x)$ and $g(x)$ are two polynomials with non-negative coefficients. Then the following statements are equivalent:*

(1) $f(x)$ interlaces $g(x)$;

(2) $f(x)$ and $g(x)$ are compatible, and $xf(x)$ and $g(x)$ are compatible.

With the above theorem, we now recall an alternative description of Theorem 2.3 when all the polynomials involved have only non-negative coefficients. It is our feeling that, for polynomials with non-negative coefficients, it is more convenient to work with interlacing than with compatibility. The following theorem is efficient for proving the mutual interlacing of the refined Eulerian polynomials satisfying recurrence relations as in (5).

Theorem 2.5 ([10, Theorem 2.4])**.** *Given a sequence of polynomials $(f_1(x), \ldots, f_m(x))$ with non-negative coefficients, let $g_1(x), \ldots, g_{m'}(x)$ be polynomials defined as in (6). If $(f_1(x), \ldots, f_m(x))$ is mutually interlacing, then so is $(g_1(x), \ldots, g_{m'}(x))$.*

Recently, Brändén [3] gave a generalization of Theorem 2.5. We would like to point out that Theorem 2.5 is also important for proving the real-rootedness of $D_n(x; q)$.

3 The real-rootedness of $D_n(x; q)$

In this section we aim to prove the real-rootedness of the q-Eulerian polynomials $D_n(x; q)$ for any positive q. Brenti [4] noted that the type D statistics neg_D and des_D can be extended to all signed permutations, and proved that

$$(1+q)D_n(x; q) = \sum_{\sigma \in B_n} q^{\text{neg}\,\sigma} x^{\text{des}_D \sigma}.$$

Let

$$T_n(x;q) = \sum_{\sigma \in B_n} q^{\operatorname{neg}\sigma} x^{\operatorname{des}_D \sigma}.$$

It is obvious that $T_n(x;q)$ has only real zeros if and only if $D_n(x;q)$ has only real zeros. In the following we shall focus on the real-rootedness of $T_n(x;q)$.

To prove that $T_n(x;q)$ has only real zeros for positive q, let us first give a proper refinement of $T_n(x;q)$. To this end, we need to interpret $T_n(x;q)$ as the generating functions of certain statistics over inversion sequences. This could be easily done by using a map $\psi : B_n \to \mathfrak{I}_n^{(2,4,\dots)}$ established by Savage and Visontai [10]. Precisely, a signed permutation $\sigma = (\sigma_1, \dots, \sigma_n) \in B_n$ under ψ is mapped to an inversion sequence $(e_1, \dots, e_n) \in \mathfrak{I}_n^{(2,4,\dots)}$ given by, for $1 \le i \le n$,

$$e_i = \begin{cases} t_i & \text{if} \quad \sigma_i > 0\,, \\ 2i - t_i - 1 & \text{if} \quad \sigma_i < 0, \end{cases}$$

where $t_i = |\{j \in [i-1] : |\sigma_j| > |\sigma_i|\}|$. The map ψ satisfies the following properties.

Lemma 3.1 ([10, Theorem 3.12])**.** *The map $\psi : B_n \to \mathfrak{I}_n^{(2,4,\dots)}$ is a bijection satisfying the following properties:*

(1) $\sigma_i < 0$ if and only if $e_i \ge i$, for any $1 \le i \le n$;

(2) $\sigma_1 + \sigma_2 < 0$ if and only if $e_1 + \frac{e_2}{2} \ge \frac{3}{2}$;

(3) $\sigma_i > \sigma_{i+1}$ if and only if $\frac{e_i}{i} < \frac{e_{i+1}}{i+1}$ for $1 \le i \le n-1$;

(4) $\sigma_{n-1} + \sigma_n > 0$ if and only if $\frac{e_{n-1}}{n-1} + \frac{e_n}{n} < \frac{n-1}{n}$.

