

Ratio Monotonicity of Polynomials Derived from Nondecreasing Sequences

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Abstract

The ratio monotonicity of a polynomial is a stronger property than log-concavity. Let $P(x)$ be a polynomial with nonnegative and nondecreasing coefficients. We prove the ratio monotone property of $P(x+1)$, which leads to the log-concavity of $P(x+c)$ for any $c \geq 1$ due to Llamas and Martínez-Bernal. As a consequence, we obtain the ratio monotonicity of the Boros-Moll polynomials obtained by Chen and Xia without resorting to the recurrence relations of the coefficients.

Keywords: log-concavity, ratio monotonicity, Boros-Moll polynomials.

1 Introduction

This paper is concerned with the ratio monotone property of polynomials derived from nonnegative and nondecreasing sequences. A sequence $\{a_k\}_{0 \leq k \leq m}$ of positive real numbers is said to be unimodal if there exists an integer $r \geq 0$ such that

$$a_0 \leq \cdots \leq a_{r-1} \leq a_r \geq a_{r+1} \geq \cdots \geq a_m,$$

and it is said to be spiral if

$$a_m \leq a_0 \leq a_{m-1} \leq a_1 \leq \cdots \leq a_{\lfloor \frac{m}{2} \rfloor}, \quad (1.1)$$

where $\lfloor \frac{m}{2} \rfloor$ stands for the largest integer not exceeding $\frac{m}{2}$. We say that a sequence $\{a_k\}_{0 \leq k \leq m}$ is log-concave if for any $1 \leq k \leq m-1$,

$$a_k^2 - a_{k+1}a_{k-1} \geq 0,$$

or equivalently,

$$\frac{a_0}{a_1} \leq \frac{a_1}{a_2} \leq \dots \leq \frac{a_{m-1}}{a_m}.$$

It is easy to see that either log-concavity or the spiral property implies unimodality, while a log-concave sequence is not necessarily spiral, and vice versa.

A stronger property, which implies both log-concavity and the spiral property, was introduced by Chen and Xia [6] and is called the ratio monotonicity. A sequence of positive real numbers $\{a_k\}_{0 \leq k \leq m}$ is said to be ratio monotone if

$$\frac{a_m}{a_0} \leq \frac{a_{m-1}}{a_1} \leq \dots \leq \frac{a_{m-i}}{a_i} \leq \dots \leq \frac{a_{m-\lfloor \frac{m-1}{2} \rfloor}}{a_{\lfloor \frac{m-1}{2} \rfloor}} \leq 1 \quad (1.2)$$

and

$$\frac{a_0}{a_{m-1}} \leq \frac{a_1}{a_{m-2}} \leq \dots \leq \frac{a_{i-1}}{a_{m-i}} \leq \dots \leq \frac{a_{\lfloor \frac{m}{2} \rfloor - 1}}{a_{m-\lfloor \frac{m}{2} \rfloor}} \leq 1. \quad (1.3)$$

Given a polynomial $P(x) = a_0 + a_1x + \dots + a_mx^m$ with positive coefficients, we say that $P(x)$ is log-concave (or ratio monotone) if $\{a_k\}_{0 \leq k \leq m}$ is log-concave (resp., ratio monotone).

Assume that $P(x)$ is a polynomial with nonnegative and nondecreasing coefficients. Boros and Moll [3] proved the unimodality of $P(x+1)$ which implies the unimodality of the Boros-Moll polynomials. They posed the conjecture that the Boros-Moll polynomials are log-concave, which was confirmed by Kauers and Paule [8]. Alvarez et al. [1] showed that $P(x+n)$ is also unimodal for any positive integer n . Wang and Yeh [12] obtained a stronger result that $P(x+c)$ is unimodal for $c > 0$. Llamas and Martínez-Bernal [9] proved that $P(x+c)$ is log-concave for $c \geq 1$.

