The Zrank Conjecture and Restricted Cauchy Matrices

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March 13, 2005

Abstract. The rank of a skew partition λ/μ, denoted $rank(\lambda/\mu)$, is the smallest number r such that λ/μ is a disjoint union of r border strips. Let $s_{\lambda/\mu}(1^t)$ denote the skew Schur function $s_{\lambda/\mu}$ evaluated at $x_1 = \cdots = x_t = 1$, $x_i = 0$ for $i > t$. The zrank of λ/μ, denoted $zrank(\lambda/\mu)$, is the exponent of the largest power of t dividing $s_{\lambda/\mu}(1^t)$. Stanley conjectured that $rank(\lambda/\mu) = zrank(\lambda/\mu)$. We show the equivalence between the validity of the zrank conjecture and the nonsingularity of restricted Cauchy matrices. In support of Stanley's conjecture we give affirmative answers for some special cases.

Keywords: zrank, rank, outside decomposition, border strip decomposition, snakes, interval sets, restricted Cauchy matrix, reduced code.

MSC2000 Subject Classification: 05E10, 15A15.

Suggested Running Title: The Zrank Conjecture

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1 Introduction

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition of an integer n, i.e., $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and $\lambda_1 + \lambda_2 + \cdots = n$. The number of positive parts of λ is called the length of λ, denoted $\ell(\lambda)$. The *Young diagram* of λ may be defined as the set of points $(i, j) \in \mathbb{Z}^2$ such that $1 \leq j \leq \lambda_i$ and $1 \leq i \leq \ell(\lambda)$. A Young diagram can also be represented in the plane by an array of squares justified from the top and left corner with $\ell(\lambda)$ rows and λ_i squares in row i. A square (i, j) in the diagram is the square in row i from the top and column j from the left. The content of (i, j), denoted $\tau((i, j))$, is given by $j - i$. The *rank* of λ, denoted $rank(\lambda)$, is the length of the main diagonal of the diagram of λ.

Given two partitions λ and μ, we say that $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all i. If $\mu \subseteq \lambda$, we define a *skew partition* λ/μ, whose Young diagram is obtained from the Young diagram of λ by peeling off the Young diagram of μ from the upper left corner.

We assume that the reader is familiar with the notation and terminology on symmetric functions in [10]. In connection with tensor products of Yangian modules, Nazarov and Tarasov [9] give a generalization of a rank to a skew partition λ/μ. Recently Stanley developed a general theory of minimal border strip decompositions and gave several simple equivalent characterizations of $rank(\lambda/\mu)$ in [11]. One of the characterizations of the rank of a skew partition λ/μ says that $rank(\lambda/\mu)$ is the smallest integer r such that the Young diagram of λ/μ is the disjoint union of r border strips. Let $s_{\lambda/\mu}(1^t)$ denote the skew Schur function $s_{\lambda/\mu}$ evaluated at $x_1 = \cdots = x_t = 1$, $x_i = 0$ for $i > t$. The *zrank* of λ/μ, denoted $zrank(\lambda/\mu)$, is the largest power of t dividing the polynomial $s_{\lambda/\mu}(1^t)$. Stanley conjectured that the equality $rank(\lambda/\mu) = zrank(\lambda/\mu)$ always holds, which we call the *zrank conjecture*.

In his combinatorial approach to the zrank conjecture in [11], Stanley defined the snake sequence and the interval sets for a skew partition λ/μ. In Section 2 for each interval set $\mathcal{I}$ of λ/μ we define an interval permutation $\sigma_{\mathcal{I}}$. Let $cr(\mathcal{I})$ be the number of crossings of $\mathcal{I}$, and let $inv(\sigma_{\mathcal{I}})$ be the number of inversions of $\sigma_{\mathcal{I}}$. We show that $cr(\mathcal{I})$ and $inv(\sigma_{\mathcal{I}})$ have the same parity.

Stanley generalized the code of a partition to the code of a skew partition, and obtained a two-line binary sequence in [11]. This sequence is called the *partition sequence* by Bessenrodt [1, 2]. Given a minimal border strip decomposition $\mathbf{D}$ of λ/μ, let $P_{\mathbf{D}}$ be the set of the contents of the lower left-hand squares of the border strips in $\mathbf{D}$, and let $Q_{\mathbf{D}}$ be the set of the contents of the upper right-hand squares. Using the partition sequence, we show that $P_{\mathbf{D}}$ and $Q_{\mathbf{D}}$ are uniquely determined by the shape of the skew partition λ/μ in Section 3, i.e., these two sets are independent of the minimal border stirp decomposition $\mathbf{D}$. For a given skew partition, we find a connection between the values of these two sets and the paired integers of the interval set.

Outside decompositions are introduced by Hamel and Goulden [7] and are used to give a unified approach to the determinantal expressions for the skew Schur funtions including the Jacobi-Trudi determinant, its dual, the Giambelli determinant and the rim ribbon determinant. For any outside decomposition, Hamel and Goulden derive a determinantal formula with strip Schur functions as entries. Their proof is based on a lattice path construction and the Gessel-Viennot methodology [5, 6]. In Section 4 we employ the determinantal formula in the case of the greedy border strip decomposition and give the evaluation of $(t^{-rank(\lambda/\mu)} s_{\lambda/\mu}(1^t))_{t=0}$. As a consequence we obtain the combinatorial description of $(t^{-rank(\lambda/\mu)} s_{\lambda/\mu}(1^t))_{t=0}$ in terms of the interval sets of λ/μ given by Stanley [11, Eq. (30)].

Based on the above results, we give an equivalent characterization of the zrank conjecture. Given two positive integer sequences, we define a *restricted Cauchy matrix* corresponding to these two sequences. The main objective of this paper is to show that the zrank conjecture holds for any skew partition if and only if all the restricted Cauchy matrices are nonsingular. We present a constructive proof for this equivalence in Section 5. Using some fundamental properties of determinants, we confirm the nonsingularity of the restricted Cauchy matrices for several special classes of skew partitions.

