The Skew Schubert Polynomials

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Abstract. We obtain a tableau definition of the skew Schubert polynomials named by Lascoux, which are defined as flagged double skew Schur functions. These polynomials are in fact Schubert polynomials in two sets of variables indexed by 321-avoiding permutations. From the divided difference definition of the skew Schubert polynomials, we construct a lattice path interpretation based on the Chen-Li-Louck pairing lemma. The lattice path explanation immediately leads to the determinantal definition and the tableau definition of the skew Schubert polynomials. For the case of a single variable set, the skew Schubert polynomials reduce to flagged skew Schur functions as studied by Wachs and by Billey, Jockusch, and Stanley. We also present a lattice path interpretation of the isobaric divided difference operators, and derive an expression for the flagged Schur function in terms of isobaric operators acting on a monomial. Moreover, we find lattice path interpretations for the Giambelli identity and the Lascoux-Pragacz identity for super-Schur functions. For the super-Lascoux-Pragacz identity, the lattice path construction is related to the code of the partition which determines the directions of the lines parallel to the y-axis in the lattice.

Keywords: Lattice path; Isobaric divided difference; Flagged double skew Schur function; Skew Schubert polynomial; Giambelli identity; Lascoux-Pragacz identity; Key polynomial; Code of a partition

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1. Introduction

The flagged double skew Schur functions (on variable sets X and Y) are called skew Schubert polynomials by Lascoux [15]; these are Schubert polynomials indexed by 321-avoiding permutations. When Y is the emptyset, the skew Schubert polynomials reduce to the flagged skew Schur functions, which have been studied by Wachs [28] and Billey et al. [3]. Note that the skew Schubert polynomials referred to in this paper are different from the objects studied by Lenart and Sottile [22] under the same name. There are many interesting specializations of skew Schubert polynomials, for example, the binomial determinants [9], the q-binomial determinants [4], and the double Schur functions [5].

This paper contains the following results.

1. A lattice path interpretation of skew Schubert polynomials based on the divided difference definition due to Lascoux and Schützenberger [14, 18, 19]. The determinantal formula of Lascoux [15] directly follows from the lattice path structure by the Gessel-Viennot argument [9, 10].

2. A tableau interpretation of skew Schubert polynomials based on the lattice path construction.

3. We introduce the notion of flagged skew Schubert polynomials from which we can get the flagged double Schur functions.

4. A lattice path interpretation of isobaric divided differences leading to an expression for the flagged Schur functions in terms of isobaric divided differences. This implies that any flagged Schur function is a key polynomial, which is also a consequence of a result of Reiner and Shimozono on the necessary and sufficient condition for a flagged skew Schur function to be a key polynomial [25].

5. Lattice path interpretations of the Giambelli identity and the Lascoux-Pragacz identity for super-Schur functions. The code of a partition is used to determine the directions of the lines parallel to the y-axis in the lattice.

The lattice path method of Gessel-Viennot [9, 10] has been extensively used in the study of Schur functions and their generalizations; see Brenti [4], Goulden and Greene [11], Hamel and Goulden [12, 13], and Stembridge [27]. A lattice path approach to the flagged double Schur functions is presented in [5]. We note that the weight of a path given in this paper for the skew Schubert polynomials is not the same as that in [5] for the flagged double Schur functions.

The Giambelli identity and the Lascoux-Pragacz identity for the super-Schur functions have been studied by various methods; see Eğecioğlu and Remmel [6], Lascoux and Pragacz [17], and Macdonald [24]. The super-Schur functions are related to the skew Schubert polynomial in the sense that they have similar tableau representations when the variable sets X and Y of the super-Schur function are indexed by integers from $-\infty$ to ∞. Besides the existing lattice path treatments of the super-Schur functions, it seems desirable to give lattice path proofs of the Giambelli identity and the Lascoux-Pragacz identity based on the new tableau representation of super-Schur functions [11, 24] rather than the supersymmetric tableau representation given by Berele and Regev [1].