In this paper, we prove that if $P(x)$ is a polynomial with nonnegative and nondecreasing coefficients, then $P(x+1)$ is ratio monotone. This property implies the log-concavity of $P(x+1)$. Note that by a criterion for log-concavity due to Brenti [5], the log-concavity of $P(x+1)$ leads to the log-concavity of $P(x+c)$ for $c \geq 1$, as established by Llamas and Martínez-Bernal [9]. The ratio monotonicity of $P(x+1)$ serves as a simple proof of the ratio monotonicity of the Boros-Moll polynomials obtained by Chen and Xia [7] without resorting to the recurrence relations of the coefficients.

2 The ratio monotone property

The main result of this paper is given below.

Theorem 2.1 *If $P(x)$ is a polynomial with nonnegative and nondecreasing coefficients, then $P(x+1)$ is ratio monotone.*

To prove Theorem 2.1, we need three lemmas. The first lemma is a special case of [6, Lemma 2.1].

Lemma 2.2 Suppose that a, b, c, d, e, f are positive real numbers satisfying

$$\frac{a}{b} \leq \frac{c}{d} \leq \frac{e}{f}.$$

Then

$$\frac{a+c}{b+d} \leq \frac{e+c}{f+d}.$$

Lemma 2.3 If $B(x)$ is a ratio monotone polynomial, so is $(x+1)B(x)$.

Proof. Let

$$B(x) = \sum_{k=0}^m a_k x^k \quad \text{and} \quad (x+1)B(x) = \sum_{k=0}^{m+1} b_k x^k.$$

For each k we have $b_k = a_{k-1} + a_k$, where a_{-1} and a_{m+1} are set to 0.

When $m = 2n$, the ratio monotonicity of $B(x)$ states that

$$\frac{a_{2n}}{a_0} \leq \frac{a_{2n-1}}{a_1} \leq \dots \leq \frac{a_{2n-i}}{a_i} \leq \dots \leq \frac{a_{n+1}}{a_{n-1}} \leq 1 \quad (2.1)$$

and

$$\frac{a_0}{a_{2n-1}} \leq \frac{a_1}{a_{2n-2}} \leq \dots \leq \frac{a_{i-1}}{a_{2n-i}} \leq \dots \leq \frac{a_{n-1}}{a_n} \leq 1. \quad (2.2)$$

In order to show that $(x+1)B(x)$ is ratio monotone, we need to verify that

$$\frac{b_{2n+1}}{b_0} \leq \frac{b_{2n}}{b_1} \leq \dots \leq \frac{b_{2n+1-i}}{b_i} \leq \dots \leq \frac{b_{n+1}}{b_n} \leq 1 \quad (2.3)$$

and

$$\frac{b_0}{b_{2n}} \leq \frac{b_1}{b_{2n-1}} \leq \dots \leq \frac{b_i}{b_{2n-i}} \leq \dots \leq \frac{b_{n-1}}{b_{n+1}} \leq 1. \quad (2.4)$$

We first consider (2.3). Since

$$\frac{a_{2n}}{a_0} \leq \frac{a_{2n-1}}{a_1},$$

we see that

$$\frac{a_{2n}}{a_0} \leq \frac{a_{2n-1} + a_{2n}}{a_1 + a_0},$$

that is,

$$\frac{b_{2n+1}}{b_0} \leq \frac{b_{2n}}{b_1}.$$

For $1 \leq i \leq n-1$, from (2.1) we deduce that

$$\frac{a_{2n+1-i}}{a_{i-1}} \leq \frac{a_{2n-i}}{a_i} \leq \frac{a_{2n-i-1}}{a_{i+1}}.$$

By Lemma 2.2, we obtain

$$\frac{a_{2n+1-i} + a_{2n-i}}{a_i + a_{i-1}} \leq \frac{a_{2n-i} + a_{2n-i-1}}{a_{i+1} + a_i},$$

or equivalently,

$$\frac{b_{2n+1-i}}{b_i} \leq \frac{b_{2n-i}}{b_{i+1}}.$$

In light of (2.1), we see that $a_{n+1} \leq a_{n-1}$, and thus we have

$$\frac{b_{n+1}}{b_n} = \frac{a_{n+1} + a_n}{a_n + a_{n-1}} \leq 1.$$

Next, we proceed to prove (2.4). Since $\frac{a_0}{a_{2n-1}} \leq \frac{a_1}{a_{2n-2}}$, we get that