2 Snake sequences and interval sets

We follow the treatment of Stanley [11] to define snake sequences and interval sets, which are helpful notions for the enumeration of the minimal border strip decompositions of a skew partition λ/μ.

Now consider the bottom-right boundary lattice path with steps $(0,1)$ or $(1,0)$ from the bottom-leftmost point of the diagram of λ/μ to the top-rightmost point. We regard this path as a sequence of edges $e_1, e_2, \ldots, e_k$. For an edge e in this path we define a certain subset S_e of squares of λ/μ, called a *snake*. If there exists no square having e as its edge, then set $S_e=\emptyset$. Let (i,j) be the unique square of λ/μ having e as its edge. If e is horizontal, then define

$$S_e = \lambda/\mu \cap \{(i,j),\ (i-1,j),\ (i-1,j-1),\ (i-2,j-1),\ (i-2,j-2),\ \ldots\}. \quad (1)$$

If e is vertical, then define

$$S_e = \lambda/\mu \cap \{(i,j),\ (i,j-1),\ (i-1,j-1),\ (i-1,j-2),\ (i-2,j-2),\ \ldots\}. \quad (2)$$

For example the nonempty snakes of the skew shape $(7,6,6,3)/(3,1)$ are shown with dashed lines through their squares in Figure 1, and the two snakes with just one square are shown with a single bullet. The *length* $\ell(S)$ of a snake S is one fewer than its number of squares. For an empty snake S, let $\ell(S)=-1$. A *right snake* is a snake of even length with form (1), and a *left snake* is a snake of even length with form (2). From the boundary lattice path we obtain a sequence $(S_{e_1}, S_{e_2}, \ldots, S_{e_k})$ of snakes. The *snake sequence* of λ/μ, denoted $SS(\lambda/\mu)$, is defined by replacing a left snake of length $2m$ with the symbol L_m in the sequence $(S_{e_1}, S_{e_2}, \ldots, S_{e_k})$, a right snake of length $2m$ with R_m, and a snake of odd length with O. From Figure 1 we see that

$$SS((7,6,6,3)/(3,1)) = L_0L_1OOOOL_2R_2R_1OR_0.$$

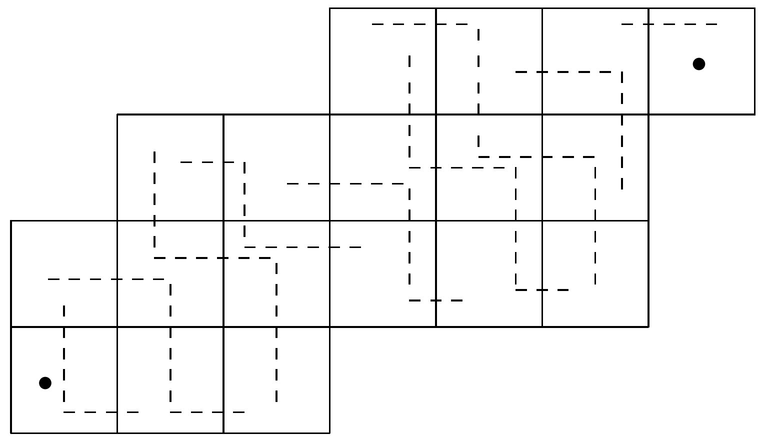

Figure 1: Snakes for the skew partition $(7, 6, 6, 3)/(3, 1)$

Let $rank(\lambda/\mu) = r$, and let $SS(\lambda/\mu) = q_1 q_2 \cdots q_k$. An *interval set* $\mathcal{I}$ of λ/μ is defined to be a collection of r ordered pairs $\{(u_1, v_1), (u_2, v_2), \ldots, (u_r, v_r)\}$ such that

1. $u_i \neq u_j$ and $v_i \neq v_j$ for $1 \leq i < j \leq r$.
2. $1 \leq u_i < v_i \leq k$ and $u_i \neq v_j$ for $1 \leq i, j \leq r$.
3. $q_{u_i} = L_s$ and $q_{v_i} = R_{s'}$ for some s and s' (depending on i).

Let $cr(\mathcal{I})$ denote the number of crossings of $\mathcal{I}$, i.e., the number of pairs (i, j) for which $u_i < u_j < v_i < v_j$. According to [11, Proposition 4.3] there exists a unique interval set $\mathcal{I}_0 = \{(w_1, y_1), (w_2, y_2), \ldots, (w_r, y_r)\}$ such that $cr(\mathcal{I}_0) = 0$. From [11] we know that $SS(\lambda/\mu)$ has exactly r left snakes, as well as r right snakes. For an interval set $\mathcal{I} = \{(u_1, v_1), (u_2, v_2), \ldots, (u_r, v_r)\}$ we may suppose its elements are ordered by $u_1 < u_2 < \cdots < u_r$. Now there exists a unique permutation σ relative to $\mathcal{I}_0$ such that for each i

$$u_i = w_i \text{ and } v_i = y_{\sigma_i}. \tag{3}$$

In this meaning each interval set $\mathcal{I}$ is associated with a permutation σ_I, which we call the *interval permutation* of $\mathcal{I}$ relative to $\mathcal{I}_0$. Given a permutation σ, let $inv(\sigma)$ denote the number of inversions of σ, i.e., the number of pairs (i, j) satisfying $i < j$ but $\sigma_i > \sigma_j$.

Proposition 2.1 *Given a skew partition λ/μ and an interval set $\mathcal{I}$ of λ/μ, let $\sigma_{\mathcal{I}}$ be the interval permutation relative to $\mathcal{I}_0$. Then*

$$cr(\mathcal{I}) \equiv inv(\sigma_{\mathcal{I}}) \ (mod\ 2). \tag{4}$$

Proof. First we give a geometric representation of $cr(\mathcal{I})$. For each interval (u_i, v_i) of $\mathcal{I}$ we draw an arc on the top of $SS(\lambda/\mu)$ which links two snakes q_{u_i} and q_{v_i}. For a given pair (i, j) with $i < j$, two arcs corresponding to (u_i, v_i) and (u_j, v_j) need to be noncrossing if $u_i < u_j < v_j < v_i$, otherwise they intersect each other only once. In this way $cr(\mathcal{I})$ is the total number of crossings of the arcs.