2. Notation and definitions

By a *partition* λ we mean a weakly decreasing sequence $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m > 0$, where m is the *length* of λ, denoted by $\ell(\lambda)$. The *Ferrers diagram* of λ is an array of cells with $\ell(\lambda)$ left justified rows and λ_i cells in row i. We denote the *conjugate* of λ by λ'. Its Ferrers diagram is the transpose of the Ferrers diagram of λ. Given two partitions λ and μ, we say $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all i. If $\mu \subseteq \lambda$, we can define a *skew partition* λ/μ whose Ferres diagram can be obtained from the Ferrers diagram of λ by peeling off the Ferrers diagram of μ from the upper left corner.

Recall that the number of diagonal cells in the Ferrers diagram is called the *rank* of λ. Suppose that the rank of λ is r. Then we write $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_r)$ where $\alpha_i = \lambda_i - i$, $\beta_j = \lambda'_j - j$. Clearly, α and β provide a unique encoding of the partition λ, which is called the *Frobenius notation*, denoted by $(\alpha|\beta)$. A partition of rank 1 is called a *hook*. One sees that the Frobenius notation provides the hook decomposition along the main diagonal of the Ferrers diagram.

Lascoux and Pragacz [17] introduced the *ribbon decomposition* of a partition λ. A *ribbon* is a skew partition whose Ferrers diagram is connected and does not contain any 2×2 squares; the *rim* of a diagram is the maximal outer ribbon of the diagram. Given a partition λ with rank r, we can decompose its Ferrers diagram into successive rims $\Theta_1, \Theta_2, \ldots, \Theta_r$ beginning from the outside, where Θ_1 is the rim of λ, Θ_2 is the rim of the partition obtained from λ by removing the rim, etc. It is clear that $\Theta_1, \Theta_2, \ldots, \Theta_r$ also provide a unique encoding of λ, which is called the ribbon decomposition of λ. Note that each of the ribbons $\Theta_1, \Theta_2, \ldots, \Theta_r$ contains a diagonal cell.

The diagonal cells of the diagram break each ribbon Θ_i into three parts: the diagonal cell (i,i), the part Θ_i^+ above (i,i) and the part Θ_i^- below (i,i). We denote by $\Theta_i^+ \& \Theta_j^-$ respectively the ribbon which are obtained by adding the diagonal cell to Θ_i^+ and Θ_j^-, and then merging the two ribbons by overlapping the diagonal cells, as shown in Figure 2.1.

Let λ/μ be a skew partition. A *semistandard Young tableau* on X of shape

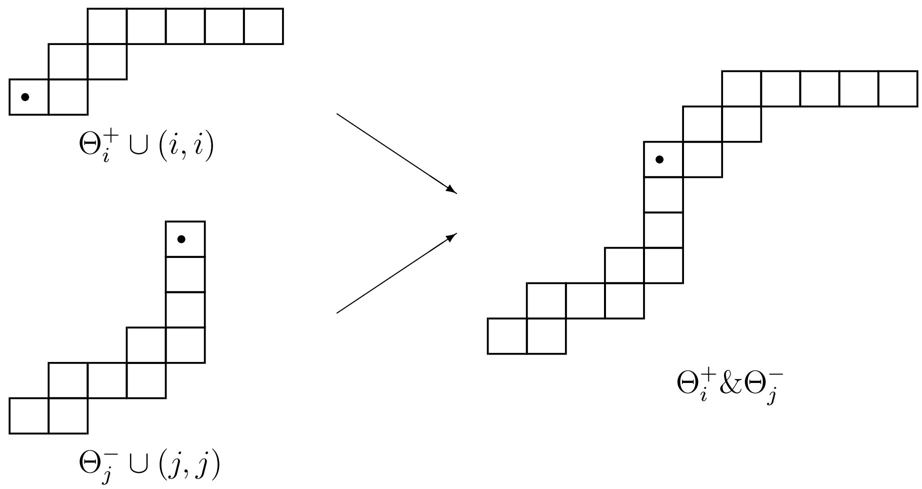


Figure 2.1 Merging of two ribbons

λ/μ is meant to be strictly increasing in each column and weakly increasing in each row. For each cell (i,j) of a tableau T in row i and column j, we denote the element in the cell by $T_{i,j}$. Let $C_{i,j} = j - i$ denote the *content* of the cell (i,j).