$$\frac{a_0}{a_{2n-1} + a_{2n}} \leq \frac{a_1 + a_0}{a_{2n-2} + a_{2n-1}},$$

that is,

$$\frac{b_0}{b_{2n}} \leq \frac{b_1}{b_{2n-1}}.$$

For $2 \leq i \leq n-1$, in view of (2.2) we find that

$$\frac{a_{i-2}}{a_{2n-i+1}} \leq \frac{a_{i-1}}{a_{2n-i}} \leq \frac{a_i}{a_{2n-i-1}}.$$

By Lemma 2.2, we have

$$\frac{a_{i-1} + a_{i-2}}{a_{2n-i+1} + a_{2n-i}} \leq \frac{a_i + a_{i-1}}{a_{2n-i} + a_{2n-i-1}},$$

which can be expressed as

$$\frac{b_{i-1}}{b_{2n-i+1}} \leq \frac{b_i}{b_{2n-i}}.$$

From (2.2) it is clear that $a_{n-2} \leq a_{n+1}$ and $a_{n-1} \leq a_n$, and hence

$$\frac{b_{n-1}}{b_{n+1}} = \frac{a_{n-1} + a_{n-2}}{a_{n+1} + a_n} \leq 1.$$

The case $m = 2n + 1$ can be dealt with in the same manner. This completes the proof. ■

The third lemma is concerned with an inequality of increasing positive sequences.

Lemma 2.4 *For any nondecreasing positive sequence $\{a_k\}_{0 \leq k \leq m}$, we have*

$$\frac{m(m+1)}{2} a_m^2 + a_m a_{m-1} \geq \left(\sum_{k=0}^{m-2} (m-1-k) a_k \right) a_{m-1} + \left(\sum_{k=0}^m a_k \right) a_{m-2}.$$

Proof. Since $0 < a_0 \leq a_1 \leq \dots \leq a_{m-1} \leq a_m$, we have

$$\begin{aligned} & \frac{m(m+1)}{2} a_m^2 + a_m a_{m-1} - \left(\sum_{k=0}^{m-2} (m-1-k) a_k \right) a_{m-1} - \left(\sum_{k=0}^m a_k \right) a_{m-2} \\ & \geq \frac{m(m+1)}{2} a_m^2 + a_m a_{m-1} - \sum_{k=0}^{m-2} (m-1-k) a_m^2 - \sum_{k=1}^m a_m^2 - a_m a_{m-1}, \end{aligned}$$

which simplifies to zero, as desired. ■

Proof of Theorem 2.1. We use induction on the degree m of $P(x)$. Let

$$P(x) = \sum_{k=0}^m a_k x^k,$$

where $0 < a_0 \leq a_1 \leq \dots \leq a_{m-1} \leq a_m$.

When $m = 2$, we have

$$P(x+1) = a_2 x^2 + (a_1 + 2a_2)x + a_0 + a_1 + a_2.$$

Note that $a_2 \leq a_0 + a_1 + a_2$, $a_0 + a_1 + a_2 \leq a_1 + 2a_2$. Therefore, the theorem holds for $m = 2$.

Now assume that the theorem holds for polynomials of degree $m-1$. We need to show that it is also true for polynomials $P(x)$ of degree m . Suppose that

$$P(x+1) = \sum_{k=0}^m a_k (x+1)^k = \sum_{k=0}^m d_k x^k. \tag{2.5}$$

We wish to prove that

$$\frac{d_m}{d_0} \leq \frac{d_{m-1}}{d_1} \leq \dots \leq \frac{d_{m-i}}{d_i} \leq \dots \leq \frac{d_{m-\lfloor \frac{m-1}{2} \rfloor}}{d_{\lfloor \frac{m-1}{2} \rfloor}} \leq 1 \tag{2.6}$$

and

$$\frac{d_0}{d_{m-1}} \leq \frac{d_1}{d_{m-2}} \leq \dots \leq \frac{d_{i-1}}{d_{m-i}} \leq \dots \leq \frac{d_{\lfloor \frac{m}{2} \rfloor - 1}}{d_{m-\lfloor \frac{m}{2} \rfloor}} \leq 1. \tag{2.7}$$

