On the Modes of Polynomials Derived from Nondecreasing Sequences

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Abstract

Wang and Yeh proved that if $P(x)$ is a polynomial with nonnegative and nondecreasing coefficients, then $P(x+d)$ is unimodal for any $d>0$. A mode of a unimodal polynomial $f(x)=a_0+a_1x+\cdots+a_mx^m$ is an index k such that a_k is the maximum coefficient. Suppose that $M_*(P,d)$ is the smallest mode of $P(x+d)$, and $M^*(P,d)$ the greatest mode. Wang and Yeh conjectured that if $d_2>d_1>0$, then $M_*(P,d_1)\geq M_*(P,d_2)$ and $M^*(P,d_1)\geq M^*(P,d_2)$. We give a proof of this conjecture.

Keywords: unimodal polynomials, the smallest mode, the greatest mode.

1 Introduction

This paper is concerned with the modes of unimodal polynomials constructed from nonnegative and nondecreasing sequences. Recall that a sequence $\{a_i\}_{0\leq i\leq m}$ is unimodal if there exists an index $0\leq k\leq m$ such that

$$a_0\leq\cdots\leq a_{k-1}\leq a_k\geq a_{k+1}\geq\cdots\geq a_m.$$

Such an index k is called a mode of the sequence. Note that a mode of a sequence may not be unique. The sequence $\{a_i\}_{0\leq i\leq m}$ is said to be spiral if

$$a_m\leq a_0\leq a_{m-1}\leq a_1\leq\cdots\leq a_{[\frac{m}{2}]}, \tag{1.1}$$

where $[\frac{m}{2}]$ stands for the largest integer not exceeding $\frac{m}{2}$. Clearly, the spiral property implies unimodality. We say that a sequence $\{a_i\}_{0\le i\le m}$ is log-concave if for $1\le k\le m-1$,

$$a_k^2 \ge a_{k+1}a_{k-1},$$

and it is ratio monotone if

$$\frac{a_m}{a_0} \le \frac{a_{m-1}}{a_1} \le \cdots \le \frac{a_{m-i}}{a_i} \le \cdots \le \frac{a_{m-[\frac{m-1}{2}]}}{a_{[\frac{m-1}{2}]}} \le 1 \tag{1.2}$$

and

$$\frac{a_0}{a_{m-1}} \le \frac{a_1}{a_{m-2}} \le \cdots \le \frac{a_{i-1}}{a_{m-i}} \le \cdots \le \frac{a_{[\frac{m}{2}]-1}}{a_{m-[\frac{m}{2}]}} \le 1. \tag{1.3}$$

It is easily checked that ratio monotonicity implies both log-concavity and the spiral property.

Let $P(x) = a_0 + a_1x + \cdots + a_mx^m$ be a polynomial with nonnegative coefficients. We say that $P(x)$ is unimodal if the sequence $\{a_i\}_{0\le i\le m}$ is unimodal. A mode of $\{a_i\}_{0\le i\le m}$ is also called a mode of $P(x)$. Similarly, we say that $P(x)$ is log-concave or ratio monotone if the sequence $\{a_i\}_{0\le i\le m}$ is log-concave or ratio monotone.

Throughout this paper $P(x)$ is assumed to be a polynomial with nonnegative and nondecreasing coefficients. Boros and Moll [2] proved that $P(x+1)$, as a polynomial of x, is unimodal. Alvarez et al. [1] showed that $P(x+n)$ is also unimodal for any positive integer n, and conjectured that $P(x+d)$ is unimodal for any $d>0$. Wang and Yeh [6] confirmed this conjecture and studied the modes of $P(x+d)$. Llamas and Martínez-Bernal [5] obtained the log-concavity of $P(x+c)$ for $c\ge 1$. Chen, Yang and Zhou [4] showed that $P(x+1)$ is ratio monotone, which leads to an alternative proof of the ratio monotonicity of the Boros-Moll polynomials [3].

Let $M_*(P,d)$ and $M^*(P,d)$ denote the smallest and the greatest mode of $P(x+d)$ respectively. Our main result is the following theorem, which was conjectured by Wang and Yeh [6].

Theorem 1.1 *Suppose that $P(x)$ is a monic polynomial of degree $m\ge 1$ with nonnegative and nondecreasing coefficients. Then for $0<d_1<d_2$, we have $M_*(P,d_1)\ge M_*(P,d_2)$ and $M^*(P,d_1)\ge M^*(P,d_2)$.*

From now on, we further assume that $P(x)$ is monic, that is $a_m=1$. For $0\le k\le m$, let

$$b_k(x) = \sum_{j=k}^{m} \binom{j}{k} a_j x^{j-k}. \tag{1.4}$$

Therefore, $b_k(x)$ is of degree $m-k$ and $b_k(0)=a_k$. For $1\le k\le m$, let

$$f_k(x) = b_{k-1}(x) - b_k(x), \tag{1.5}$$

which is of degree $m-k+1$. Let $f_k^{(n)}(x)$ denote the n-th derivative of $f_k(x)$.

Our proof of Theorem 1.1 relies on the fact that $f_k(x)$ has at most one real zero on $(0,+\infty)$. In fact, the derivative $f_k^{(n)}(x)$ of order $n\le m-k$ has the same property. We establish this property by induction on n.

2 Proof of Theorem 1.1

To prove Theorem 1.1, we need the following three lemmas.