We now come to the definition of flagged Schur functions. Let λ be a partition of length m, and b a flag sequence of the same length such that $0 < b_1 \le b_2 \le \cdots \le b_m = n$. The *flagged Schur function* with shape λ and flag b is defined as

$$s_\lambda(b) = \det(h_{\lambda_i - i + j}(b_i))_{m\times m}, \tag{2.1}$$

where $h_{\lambda_i-i+j}(b_i) = h_{\lambda_i-i+j}(x_1, x_2, \ldots, x_{b_i})$. When $b_i = n$ for all i, $s_\lambda(b)$ is the usual Schur function $s_\lambda(x_1, x_2, \ldots, x_n)$ with n variables.

The double form of the flagged Schur function can be used to define the skew Schubert polynomials. Let J, I be two weakly increasing codes of length n with $I \le J$ (that is, $I_k \le J_k$ for all k); then $\lambda = (J_n - I_1, J_n - I_2, \cdots, J_n - I_n)$ and $\mu = (J_n - J_1, J_n - J_2, \cdots, J_n - J_n)$ are two partitions and $\lambda \ge \mu$. Let $< J/I >= (0^{I_1}, J_1 - I_1, 0^{I_2 - I_1}, J_2 - I_2, 0^{I_3 - I_2}, J_3 - I_3, \ldots, J_n - I_n)$. Then by Theorem 2.1 in [3], $\langle J/I \rangle$ is the code of some 321-avoiding permutation by adding sufficient zeros at the end and conversely the code of every 321-avoiding permutation must have the form $\langle J/I \rangle$. It is clear that all Grassmannian permutations are 321-avoiding, so the skew Schubert polynomials are generalizations of double Schur functions. Suppose that the permutation with code $\langle J/I \rangle$ can be taken as a permutation in a symmetric group of order m throughout this paper. Then we denote the skew Schubert polynomial with respect to code $\langle J/I\rangle$ by $\mathbb{G}_{\langle J/I\rangle}(X_m, Y_m)$, where $X_m = \{x_1, \ldots, x_m\}$, $Y_m = \{y_1, \ldots, y_m\}$. With the technique of divided differences, Lascoux gave a determinantal definition of the skew Schubert polynomial:

Proposition 2.1 *Let J, I be two weakly increasing codes of length n corresponding to some Grassmannian permutations with $I \leq J$, and $\lambda = (J_n - I_1, J_n - I_2, \cdots, J_n - I_n)$, $\mu = (J_n - J_1, J_n - J_2, \cdots, J_n - J_n)$ be two partitions with $\lambda \geq \mu$; then*

$$\mathbb{G}_{\langle J/I\rangle}(X_m, Y_m) = \det(h_{\lambda_i-\mu_j-i+j}(X_{\widehat{\phi}_i} - Y_{\widehat{\phi}_i+\lambda_i-\mu_j-i+j-1}))_{n\times n}, \tag{2.2}$$

where $\widehat{\phi}_i = I_i + i = J_n - \lambda_i + i$ $(i = 1, 2, \cdots, n)$ is the flag of the 321*-avoiding permutation with code $\langle J/I\rangle$.*