To determine the inversions of $\sigma_{\mathcal{I}}$, we first replace q_{w_i} with F_i and q_{y_i} with G_i in $SS(\lambda/\mu)$ for each i. Clearly $\sigma_{\mathcal{I}}$ corresponds to a bijection from $\{F_1, F_2, \ldots, F_r\}$ to $\{G_1, G_2, \ldots, G_r\}$. Now reorder the snakes of $SS(\lambda/\mu)$, while keeping the links between the arcs and their vertices as we draw above, such that $F_1F_2\cdots F_rG_rG_{r-1}\cdots G_1$ appears as a subsequence. Then it is natural to see that $inv(\sigma_{\mathcal{I}})$ is the number of crossings of the arcs after reordering.

In such a reordering of the snake sequence $SS(\lambda/\mu)$ we need to move $G_1, G_2, \cdots, G_r$ step by step towards the right. In the i-th step we need to move G_i across over the pairs (F_j, G_j) for some $j > i$ (j may not exist), and this will preserve the number of crossings or change it by an even number. Thus we obtain the desired result. ∎

For example, let $\lambda/\mu = (8, 8, 7, 4)/(4, 1, 1)$. Figure 2 shows the snake sequence $SS((8, 8, 7, 4)/(4, 1, 1))$, from which we see that

$$\mathcal{I}_0 = \{(1, 12), (3, 11), (4, 5), (8, 9)\}.$$

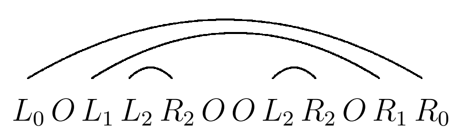

Figure 2: Parenthesization of the snake sequence $SS((8, 8, 7, 4)/(4, 1, 1))$

Now consider an interval set $\mathcal{I} = \{(1, 9), (3, 12), (4, 5), (8, 11)\}$, and we see that $\sigma_{\mathcal{I}} = [4, 1, 3, 2]$. The crossings of $\mathcal{I}$ are shown in Figure 3, where we relabel the snakes in the way as described in the proof of Proposition 2.1. Figure 4 shows the crossings after moving G_3, which are increased by 2. It is clear that

$$cr(\mathcal{I}) = 2, \quad inv(\pi) = 4, \quad cr(\mathcal{I}) \equiv inv(\pi) \ (mod\ 2). \tag{5}$$

Figure 3: Before moving G_3

Figure 4: After moving G_3

3 Minimal border strip decompositions

First we define the *reduced code* of a skew partition λ/μ, denoted $c(\lambda/\mu)$. The reduced code $c(\lambda/\mu)$ is also known as the *partition sequence* of λ/μ [1, 2]. Now consider the two boundary lattice paths of the diagram of λ/μ with steps $(0,1)$ or $(1,0)$ from the bottom-leftmost point to the top-rightmost point. Replace each step $(0,1)$ by 1 and $(1,0)$ by 0, and then we obtain two binary sequences by reading the lattice paths from the bottom-left corner to the top-right corner. Denote the top-left binary sequence by $f_1, f_2, \ldots, f_k$, and the bottom-right binary sequence by $g_1, g_2, \ldots, g_k$. The reduced code $c(\lambda/\mu)$ is defined by the two-line array

$$\begin{array}{cccc} f_1 & f_2 & \cdots & f_k \\ g_1 & g_2 & \cdots & g_k \end{array}.$$

Figure 5 shows that the reduced code $c((5,4,3,2)/(2,1,1))$ is

$$\begin{array}{ccccccccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}.$$

A *diagonal* with content j of λ/μ, denoted $d_j(\lambda/\mu)$, is the set of all the squares in λ/μ having content j. Suppose the length of $c(\lambda/\mu)$ is k. It is obvious that λ/μ has $k-1$ diagonals. Let ϵ be the smallest content of λ/μ. For each $i : 1 \leq i \leq k-1$ putting the diagonal $d_{\epsilon+i-1}$ between the i-th column and $(i+1)$-th column of $c(\lambda/\mu)$, thus we establish the connection between the diagonals of λ/μ and the reduced code $c(\lambda/\mu)$.

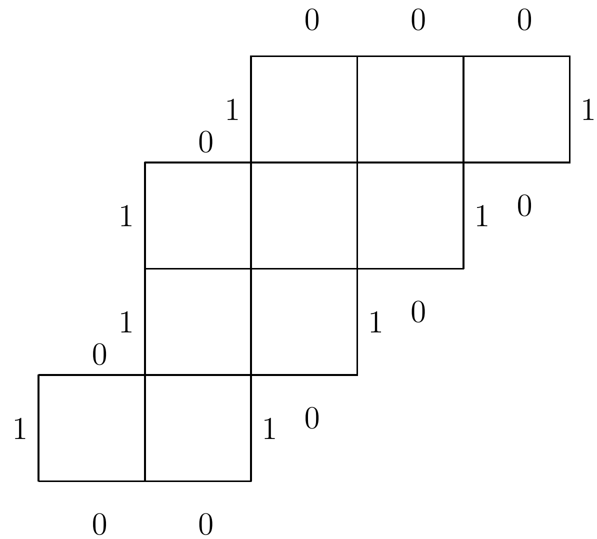

Figure 5: Constructing the reduced code of $(5, 4, 3, 2)/(2, 1, 1)$

Recall that a skew partition λ/μ is called to be *connected* if the interior of the Young diagram of λ/μ is a connected set. A *border strip* is a connected skew partition with no 2×2 square. Let the size of a border strip B be the number of squares of B, and let the *height* $ht(B)$ of B be one less than its number of rows. We say that $B \subset \lambda/\mu$ is a border strip of λ/μ if $\lambda/\mu - B$ is a skew partition ν/μ. A border strip B of λ/μ is said to be *maximal* if there does not exist another different border strip $B' \subset \lambda/\mu$ such that $B \subset B'$. A *border strip decomposition* [10] of λ/μ is a partition of the squares of λ/μ into pairwise disjoint border strips. A *greedy border strip decomposition* of λ/μ is obtained by starting with λ/μ and successively removing the maximal border strip. A border strip decomposition is *minimal* if there does not exist a border strip decomposition with fewer border strips.