Let

$$Q(x) = \sum_{k=0}^{m-1} a_{k+1} x^k.$$

Then

$$P(x+1) = a_0 + (x+1)Q(x+1).$$

By the induction hypothesis and Lemma 2.3, we deduce that the polynomial

$$(x+1)Q(x+1) = d_0 - a_0 + \sum_{k=1}^m d_k x^k$$

is ratio monotone. It follows that

$$\frac{d_m}{d_0 - a_0} \leq \frac{d_{m-1}}{d_1} \leq \dots \leq \frac{d_{m-i}}{d_i} \leq \dots \leq \frac{d_{m-\lfloor \frac{m-1}{2} \rfloor}}{d_{\lfloor \frac{m-1}{2} \rfloor}} \leq 1 \quad (2.8)$$

and

$$\frac{d_0 - a_0}{d_{m-1}} \leq \frac{d_1}{d_{m-2}} \leq \dots \leq \frac{d_{i-1}}{d_{m-i}} \leq \dots \leq \frac{d_{\lfloor \frac{m}{2} \rfloor - 1}}{d_{m-\lfloor \frac{m}{2} \rfloor}} \leq 1. \quad (2.9)$$

Clearly, (2.6) follows from (2.8). To prove (2.7), it remains to show that

$$\frac{d_0}{d_{m-1}} \leq \frac{d_1}{d_{m-2}}.$$

From (2.5), we see that

$$d_0 = \sum_{k=0}^m a_k, \quad d_{m-1} = a_{m-1} + ma_m,$$

and

$$d_1 = \sum_{k=0}^m ka_k, \quad d_{m-2} = a_{m-2} + (m-1)a_{m-1} + \binom{m}{2}a_m.$$

Consequently, it suffices to show that

$$\frac{\sum_{k=0}^m a_k}{a_{m-1} + ma_m} \leq \frac{\sum_{k=0}^m ka_k}{a_{m-2} + (m-1)a_{m-1} + \binom{m}{2}a_m},$$

or equivalently,

$$\begin{aligned} & \left(\sum_{k=0}^m ka_k \right) a_{m-1} + \left(\sum_{k=0}^m mka_k \right) a_m - \left(\sum_{k=0}^m a_k \right) a_{m-2} \\ & - \left(\sum_{k=0}^m (m-1)a_k \right) a_{m-1} - \left(\sum_{k=0}^m \binom{m}{2} a_k \right) a_m \geq 0. \end{aligned}$$

The left hand side of the above inequality can be simplified to

$$\left(\sum_{k=0}^m \frac{2k - m + 1}{2} a_k \right) ma_m + \left(\sum_{k=0}^m (k - m + 1) a_k \right) a_{m-1} - \left(\sum_{k=0}^m a_k \right) a_{m-2},$$

which can be rewritten as a sum of

$$\left(\sum_{k=0}^{m-1} \frac{2k - m + 1}{2} a_k \right) ma_m \quad (2.10)$$

and

$$\frac{m(m+1)}{2}a_m^2 + a_m a_{m-1} - \left(\sum_{k=0}^{m-2} (m-1-k) a_k \right) a_{m-1} - \left(\sum_{k=0}^m a_k \right) a_{m-2}. \quad (2.11)$$

By Lemma 2.4, the sum in (2.11) is nonnegative. The sum in (2.10) is also nonnegative, since

$$\begin{aligned} \sum_{k=0}^{m-1} \frac{2k-m+1}{2} a_k &= \sum_{k=\lfloor \frac{m-1}{2} \rfloor + 1}^{m-1} \frac{2k-m+1}{2} a_k - \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \frac{m-1-2k}{2} a_k \\ &= \sum_{k=0}^{m-2-\lfloor \frac{m-1}{2} \rfloor} \frac{m-1-2k}{2} a_{m-1-k} - \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \frac{m-1-2k}{2} a_k \\ &= \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \frac{m-1-2k}{2} (a_{m-1-k} - a_k), \end{aligned}$$

which is nonnegative, and thus the proof is complete. \blacksquare

Theorem 2.1 leads to the following result of Llamas and Martínez-Bernal [9], since the ratio monotonicity implies log-concavity of $P(x+1)$ and the log-concavity of $P(x+1)$ implies the log-concavity of $P(x+c)$ for $c \geq 1$ by a criterion of Brenti [4, 5].