For an inversion sequence $\mathbf{e} = (e_1, \dots, e_n) \in \mathfrak{I}_n^{(2,4,\dots)}$, let

$$\operatorname{exc} \mathbf{e} = \sum_{i=1}^{n} \chi\,(e_i \ge i)\,.$$

By (1) of Lemma 3.1, we see that $\operatorname{exc} \mathbf{e} = \operatorname{neg} \psi^{-1}(\mathbf{e})$. By (2) and (3) of Lemma 3.1, we have $\operatorname{asc}_D \mathbf{e} = \operatorname{des}_D \psi^{-1}(\mathbf{e})$. The following result is therefore immediate.

Lemma 3.2. *For any $n \ge 2$, we have*

$$T_n(x;q) = \sum_{\mathbf{e} \in \mathfrak{I}_n^{(2,4,\dots)}} q^{\operatorname{exc} \mathbf{e}} x^{\operatorname{asc}_D \mathbf{e}}.$$

Now we can give a refinement of $T_n(x;q)$. Let

$$T_{n,i}(x;q) \;=\; \sum_{\mathbf{e}\in\mathfrak{I}_n^{(2,4,6,\dots)}} \chi(e_n=i)\, q^{\operatorname{exc}\mathbf{e}} x^{\operatorname{asc}_D \mathbf{e}}.$$

It is clear that

$$T_n(x;q)=\sum_{i=0}^{2n-1} T_{n,i}(x;q), \qquad T_{n,i}(x;1)=T_{n,i}(x).$$

These polynomials $T_{n,i}(x;q)$ satisfy the following recurrence relation.

Lemma 3.3. *For $n\geq 3$ and $0\leq i\leq 2n-1$, we have*

$$T_{n,i}(x;q) \;=\; q^{\chi(i\geq n)}\Bigg(x\sum_{j=0}^{\lceil\frac{n-1}{n}i\rceil-1} T_{n-1,j}(x;q)+\sum_{j=\lceil\frac{n-1}{n}i\rceil}^{2n-3} T_{n-1,j}(x;q)\Bigg), \tag{9}$$

with the initial conditions that $T_{2,0}(x;q)=1+q$, $T_{2,1}(x;q)=(1+q)x$, $T_{2,2}(x;q)=(q+q^2)x$, and $T_{2,3}(x;q)=(q+q^2)x^2$. In particular, $T_{n,0}(x;q) \;=\; T_{n-1}(x;q)$.

Proof. For the initial values, it is easy to verify. For $\mathbf{e}=(e_1,\dots,e_{n-1},i)\in\mathfrak{I}_n^{(2,4,6,\dots)}$, it is clear that

$$\operatorname{exc}\mathbf{e}=\operatorname{exc}(e_1,\dots,e_{n-1})+\chi(i\geq n)\,.$$

Moreover, we have that $n-1\in \operatorname{Asc}_D\mathbf{e}$ if and only if

$$\frac{e_{n-1}}{n-1}<\frac{i}{n}\,,$$

that is, $e_{n-1}<\frac{n-1}{n}i$. This completes the proof. ∎

To show the real-rootedness of $T_n(x;q)$, we further need to prove that the sequence $(T_{n,i}(x;q))_{i=0}^{2n-1}$ is mutually interlacing. With the above recurrence relation, it is desirable to give an induction proof as done by Savage and Visontai for the polynomials $T_{n,i}(x)$. For the basis step of the induction, Savage and Visontai showed that the sequence $(T_{4,i}(x))_{i=0}^{7}$ is mutually interlacing by numerical calculations. It is hoped that for any positive q the sequence $(T_{4,i}(x;q))_{i=0}^{7}$ is also mutually interlacing. In fact, this is true, as shown in Lemma 3.6. However, due to the additional parameter q, we can not directly follow the way of Savage and Visontai to verify the interlacing. To fix this, we shall use the Hermite–Biehler theorem and the Routh–Hurwitz criterion for stability of complex polynomials, as illustrated below.