Lemma 2.1 *For any* $0 \le k \le m$*, we have* $b_k'(x) = (k+1)b_{k+1}(x)$*.*

Proof. Let $B_{j,k}(x)$ denote the summand of $b_k(x)$. It is readily checked that

$$B'_{j,k}(x) = (k+1)B_{j,k+1}(x).$$

The result immediately follows. ∎

Lemma 2.2 *For* $n \ge 1$ *and* $1 \le k \le m$*, we have*

$$f_k^{(n)}(x) = (k+n-1)_n b_{k+n-1}(x) - (k+n)_n b_{k+n}(x), \qquad (2.1)$$

where $(m)_j = m(m-1)\cdots(m-j+1)$*.*

Proof. Use induction on n. For $n = 1$, we have

$$f_k^{(n)}(x) = f_k'(x) = kb_k - (k+1)b_{k+1}.$$

Assume that the lemma holds for $n = j$, namely,

$$f_k^{(j)}(x) = (k+j-1)_j b_{k+j-1}(x) - (k+j)_j b_{k+j}(x).$$

Therefore,

$$\begin{aligned} f_k^{(j+1)}(x) &= (k+j-1)_j b'_{k+j-1}(x) - (k+j)_j b'_{k+j}(x) \\ &= (k+j)(k+j-1)_j b_{k+j}(x) - (k+j+1)(k+j)_j b_{k+j+1}(x) \\ &= (k+j)_{j+1} b_{k+j}(x) - (k+j+1)_{j+1} b_{k+j+1}(x). \end{aligned}$$

This completes the proof. ∎

Lemma 2.3 *For* $1 \le k \le m$ *and* $0 \le n \le m-k$*, the polynomial* $f_k^{(n)}(x)$ *has at most one real zero on the interval* $(0, +\infty)$*. In particular,* $f_k(x)$ *has at most one real zero on the interval* $(0, +\infty)$*.*

Proof. Use induction on n from $m-k$ to 0. First, we consider the case $n = m-k$. Recall that

$$f_k(x) = \sum_{j=k-1}^{m} \binom{j}{k-1} a_j x^{j-k+1} - \sum_{j=k}^{m} \binom{j}{k} a_j x^{j-k}.$$

Thus $f_k(x)$ is a polynomial of degree $m-k+1$. Note that

$$f_k^{(m-k)}(x) = (m-k+1)!\binom{m}{k-1} a_m x + \left[\binom{m-1}{k-1} a_{m-1} - \binom{m}{k} a_m\right](m-k)!.$$

Clearly, $f_k^{(m-k)}(x)$ has at most one real zero x_0 on $(0, +\infty)$. So the lemma is true for $n = m - k$.

Suppose that the lemma holds for $n = j$, where $m - k \geq j \geq 1$. We proceed to show that $f_k^{(j-1)}(x)$ has at most one real zero on $(0, +\infty)$. From the inductive hypothesis it follows that $f_k^{(j)}(x)$ has at most one real zero on $(0, +\infty)$. In light of (2.1), it is easy to verify that $f_k^{(j)}(+\infty) > 0$ and

$$f_k^{(j)}(0) = (k + j - 1)_j a_{k+j-1} - (k + j)_j a_{k+j} \leq 0.$$

It follows that either the polynomial $f_k^{(j-1)}(x)$ is increasing on the entire interval $(0, +\infty)$, or there exists a positive real number r such that $f_k^{(j-1)}(x)$ is decreasing on $(0, r]$ and increasing on $(r, +\infty)$. Again by (2.1) we find $f_k^{(j-1)}(+\infty) > 0$ and

$$f_k^{(j-1)}(0) = (k + j - 2)_{j-1} a_{k+j-2} - (k + j - 1)_{j-1} a_{k+j-1} \leq 0.$$

So we conclude that $f_k^{(j-1)}(x)$ has at most one real zero on $(0, +\infty)$. This completes the proof. ∎

Proof of Theorem 1.1. In view of (1.4), we have

$$P(x + d) = \sum_{k=0}^{m} a_k (x + d)^k = \sum_{k=0}^{m} b_k(d) x^k.$$

Let us first prove that $M^*(P, d_1) \geq M^*(P, d_2)$. Suppose that $M^*(P, d_1) = k$. If $k = m$, then the inequality $M^*(P, d_1) \geq M^*(P, d_2)$ holds. For the case $0 \leq k < m$, it suffices to verify that $b_k(d_2) > b_{k+1}(d_2)$. By Lemma 2.2, $f_{k+1}(x)$ has at most one real zero on $(0, +\infty)$. Note that

$$f_{k+1}(0) \leq 0 \quad \text{and} \quad f_{k+1}(+\infty) > 0.$$

From $M^*(P, d_1) = k$ it follows that $b_k(d_1) > b_{k+1}(d_1)$, that is $f_{k+1}(d_1) > 0$. Therefore, $f_{k+1}(d_2) > 0$, that is, $b_k(d_2) > b_{k+1}(d_2)$.

Similarly, it can be seen that $M_*(P, d_1) \geq M_*(P, d_2)$. Suppose that $M_*(P, d_2) = k$. If $k = 0$, then we have $M_*(P, d_1) \geq M_*(P, d_2)$. If $0 < k \leq m$, it is necessary to show that $b_{k-1}(d_1) < b_k(d_1)$. Again, by Lemma 2.2, we know that $f_k(x)$ has at most one real zero on $(0, +\infty)$. From $M_*(P, d_2) = k$, it follows that $b_{k-1}(d_2) < b_k(d_2)$, that is $f_k(d_2) < 0$. By the boundary conditions

$$f_k(0) \leq 0 \quad \text{and} \quad f_k(+\infty) > 0,$$

we obtain $f_k(d_1) < 0$, that is $b_{k-1}(d_1) < b_k(d_1)$. This completes the proof. ∎

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