Next we introduce the determinantal formulas for the super-Schur functions. For convenience, we suppose the super symmetric functions are over two countably infinite sets of variables X and Y, where $X = \{\ldots, x_{-1}, x_0, x_1, \ldots\}$ and $Y = \{\ldots, y_{-1}, y_0, y_1, \ldots\}$. Macdonald [24] and Goulden and Greene [11] give a tableau description of the super-Schur functions $S_\lambda(X, Y)$, which can be generalized to super-Schur functions of skew shape. Given a skew partition λ/μ, we have

$$S_{\lambda/\mu}(X, Y) = \sum_T \prod_{(i,j)\in T} (x_{T_{i,j}} - y_{T_{i,j}+C_{i,j}}) \tag{2.3}$$

summed over all semistandard tableaux T filled with $\{\ldots, -1, 0, 1, \ldots\}$ of shape λ/μ. There are four important determinantal formulas for super-Schur functions. Besides the Jacobi-Trudi formula and its dual form for the super-Schur functions, known as the Nägelsbach-Kostka formula, there are two others:

- The Giambelli formula

$$S_\lambda(X, Y) = \det(S_{(\alpha_i|\beta_j)}(X, Y))_{r\times r}. \tag{2.4}$$

- The Lascoux-Pragacz formula

$$S_\lambda(X, Y) = \det(S_{\Theta_i^+ \& \Theta_j^-}(X, Y))_{r\times r}, \tag{2.5}$$

where λ is a partition of length m and rank r, and $S_{(\alpha_i|\beta_j)}$ and $S_{\Theta_i^+\&\Theta_j^-}$ are given by Eq. (2.3) and are called the hook Schur function and the ribbon Schur function respectively [17].

3. Skew Schubert polynomials

In this section we obtain a tableau representation of the skew Schubert polynomials. This is achieved via a lattice path interpretation of the skew Schubert polynomials based on the divided difference definition. Once the lattice path construction is accomplished, the tableau definition and the determinantal formula both become immediate consequences. Moreover one also obtains a new tableau interpretation of the double Schur functions different from that given in [5]. All the lattice paths of this paper are in the two dimensional integer lattice $\mathbb{Z} \times \mathbb{Z}$, namely, the set of lattice points $\{(i,j) | i, j \in \mathbb{Z}\}$.

A lattice path is defined in the usual sense, which is a directed path in the integer lattice. In practice, we will assign a direction to each line in the integer lattice so that at each point the choices of next move are specified. In other words we will work with the directed integer lattice (or a region of the integer lattice in which each line has a direction). By a weight function we mean an assignment of weights to each (directed) edge in the lattice. Then the weight of a lattice path is taken as the product of the weights of the steps. For an m-tuple of paths $(P_1, P_2, \ldots, P_m)$, the weight is defined to be the product of all the weights. Given two tuples of lattice points $A = (A_1, A_2, \ldots, A_m)$ and $B = (B_1, B_2, \ldots, B_m)$, let $GF(A, B)$ represent the generating function (sum of weights) of all tuples $(P_1, P_2, \ldots, P_m)$ of non-intersecting lattice paths, where each P_i is from A_i to B_i. Such an m-tuple of non-intersecting lattice paths is called a lattice path configuration from A to B.

To give a divided difference definition of the skew Schubert polynomials, we recall the definition of Schubert polynomials. Let w be a permutation on $\{1, 2, \ldots, n\}$, and let the *length* of w be the inversion number of w, denoted by $\ell(w)$. Let σ_i be the permutation which interchanges i and $i+1$, and let w_0 be the longest permutation $[n, n-1, \ldots, 1]$. Given a function $g(x_1, x_2, \ldots, x_n)$, the simple transposition operator σ_i is defined by

$$\sigma_i g(x_1, x_2, \ldots, x_n) = g(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n),$$

and the *divided difference* operator ∂_i is defined by

$$\partial_i g = \frac{g - \sigma_i g}{x_i - x_{i+1}}.$$

Let

$$\Delta(X_n, Y_n) = \prod_{i+j \leq n} (x_i - y_j),$$

where $X_n = \{x_1, x_2, \ldots, x_n\}$ and $Y_n = \{y_1, y_2, \ldots, y_n\}$. Then the Schubert polynomials in two sets of variables X_n and Y_n can be recursively defined as follows [14, 18, 19, 23]:

$$\mathfrak{S}_w(X_n, Y_n) = \begin{cases} \Delta(X_n, Y_n), & \text{if } w = w_0 \\ \partial_i \mathfrak{S}_{w\sigma_i}(X_n, Y_n), & \text{if } \ell(w\sigma_i) = \ell(w) + 1. \end{cases}$$