The Hermite–Biehler theorem presents necessary and sufficient conditions for the stability of a polynomial in terms of certain interlacing conditions. Recall that a complex polynomial $p(z)$ is said to be Hurwitz stable (respectively, weakly Hurwitz stable) if

$p(z) \neq 0$ whenever $\Re(z) \geq 0$ (respectively, $\Re(z) > 0$), where $\Re(z)$ denotes the real part of z. Suppose that $p(z) = \sum_{k=0}^{n} a_{n-k} z^k$. Let

$$p^E(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} a_{n-2k} z^k \quad \text{and} \quad p^O(z) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n-1-2k} z^k. \tag{10}$$

The Hermite–Biehler theorem can be stated as follows. It establishes a connection between the interlacing property between $p^E(z)$ and $p^O(z)$ and the stability of $p(z)$.

Theorem 3.4 ([2, Theorem 4.1])**.** *Let $p(z)$ be a polynomial with real coefficients, and let $p^E(z)$ and $p^O(z)$ be defined as in (10). Suppose that $p^E(z)p^O(z) \not\equiv 0$. Then $p(z)$ is Hurwitz stable (resp. weakly Hurwitz stable) if and only if $p^E(z)$ and $p^O(z)$ have only negative (resp. nonpositive) zeros, and moreover $p^O(z) \prec p^E(z)$ (resp. $p^O(z) \preceq p^E(z)$).*

Therefore, to prove the mutual interlacing of the sequence $(T_{4,i}(x;q))_{i=0}^{7}$ for positive q, it suffices to show that the polynomial

$$zT_{4,i}(z^2;q) + T_{4,j}(z^2;q)$$

is weakly Hurwitz stable for any $0 \leq i < j \leq 7$. A useful criterion for determining stability was given by Hurwitz [8], which we shall explain below. Given a polynomial $p(z) = \sum_{k=0}^{n} a_{n-k} z^k$, for any $1 \leq k \leq n$ let

$$\Delta_k(p) = \det \begin{pmatrix} a_1 & a_3 & a_5 & \dots & a_{2k-1} \\ a_0 & a_2 & a_4 & \dots & a_{2k-2} \\ 0 & a_1 & a_3 & \dots & a_{2k-3} \\ 0 & a_0 & a_2 & \dots & a_{2k-r} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_k \end{pmatrix}_{k \times k}.$$

These determinants are known as the Hurwitz determinants of $p(z)$. Hurwitz showed that the stability of $p(z)$ is uniquely determined by the signs of $\Delta_k(p)$.

Theorem 3.5 ([8])**.** *Suppose that $p(z) = \sum_{k=0}^{n} a_{n-k} z^k$ is a real polynomial with $a_0 > 0$. Then $p(z)$ is Hurwitz stable if and only if the corresponding Hurwitz determinants $\Delta_k(p) > 0$ for any $1 \leq k \leq n$.*

The above result is usually called the Routh-Hurwitz stability criterion since it is equivalent to the Routh test, for more historical background see [9, p. 393]. With this criterion, we are able to prove the mutually interlacing propertpy of $(T_{4,i}(x;q))_{i=0}^{7}$.

Lemma 3.6. *For any positive q, the sequence $(T_{4,i}(x;q))_{i=0}^{7}$ is mutually interlacing.*

Proof. By Lemma 3.3, it is easy to compute that

$$\begin{aligned}
T_{4,0}(x;q) &= (q+1)\,(q^2x^3 + (4q^2+6q+1)x^2 + (q^2+6q+4)x + 1),\\
T_{4,1}(x;q) &= (q+1)\,((q^2+q)x^3 + (4q^2+6q+2)x^2 + (q^2+5q+4)x),\\
T_{4,2}(x;q) &= (q+1)\,((q^2+2q)x^3 + (4q^2+6q+4)x^2 + (q^2+4q+2)x),\\
T_{4,3}(x;q) &= (q+1)\,((q^2+3q+1)x^3 + (4q^2+6q+4)x^2 + (q^2+3q+1)x),\\
T_{4,4}(x;q) &= (q+1)\,(q(q^2+3q+1)x^3 + q(4q^2+6q+4)x^2 + q(q^2+3q+1)x),\\
T_{4,5}(x;q) &= (q+1)\,(q(2q^2+4q+1)x^3 + q(4q^2+6q+4)x^2 + q(2q+1)x),\\
T_{4,6}(x;q) &= (q+1)\,(q(4q^2+5q+1)x^3 + q(2q^2+6q+4)x^2 + q(q+1)x),\\
T_{4,7}(x;q) &= (q+1)\,(q^3x^4 + q(4q^2+6q+1)x^3 + q(q^2+6q+4)x^2 + qx).
\end{aligned}$$