As two equivalent characterizations of the rank of a skew partition λ/μ, the number of border strips in a minimal border strip decomposition of λ/μ and the number of $\substack{1\\0}$ columns of $c(\lambda/\mu)$ are equal [11, Proposition 2.2]. Immediately, it follows that a greedy border strip decomposition is minimal, since as we successively remove the maximal border strips from λ/μ one column $\substack{1\\0}$ of $c(\lambda/\mu)$ changes into $\substack{1\\1}$, while one column $\substack{0\\1}$ changes into $\substack{0\\0}$.

Suppose $rank(\lambda/\mu) = r$. Given a minimal border strip decomposition $\mathbf{D} = \{B_1, B_2, \ldots, B_r\}$ of λ/μ, let

$$P_{\mathbf{D}} = \{\tau(init(B_1)), \tau(init(B_2)), \ldots, \tau(init(B_r))\}$$

and

$$Q_{\mathbf{D}} = \{\tau(fin(B_1)), \tau(fin(B_2)), \ldots, \tau(fin(B_r))\},$$

where $init(B_i)$ is the lower left-hand square of B_i and $fin(B_i)$ is the upper right-hand square. The following proposition shows that $P_{\mathbf{D}}$ and $Q_{\mathbf{D}}$ are independent of the minimal border strip decomposition $\mathbf{D}$.

Proposition 3.1 *Let $\mathcal{I}_0 = \{(w_1, y_1), (w_2, y_2), \ldots, (w_r, y_r)\}$ be the interval set of λ/μ with $cr(\mathcal{I}_0) = 0$. Let ϵ be the smallest value among the contents of the squares of λ/μ. Let $\mathbf{D}$ be a minimal border strip decomposition of λ/μ. Then we have*

$$P_{\mathbf{D}} = \{\epsilon + w_i - 1 \,|\, 1 \leq i \leq r\} \text{ and } Q_{\mathbf{D}} = \{\epsilon + y_i - 2 \,|\, 1 \leq i \leq r\}. \quad (6)$$

Proof. By [11, Proposition 2.1], removing a border strip B of size p from λ/μ is equivalent to choosing i with the i-th column being $\substack{1\\0}$ and the $(i+p)$-th column being $\substack{0\\1}$, and then replacing the i-th column with $\substack{1\\1}$ and the $(i+p)$-th column with $\substack{0\\0}$. Moreover, the lower left-hand square of B lies on the diagonal d_i, and the upper right-hand square of B lies on the diagonal d_{i+p-1}. Therefore $\tau(init(B)) = \epsilon + i - 1$ and $\tau(fin(B)) = \epsilon + i + p - 2$. It follows that $P_{\mathbf{D}}$ and $Q_{\mathbf{D}}$ are determined by the indices of columns $\substack{1\\0}$ and $\substack{0\\1}$ of $c(\lambda/\mu)$ respectively. Since $\{w_i\}$ is just the set of indices of columns $\substack{1\\0}$ of $c(\lambda/\mu)$, and $\{y_i\}$ is the set of indices of $\substack{0\\1}$, the desired result immediately follows. ∎

4 Giambelli-type determinant formulas for the skew Schur function

Let λ/μ be a skew diagram. A border strip decomposition of λ/μ is said to be an *outside decomposition* if every strip in the decomposition has an initial square on the left or bottom perimeter of the diagram and a terminal square on the right or top perimeter, see Figure 6. Clearly, a greedy border strip decomposition of λ/μ is an outside decomposition.

a. A border strip decomposition   b. An outside decomposition

Figure 6: Border strip decompositions

The notion of the cutting strip of an outside decomposition is introduced by Chen, Yan and Yang [3]. They use it to give a transformation theorem on the Giambelli-type determinant formulas for the skew Schur function. Now we show how to construct a cutting strip for an edgewise connected skew partition λ/μ. Suppose that λ/μ has k diagonals, the cutting strip of an outside decomposition is defined to be a border strip of length k. First we assign a direction to each square in the diagram. Starting with the bottom-left corner of a strip, we say that a square of a strip goes upwards (resp. rightwards) if the next square in the strip lies on its top (resp. to its right). Notice that the strips in any outside decomposition of λ/μ are nested in the sense that the squares in the same diagonal of λ/μ all go upwards or all go rightwards. Based on this property, the cutting strip ϕ of an outside decomposition $\mathbf{D}$ of λ/μ is defined as follows: for $i = 1, 2, \ldots, k-1$ the i-th square in ϕ keeps the same direction as the i-th diagonal of λ/μ with respect to $\mathbf{D}$. For any two integers p, q a strip $[p, q]$ is defined by the following rule: if $p \leq q$, then let $[p, q]$ be the segment of ϕ from the square with content p to the square with content q; if $p = q + 1$, then let $[p, q]$ be the empty strip; if $p > q + 1$, then $[p, q]$ is undefined. Using the above notation, Hamel and Goulden's theorem on the Giambelli-type formulas for the skew Schur function can be formulated as

Theorem 4.1 ([7, Theorem 3.1]) *For an outside decomposition* $\mathbf{D}$ *with* k *border strips* $B_1, B_2, \ldots, B_k$*, we have*

$$s_{\lambda/\mu} = \det\left(s_{[\tau(init(B_i)),\tau(fin(B_j))]}\right)_{i,j=1}^{k}. \tag{7}$$

By choosing the outside decomposition whose border strips are the rows of diagram of λ/μ in the above theorem, we obtain the Jacobi-Trudi identity for the skew Schur function, which asserts that

$$s_{\lambda/\mu} = \det\left(h_{\lambda_i-\mu_j-i+j}\right)_{i,j=1}^{\ell(\lambda)}, \tag{8}$$

where h_k denotes the k-th complete symmetric function, $h_0 = 1$ and $h_k = 0$ for $k < 0$.