Corollary 2.5 *If $P(x)$ is a polynomial with nonnegative and nondecreasing coefficients, then for any $c \geq 1$ the polynomial $P(x+c)$ is log-concave and has no internal zero coefficients.*

Theorem 2.1 also serves as a simple proof of the ratio monotonicity of the Boros-Moll polynomials $P_m(x)$, which were introduced by Boros and Moll [2] in their study of the following quartic integral

$$\int_0^{+\infty} \frac{1}{(t^4 + 2xt^2 + 1)^{m+1}} dt = \frac{\pi}{2^{m+3/2}(x+1)^{m+1/2}} P_m(x).$$

Let

$$c_k(m) = 2^{-2m+k} \binom{2m-2k}{m-k} \binom{m+k}{k}.$$

Boros and Moll showed that

$$P_m(x) = \sum_{k=0}^m c_k(m) (x+1)^k. \quad (2.12)$$

They also observed that, for $0 \leq k \leq m-1$,

$$\frac{c_k(m)}{c_{k+1}(m)} = \frac{(2m-2k-1)(k+1)}{(m-k)(m+k+1)} < 1.$$

Thus, $P_m(x - 1)$ is a polynomial with nonnegative and nondecreasing coefficients. Boros and Moll [2] proved that $P_m(x)$ is unimodal for any $m \geq 0$, and Moll [10] conjectured that $P_m(x)$ is log-concave for any m . This conjecture was confirmed by Kauers and Paule [8]. The ratio monotonicity of $P_m(x)$ was established by Chen and Xia and the proof is quite involved and heavily depends on inequalities on the coefficients. The proof of Theorem 2.1 shows that the log-concavity and ratio monotonicity only depend on the nondecreasing property of the coefficients of $P_m(x - 1)$.

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References

- [1] J. Alvarez, M. Amadis, G. Boros, D. Karp, V.H. Moll and L. Rosales, An extension of a criterion for unimodality, *Electron. J. Combin.* 8 (2001), #R30.
- [2] G. Boros and V.H. Moll, A sequence of unimodal polynomials, *J. Math. Anal. Appl.* 237 (1999), 272–285.
- [3] G. Boros and V.H. Moll. A criterion for unimodality, *Electron. J. Combin.* 6 (1999), #R10.
- [4] F. Brenti, Unimodal, log-concave, and Pólya frequency sequences in combinatorics, *Mem. Amer. Math. Soc.* 413 (1989), 1–106.
- [5] F. Brenti, Log-concave and unimodal sequences in algebra, combinatorics and geometry: an update, *Contemp. Math.* 178 (1994), 71–89.
- [6] W.Y.C. Chen and E.X.W. Xia, The ratio monotonicity of the q -derangement numbers, *Discrete Math.*, to appear.
- [7] W.Y.C. Chen and E.X.W. Xia, The ratio monotonicity of the Boros-Moll polynomials, *Math. Comput.* 78 (2009), 2269–2282.
- [8] M. Kauers and P. Paule, A computer proof of Moll’s log-concavity conjecture, *Proc. Amer. Math. Soc.* 135 (2007), 3847–3856.
- [9] A. Llamas, J. Martínez-Bernal, Nested log-concavity, *Commun. Algebra* 38 (2010), 1968–1981.
- [10] V.H. Moll, The evaluation of integrals: A personal story, *Notices Amer. Math. Soc.* 49 (2002), 311–317.
- [11] R.P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics and geometry, *Ann. New York Acad. Sci* 576 (1989), 500–535.
- [12] Y. Wang and Y.-N. Yeh, Proof of a conjecture on unimodality, *European J. Combin.* 26 (2005), 617–627.