Note that

$$T_{4,4}(x;q) = qT_{4,3}(x;q) \quad \text{and} \quad T_{4,7}(x;q) = qxT_{4,0}(x;q).$$

Thus, it suffices to show that $T_{4,i}(x;q) \preceq T_{4,j}(x;q)$ for any $q > 0$ and $i < j$ with $i, j \in \{0,1,2,3,5,6\}$. For the case of $q = 1$, the interlacing property has already been obtained by Savage and Visontai [10]. In the following we shall assume that $q \neq 1$.

By Theorem 3.4, we only need to prove that

$$T_{4,j}(z^2;q) + z\,T_{4,i}(z^2;q)$$

is weakly Hurwitz stable for any positive $q \neq 1$ and $i < j$ with $i, j \in \{0,1,2,3,5,6\}$. In fact, all these polynomials are Hurwitz stable up to a power of z. Let

$$C_{i,j}(z) = \frac{T_{4,j}(z^2;q) + z\,T_{4,i}(z^2;q)}{z^{m_{i,j}}(q+1)},$$

where $m_{i,j}$ is the largest nonnegative integer k such that

$$z^k \mid (T_{4,j}(z^2;q) + zT_{4,i}(z^2;q)).$$

We proceed to show that $C_{i,j}(z)$ is Hurwitz stable for any $i < j$ with $i, j \in \{0,1,2,3,5,6\}$. By Theorem 3.5, we only need to show that all the Hurwitz determinants of $C_{i,j}(z)$ are positive. It is easy to compute these Hurwitz determinants with the aid of a computer. We would like to mention that most of these determinants are nonzero polynomials of q with only nonnegative coefficients except for $(i,j) \in \{(0,1),(0,6),(1,6)\}$. Therefore, if $i < j$ and $(i,j) \notin \{(0,1),(0,6),(1,6)\}$, then the corresponding Hurwitz determinants of $C_{i,j}(z)$ must be positive for any positive q, hence establishing its Hurwitz stability. In the following, we shall separately check the sign of the Hurwitz determinants of $C_{i,j}(z)$ for $(i,j) \in \{(0,1),(0,6),(1,6)\}$.

For $(i,j) = (0,1)$, the testing polynomial is

$$\begin{aligned}
C_{0,1}(z) = {} & q^2z^6 + (q^2+q)z^5 + (4q^2+6q+1)z^4 + (4q^2+6q+2)z^3\\
& + (q^2+6q+4)z^2 + (q^2+5q+4)z + 1,
\end{aligned}$$

and the corresponding Hurwitz determinants are

$$\begin{aligned}
\Delta_1 &= q(q+1), & \Delta_2 &= q(4q^2+5q+1),\\
\Delta_3 &= 2q(q+1)^2(7q^2+4q+1), & \Delta_4 &= 4q(q+1)^2(3q^3+q^2+q+1),\\
\Delta_5 &= 12q(q+1)^3(q^2-1)^2, & \Delta_6 &= 12q(q+1)^3(q^2-1)^2.
\end{aligned}$$

It is obvious that they are all positive for any positive $q \neq 1$. This means that $C_{0,1}(z)$ is Hurwitz stable for any positive $q \neq 1$.