Let $y(\lambda/\mu) = (t^{-rank(\lambda/\mu)} s_{\lambda/\mu}(1^t))_{t=0}$. The zrank conjecture says that $y(\lambda/\mu) \neq 0$ for any skew partition λ/μ. Now we give the evaluation of $y(\lambda/\mu)$ by using Theorem 4.1. First we consider the case when λ/μ is a border strip. In this case we have $rank(\lambda/\mu) = 1$, $\mu_i = \lambda_{i+1} - 1$ for $i \leq \ell(\lambda) - 1$ and $\mu_{\ell(\lambda)} = 0$.

Lemma 4.2 *For a border strip* λ/μ *we have*

$$y(\lambda/\mu) = \frac{(-1)^{\ell(\lambda)+1}}{\lambda_1 + \ell(\lambda) - 1}. \tag{9}$$

Proof. Notice that for a strip λ/μ the matrix $\left(h_{\lambda_i-\mu_j-i+j}\right)$ appearing in the Jacobi-Trudi identity has the following property: the diagonal of the matrix below the main diagonal is filled with 1's, and the entries below 1's are 0's. An obvious fact is

$$h_k(1^t) = \frac{t(t+1)\cdots(t+k-1)}{k!}. \tag{10}$$

Note that in the expansion of the determinant det $\left(h_{\lambda_i-\mu_j-i+j}\right)_{i,j=1}^{\ell(\lambda)}$ all nonzero terms have a factor t^2 except the one taking all 1's and $h_{\lambda_1-1+\ell(\lambda)}$. It immediately leads to the desired result. ∎

In order to compute $y(\lambda/\mu)$ for a general skew partition λ/μ, we now consider the greedy border strip decomposition $\mathbf{D_0}$ of λ/μ. Suppose $rank(\lambda/\mu) = r$, then $\mathbf{D_0}$ has r border strips since it is minimal. We can apply Theorem 4.1 to $\mathbf{D_0}$ because it is also an outside decomposition. We may impose a canonical order on the strips $B_1, B_2, \ldots, B_r$ of $\mathbf{D_0}$ by the contents of their lower left-hand squares such that $\tau(init(B_i)) < \tau(init(B_{i+1}))$ for $i < r$. Notice that $ht(B_1) + ht(B_2) + \cdots + ht(B_r)$, the sum of the heights of border strips in $\mathbf{D}_0$, is uniquely determined by the shape λ/μ. Therefore $z(\lambda/\mu) = ht(B_1) + ht(B_2) + \cdots + ht(B_r)$ is well defined. Let $\mathcal{I}_0 = \{(w_1, y_1), (w_2, y_2), \ldots, (w_r, y_r)\}$ be the interval set of λ/μ with $cr(\mathcal{I}_0) = 0$. Then by Proposition 3.1 and the properties of $\mathbf{D}_0$ and $\mathcal{I}_0$ we have

$$\tau(init(B_i)) = \epsilon + w_i - 1 \text{ and } \tau(fin(B_i)) = \epsilon + y_i - 2, \tag{11}$$

where ϵ is the smallest value among the contents of the squares of λ/μ.

Theorem 4.3 *Let λ/μ be a skew partition with $rank(\lambda/\mu) = r$, and let $\mathcal{I}_0$ be the noncrossing interval set $\{(w_1, y_1), (w_2, y_2), \ldots, (w_r, y_r)\}$ of λ/μ. Then*

$$y(\lambda/\mu) = (-1)^{z(\lambda/\mu)} \det(d_{ij})_{i,j=1}^{r}, \tag{12}$$

where

$$d_{ij} = \begin{cases} \dfrac{1}{y_j - w_i}, & y_j > w_i \\ 0, & y_j < w_i \end{cases}$$

Remark. Stanley notes that one can also get a matrix for $y(\lambda/\mu)$ by taking the Jacobi-Trudi matrix (the matrix appearing in the Jacobi-Trudi determinant formula of $s_{\lambda/\mu}$) for the skew Schur function $s_{\lambda/\mu}$, deleting all rows and columns that contain a 1, and then substituting $1/i$ for h_i. This matrix coincides with the matrix $(d_{ij})_{i,j=1}^{r}$ defined in (12), except possibly for the permutation of columns and rows. We refer the readers to [3] to check this coincidence.

Proof. Take the greedy outside decomposition $\mathbf{D}_0 = \{B_1, B_2, \ldots, B_r\}$ of λ/μ, and let ϕ_0 be the cutting strip corresponding to $\mathbf{D}_0$. By Theorem 4.1 we have

$$s_{\lambda/\mu} = \det\left(s_{[\tau(init(B_i)),\tau(fin(B_j))]}\right)_{i,j=1}^{r}. \tag{13}$$

Suppose the square with content $\tau(init(B_i))$ lies in the k_1^i-th row of ϕ_0, and the square with content $\tau(fin(B_j))$ lies in the k_2^j-th row. Applying Lemma 4.2, we get

$$(t^{-1}s_{[\tau(init(B_i)),\tau(fin(B_j))]})_{t=0} = \frac{(-1)^{k_1^i - k_2^j}}{\tau(fin(B_j)) + 1 - \tau(init(B_i))} \tag{14}$$

if $[\tau(init(B_i)), \tau(fin(B_j))]$ is a substrip of ϕ_0, otherwise 0. Remark that $[\tau(init(B_i)), \tau(fin(B_j))]$ can never be an empty strip for the greedy border strip decomposition. Using (11) we write (14) as

$$(t^{-1}s_{[\tau(init(B_i)),\tau(fin(B_j))]})_{t=0} = \frac{(-1)^{k_1^i - k_2^j}}{y_j - w_i} \tag{15}$$

for $y_j > w_i$, or 0 for $y_j < w_i$. Now

$$y(\lambda/\mu) = (t^{-r}s_{\lambda/\mu}(1^t))_{t=0} = \det\left((t^{-1}s_{[\tau(init(B_i)),\tau(fin(B_j))]})_{t=0}\right)_{i,j=1}^{r}. \tag{16}$$

Extract the signs from the determinant, and we get

$$y(\lambda/\mu) = (-1)^{(k_1^1+\cdots+k_1^r)-(k_2^1+\cdots+k_2^r)}\det(d_{ij})_{i,j=1}^{r} = (-1)^{z(\lambda/\mu)}\det(d_{ij})_{i,j=1}^{r}.$$

Thus we accomplish the proof. ∎

From Theorem 4.3 and Proposition 2.1 there follows

Corollary 4.4 *[11, Equation (30)]*

$$y(\lambda/\mu) = (-1)^{z(\lambda/\mu)} \sum_{\mathcal{I}=\{(u_1,v_1),\ldots,(u_r,v_r)\}} \frac{(-1)^{cr(\mathcal{I})}}{\prod_{i=1}^{r}(v_i - u_i)}, \tag{17}$$

summed over all interval sets $\mathcal{I}$ of λ/μ.

5 An equivalent description of the zrank conjecture

We begin this section with the definition of a restricted Cauchy matrix. Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be two integer sequences. Suppose a

is strictly decreasing and b is strictly increasing. For any i, j they satisfy $a_i > b_{n+1-i}$ and $a_i \neq b_j$. We define a matrix $RC(a,b) = (rc_{ij})_{i,j=1}^n$ by setting

$$rc_{ij} = \begin{cases} \dfrac{1}{a_i - b_j}, & \text{if } a_i > b_j \\ 0, & \text{if } a_i < b_j \end{cases}.$$

Definition 5.1 *A matrix M is called a* restricted Cauchy matrix *if there exist two integer sequences a and b satisfying the above condition such that $M = RC(a,b)$.*

For a matrix M we say it is *singular* if $\det(M) = 0$, otherwise *nonsingular*. We now come to our main theorem

Theorem 5.2 *The following two statements are equivalent:*

(i) The zrank conjecture is true for all the skew partitions.

(ii) All the restricted Cauchy matrices are nonsingular.

Proof. Suppose (ii) is true. For a skew partition λ/μ, consider the non-crossing interval set $\mathcal{I}_0 = \{(w_1, y_1), (w_2, y_2), \ldots, (w_r, y_r)\}$ of λ/μ. Clearly, $w_i \neq y_j$ for $1 \leq i, j \leq r$. Let $w = (w'_1, w'_2, \ldots, w'_r)$ be the increasing reordering of $(w_1, w_2, \ldots, w_r)$, and let $y = (y'_1, y'_2, \ldots, y'_r)$ be the decreasing reordering of $(y_1, y_2, \ldots, y_r)$. For $1 \leq i \leq r$ we have $y'_i > w'_{r+1-i}$ since the number of $\substack{1\\0}$ columns in the first ℓ columns of the reduced code $c(\lambda/\mu)$ is bigger than, or equal to the number of $\substack{0\\1}$ columns for $1 \leq \ell \leq k$, where k is the length of $c(\lambda/\mu)$. Notice that the determinant $\det(d_{ij})_{i,j=1}^r$ appearing in (12) is equal to the determinant of the restricted Cauchy matrix $RC(y,w)$ up to a sign. By Theorem 4.3 we know that

$$y(\lambda/\mu) \neq 0 \Leftrightarrow \det(d_{ij})_{i,j=1}^r \neq 0 \Leftrightarrow \det(RC(y,w)) \neq 0.$$

Since the matrix $RC(y,w)$ is nonsingular, $rank(\lambda/\mu) = zrank(\lambda/\mu)$.

Now we show the proof direction from (i) to (ii). Fix a restricted Cauchy matrix $RC(a,b)$ of order r. Without loss of generality, we suppose a and b are positive integer sequences. Now let λ be the partition with parts $\lambda_i = a_i - r + i$, and let μ be the partition with parts $\mu_i = b_{r+1-i} - r + i$. From $a_i > b_{r+1-i}$ we deduce $\lambda_i > \mu_i$ for all i. Thus we can construct a skew diagram λ/μ. Observe that the Jacobi-Trudi matrix $(h_{\lambda_i - \mu_j - i + j})$ of $s_{\lambda/\mu}$ doesn't have a column containing 1 since

$$\lambda_i - \mu_j - i + j = a_i - b_{r+1-j} \neq 0, \text{for } 1 \leq i, j \leq r.$$

It follows that $rank(\lambda/\mu) = r$ from [11, Proposition]. Now

$$y(\lambda/\mu) = (t^{-r} s_{\lambda/\mu}(1^t))_{t=0} = \det\left((t^{-1} h_{\lambda_i - \mu_j - i + j}(1^t))_{t=0}\right)_{i,j=1}^{r},$$

which is the determinant $\det(RC(a,b))$ up to a sign. If the zrank conjecture is true for λ/μ, then $y(\lambda/\mu) \neq 0$, which shows $RC(a,b)$ is nonsingular. ∎

A square $n \times n$ matrix M is called *reducible* if M has the block form $\begin{pmatrix} M_1 & M_2 \\ M_3 & 0 \end{pmatrix}$ by permuting rows and columns, where both M_2 and M_3 are square matrices. A square matrix which is not reducible is said to be *irreducible*. Theorem 5.2 shows that to prove or disprove the zrank conjecture is equivalent to determining the singularity of restricted Cauchy matrices $RC(a,b) = (rc_{ij})$. Clearly, for a reducible restricted Cauchy matrix $RC(a,b) = \begin{pmatrix} M_1 & M_2 \\ M_3 & 0 \end{pmatrix}$ both M_2 and M_3 can be regarded as the restricted Cauchy matrices, thus $\det(RC(a,b))$ is equal to $\det(M_2)\det(M_3)$ up to a sign. Hence we only need to deal with the case of the irreducible restricted Cauchy matrices.