For $(i, j) = (0, 6)$, the testing polynomial is

$$\begin{aligned}
C_{0,6}(z) =& q^2z^6 + (4q^3+5q^2+q)z^5 + (4q^2+6q+1)z^4 + (2q^3+6q^2+4q)z^3\\
&+ (q^2+6q+4)z^2 + (q^2+q)z + 1,
\end{aligned}$$

and the corresponding Hurwitz determinants are

$$\begin{aligned}
&\Delta_1 = q(4q^2+5q+1), \qquad \Delta_2 = q(14q^4+38q^3+34q^2+11q+1),\\
&\Delta_3 = 4q^3(q+1)^2(3q^3+q^2+q+1),\\
&\Delta_4 = 2q^3(q+1)^2(6q^5+12q^4-12q^3-11q^2+10q+7),\\
&\Delta_5 = 12q^5(q+1)^3(q^2-1)^2, \quad \Delta_6 = 12q^5(q+1)^3(q^2-1)^2.
\end{aligned}$$

For any positive $q \neq 1$, it is clear that all Hurwitz determinants are positive except for Δ_4. For any $q > 0$, we can verify that $\Delta_4 > 0$ by numerical analysis with the aid of a computer. Thus, $C_{0,6}(z)$ is Hurwitz stable for any positive $q \neq 1$.

For $(i, j) = (1, 6)$, the testing polynomial is

$$\begin{aligned}
C_{1,6}(z) =& (q+1)qz^5 + (q+1)(4q^2+q)z^4 + (q+1)(4q+2)z^3\\
&+ (q+1)(2q^2+4q)z^2 + (q+1)(q+4)z + (q+1)q,
\end{aligned}$$

and the corresponding Hurwitz determinants

$$\begin{aligned}
\Delta_1 &= q(4q^2+5q+1), & \Delta_2 &= 2q(q+1)^2(7q^2+4q+1),\\
\Delta_3 &= 4q^2(q+1)^3(3q^3+q^2+q+1), & \Delta_4 &= 12q^2(q+1)^4(q^2-1)^2,\\
\Delta_5 &= 12q^3(q+1)^5(q^2-1)^2
\end{aligned}$$

are positive for any positive $q \neq 1$. Thus, $C_{1,6}(z)$ is Hurwitz stable for any positive $q \neq 1$.

Combining the above cases, we get the stability of $C_{i,j}(z)$ for any $i < j$ and $i, j \in \{0, 1, 2, 3, 5, 6\}$, which implies the mutual interlacing of the sequence $(T_{4,i}(x; q))_{i=0}^{7}$. This completes the proof. □

Now we can prove the mutual interlacing of $(T_{n,i}(x; q))_{i=0}^{2n-1}$ for general n.

Proposition 3.7. *For $n \geq 4$ and any positive q, the sequence of polynomials $(T_{n,i}(x; q))_{i=0}^{2n-1}$ is mutually interlacing.*

Proof. We use induction on n. When $n = 4$, the statement is true by Lemma 3.6. Note that $(T_{n,i}(x;q))_{i=0}^{2n-1}$ is mutually interlacing if and only if $(q^{-\chi(i\geq n)}T_{n,i}(x;q))_{i=0}^{2n-1}$ is mutually interlacing. So, it suffices to prove that the sequence of polynomials

$$\left(q^{-\chi(i\geq n)}T_{n,i}(x;q)\right)_{i=0}^{2n-1}$$

is mutually interlacing. By the recurrence (9), it is easy to see that the polynomials $q^{-\chi(i\geq n)}T_{n,i}(x;q)$ satisfy the conditions required in Theorem 2.5. By induction, the desired result immediately follows. This completes the proof. □

The main result of this section is as follows, which gives an affirmative answer to Problem 1.1.

Theorem 3.8. *For $n \geq 2$ and any positive q, the polynomial $D_n(x;q)$ has only real zeros.*

Proof. We shall prove that for $n \geq 2$ and $q > 0$ the polynomial $T_n(x;q)$ has only real zeros. This is true for $n = 2$, since $T_2(x;q) = (1+q)(1+x)(1+qx)$. By Proposition 3.7, we know that $(T_{n,i}(x;q))_{i=0}^{2n-1}$ is mutually interlacing for $n \geq 4$ and $q > 0$. This also implies that $T_{n,0}(x;q)$ is real-rooted for any $n \geq 4$ and positive q. Then by the equality $T_{n,0}(x;q) = T_{n-1}(x;q)$ in Lemma 3.3, we obtain the desired result for $n \geq 3$. This completes the proof. □

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