6 Restricted Cauchy matrices for special patterns

In this section we prove the nonsingularity of the restricted Cauchy matrix $RC(a,b) = (rc_{ij})_{i,j=1}^{r}$ for some special cases.

Case I. For all i, j we have $rc_{ij} \neq 0$.

Now $(rc_{ij})_{i,j=1}^{r}$ is a Cauchy matrix. It was shown by Cauchy [8] that

$$\det\left(\frac{1}{a_i - b_j}\right)_{i,j=1}^{r} = \prod_{i<j}(a_i - a_j)\prod_{i<j}(b_j - b_i)\prod_{i,j}\frac{1}{a_i - b_j}. \tag{18}$$

Thus we have

$$\det(rc_{ij})_{i,j=1}^{r} > 0.$$

Following the proof of [11, Theorem 3.2 (b)], we immediately get

Proposition 6.1 *For a connected skew diagram λ/μ, if every row of the Jacobi-Trudi matrix that contains a 0 also contains a 1, then the matrix $(d_{ij})_{i,j=1}^{r}$ appearing in (12) must satisfy that $d_{ij} \neq 0$ for all i, j.*

Theorem 4.3 and Proposition 6.1 yield another proof of [11, Theorem 3.2]. Some skew partition doesn't have the property stated in the above proposition, but its $(d_{ij})_{i,j=1}^{r}$ still corresponds to a Cauchy matrix. For instance taking $\lambda/\mu = (8,8,7,7,7,6,1)/(5,5,3,3,2)$, notice that its Jacobi-Trudi matrix is

$$s_{(8,8,7,7,7,6,1)/(5,5,3,3,2)} = \begin{vmatrix} h_3 & h_4 & h_7 & h_8 & h_{10} & h_{13} & h_{14} \\ h_2 & h_3 & h_6 & h_7 & h_9 & h_{12} & h_{13} \\ 1 & h_1 & h_4 & h_5 & h_7 & h_{10} & h_{11} \\ 0 & 1 & h_3 & h_4 & h_6 & h_9 & h_{10} \\ 0 & 0 & h_2 & h_3 & h_5 & h_8 & h_9 \\ 0 & 0 & 1 & h_1 & h_3 & h_6 & h_7 \\ 0 & 0 & 0 & 0 & 0 & 1 & h_1 \end{vmatrix}.$$

Case II. For all $(i,j) \neq (r,r)$ we have $rc_{ij} \neq 0$, but $rc_{rr} = 0$.

Let

$$M = \prod_{i=1}^{r-1} \frac{(a_r - a_i)(b_i - b_r)}{(a_r - b_i)(a_i - b_r)}.$$

Since $b_r > a_r$, it is easy to show that $M > 1$. We see that the restricted Cauchy matrix in this case has the shape:

$$(rc_{ij})_{i,j=1}^r = \begin{pmatrix} \frac{1}{a_1 - b_1} & \cdots & \frac{1}{a_1 - b_{r-1}} & \frac{1}{a_1 - b_r} \\ \frac{1}{a_2 - b_1} & \cdots & \frac{1}{a_2 - b_{r-1}} & \frac{1}{a_2 - b_r} \\ \vdots & \cdots & \vdots & \vdots \\ \frac{1}{a_{r-1} - b_1} & \cdots & \frac{1}{a_{r-1} - b_{r-1}} & \frac{1}{a_{r-1} - b_r} \\ \frac{1}{a_r - b_1} & \cdots & \frac{1}{a_r - b_{r-1}} & 0 \end{pmatrix}.$$

Then we have

$$\begin{aligned} \det(rc_{ij})_{i,j=1}^r &= \prod_{\substack{i,j=1 \\ i<j}}^{r} (a_i - a_j)(b_j - b_i) \prod_{i,j=1}^{r} \frac{1}{a_i - b_j} \\ &\quad - \frac{1}{a_r - b_r} \prod_{\substack{i,j=1 \\ i<j}}^{r-1} (a_i - a_j)(b_j - b_i) \prod_{i,j=1}^{r-1} \frac{1}{a_i - b_j} \\ &= \frac{1}{a_r - b_r} \prod_{\substack{i,j=1 \\ i<j}}^{r-1} (a_i - a_j)(b_j - b_i) \prod_{i,j=1}^{r-1} \frac{1}{a_i - b_j} (M - 1). \end{aligned}$$

It follows immediately that

$$\det(rc_{ij})_{i,j=1}^r < 0.$$

Case III. $rc_{ij} \neq 0$ except for rc_{rr}, $rc_{r,r-1}$ and $rc_{r-1,r}$.

In this case, we have $a_r > b_{r-2}$ but $a_r < b_{r-1}$, $a_{r-2} > b_r$ but $a_{r-1} < b_r$. Recall that the *rank* of a matrix is the number of linearly independent rows or columns of the matrix. For a matrix $M = (m_{ij})_{i,j=1}^r$, let M^* be the matrix $(M_{ji})_{i,j=1}^r$, where M_{ij} is the cofactor of m_{ij} in the expansion $\det(M) = \sum_{i=1}^r m_{ij}M_{ij}$. The following property is known

$$rank(M^*) = \begin{cases} r, & \text{if } rank(M) = r \\ 1, & \text{if } rank(M) = r-1 \\ 0, & \text{if } rank(M) < r-1 \end{cases} \tag{19}$$

Now consider the rank of $RC^* = (RC_{ij})_{i,j=1}^r$. We see that the minor $RC_{r,r}$ is the determinant of the submatrix of $RC(a,b)$ with row r and column r crossed out, which is the restricted Cauchy matrix of type **I**, and the underlying matrices of $RC_{r-1,r-1}$, $RC_{r,r-1}$, $RC_{r-1,r}$ are the restricted Cauchy matrices of type **II**. Thus we have

$$RC_{r,r} > 0, \quad RC_{r-1,r-1} < 0, \quad RC_{r,r-1} > 0 \quad \text{and} \quad RC_{r-1,r} > 0.$$

This shows that $rank(RC^*) \geq 2$. Hence $rank(RC) = r$ due to (19), namely $\det(rc_{ij})_{i,j=1}^r \neq 0$.

Case IV. For all $i \leq r$, $j \leq r-1$ we have $rc_{ij} \neq 0$; $rc_{1r} \neq 0$, and $rc_{2r} \neq 0$; $rc_{ir} = 0$ if $i > 2$.

In this case, the restricted Cauchy matrix has the shape

$$(rc_{ij})_{i,j=1}^r = \begin{pmatrix} \frac{1}{a_1 - b_1} & \cdots & \frac{1}{a_1 - b_{r-1}} & \frac{1}{a_1 - b_r} \\ \frac{1}{a_2 - b_1} & \cdots & \frac{1}{a_2 - b_{r-1}} & \frac{1}{a_2 - b_r} \\ \frac{1}{a_3 - b_1} & \cdots & \frac{1}{a_3 - b_{r-1}} & 0 \\ \vdots & \cdots & \vdots & \vdots \\ \frac{1}{a_r - b_1} & \cdots & \frac{1}{a_r - b_{r-1}} & 0 \end{pmatrix}.$$

Expand the above determinant along the last column, and we get

$$\begin{aligned}\det(rc_{ij})_{i,j=1}^r &= (-1)^{r+1}\frac{1}{a_1-b_r}\prod_{2\le i<j\le r}(a_i-a_j)\prod_{1\le i<j\le r-1}(b_j-b_i)\prod_{\substack{2\le i\le r\\1\le j\le r-1}}\frac{1}{a_i-b_j}\\&\quad+(-1)^{r+2}\frac{1}{a_2-b_r}\prod_{\substack{i\ne 2,j\ne 2\\1\le i<j\le r}}(a_i-a_j)\prod_{1\le i<j\le r-1}(b_j-b_i)\prod_{\substack{i\ne 2\\1\le j\le r-1}}\frac{1}{a_i-b_j}\\&= (-1)^{r+1}\prod_{1\le i<j\le r}(a_i-a_j)\prod_{1\le i<j\le r-1}(b_j-b_i)\prod_{\substack{i,j=1\\j\ne r}}^{r}\frac{1}{a_i-b_j}N,\end{aligned}$$

where

$$N=\frac{f(a_1)-f(a_2)}{a_1-a_2}$$

and

$$f(x)=\frac{(x-b_1)(x-b_2)\cdots(x-b_{r-1})}{(x-b_r)(x-a_3)\cdots(x-a_r)}.$$

Let $\delta=a_1-a_2$, now

$$\begin{aligned}\frac{f(a_1)}{f(a_2)} &= \frac{\dfrac{(a_1-b_1)(a_1-b_2)\cdots(a_1-b_{r-1})}{(a_1-b_r)(a_1-a_3)\cdots(a_1-a_r)}}{\dfrac{(a_2-b_1)(a_2-b_2)\cdots(a_2-b_{r-1})}{(a_2-b_r)(a_2-a_3)\cdots(a_2-a_r)}}\\&= \frac{\dfrac{(a_1-b_1)}{(a_2-b_1)}\dfrac{(a_1-b_2)}{(a_2-b_2)}\cdots\dfrac{(a_1-b_{r-1})}{(a_2-b_{r-1})}}{\dfrac{(a_1-b_r)}{(a_2-b_r)}\dfrac{(a_1-a_3)}{(a_2-a_3)}\cdots\dfrac{(a_1-a_r)}{(a_2-a_r)}}\\&= \frac{\dfrac{(\delta+a_2-b_1)}{(a_2-b_1)}\dfrac{(\delta+a_2-b_2)}{(a_2-b_2)}\cdots\dfrac{(\delta+a_2-b_{r-1})}{(a_2-b_{r-1})}}{\dfrac{(\delta+a_2-b_r)}{(a_2-b_r)}\dfrac{(\delta+a_2-a_3)}{(a_2-a_3)}\cdots\dfrac{(\delta+a_2-a_r)}{(a_2-a_r)}}.\end{aligned}$$

Let $s\in\{b_1,\ldots,b_{r-1}\}$ and $s'\in\{a_3,\ldots,a_r,b_r\}$, then we have $s<s'$ and

$$\frac{(\delta+a_2-s)}{(a_2-s)}<\frac{(\delta+a_2-s')}{(a_2-s')}. \tag{20}$$

It follows that $f(a_1)<f(a_2)$, namely $N<0$. Thus we have $\det(rc_{ij})_{i,j=1}^r>0$ if r is even and $\det(rc_{ij})_{i,j=1}^r<0$ if r is odd.

Case V. For all $i\le r-1$, $j\le r$ we have $rc_{ij}\ne 0$; $rc_{r1}\ne 0$, and $rc_{r2}\ne 0$; $rc_{ri}=0$ if $i>2$. This is the same as **Case IV.**

Acknowledgments. This work was done under the auspices of the 973 Project on Mathematical Mechanization, the Ministry of Education, the Ministry of Science and Technology, and the National Science Foundation of China. We thank Professor Richard Stanley for bringing this problem to our attention and for valuable comments. We also thank Professor William Y.C. Chen for his efforts in helping us complete this paper.

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