It can be shown that the Schubert polynomials are well defined, because ∂_i satisfies the braid relations:

$$\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}, \quad \partial_i \partial_j = \partial_j \partial_i,$$

where $|i - j| > 1$. For any permutation $w = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$, where $k = \ell(w)$, we can define the operator ∂_w as follows:

$$\partial_w(g) = \partial_{i_k}\partial_{i_{k-1}}\cdots\partial_{i_1}(g), \tag{3.6}$$

where the operators are applied from right to left.

We now define the weight function W_d which will be used throughout this section. For a vertical step from (i, j) to $(i, j + 1)$ satisfying $i + j \geq 0$, the weight is $x_i - y_{i+j}$; for a vertical step from (i, j) to $(i, j + 1)$ satisfying $i+j < 0$, the weight is $x_i - y_{-(i+j)}$; for a horizontal step from (i, j) to $(i+1, j)$, the weight is 1.

We need the Pairing Lemma of Chen, Li, and Louck [5]:

Lemma 3.1 ([5, Lemma 4.4]) *Given the above weight function W_d, and two sequences of lattice points $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, B_2, \ldots, B_n)$ with $A_i = (q, k_i)$ for each i and $B_1 = (q, p)$ and $B_i = (q + 1, t_i)$ for $i \geq 2$, suppose that $p > k_1 > k_2 > \cdots > k_n$, $p - 1 > t_2 > \cdots > t_n$, and $k_i \leq t_i$ for $i \geq 2$. If all the lattice paths lie below the line $y = -x$ or above the line $y = -x$, then we have*

$$\partial_q(GF(A, B)) = GF(A, B'), \tag{3.7}$$

where B' is obtained from B by replacing B_1 with $(q + 1, p - 1)$.

The pairing lemma can be utilized to give a lattice path construction for the skew Schubert polynomials based on the divided difference definition. Suppose that w_J is the Grassmannian permutation with code J and w_I is the Grassmannian permutation with code I. Let

$$\partial^I = (\partial_{I_1}\cdots\partial_1)(\partial_{I_2+1}\cdots\partial_2)\cdots(\partial_{I_n+n-1}\cdots\partial_n),$$

then $\partial^I(\mathfrak{S}_{w_I}) = 1$, and $\mathbb{G}_{\langle J/I\rangle} = \partial^I \mathfrak{S}_{w_J}$ if $\langle J/I\rangle$ is the skew code of some 321-avoiding permutation. Chen, Li, and Louck [5] showed that $\mathfrak{S}_{w_J}$ equals

$GF(A,B)$ for $A_k = (k, -k+1)$ and $B_k = (n, J_{n-k+1} - k + 1)$ with respect to the weight function W_d. Now we construct an involution φ on lattice points such that $\varphi((i,j)) = (n+1-i, -n-j)$; correspondingly, a step from (i,j) to $(i,j+1)$ is mapped to a step from $\varphi(i,j+1)$ to $\varphi(i,j)$, and a step from (i,j) to $(i+1,j)$ is mapped to a step from $\varphi(i+1,j)$ to $\varphi(i,j)$. If $i+j \geq 0$, then we have

$$\begin{aligned} W_d((i,j) \to (i,j+1)) &= x_i - y_{i+j}, \qquad (3.8)\\ W_d(\varphi((i,j+1)) \to \varphi((i,j))) &= x_{n+1-i} - y_{i+j}. \qquad (3.9) \end{aligned}$$

If $i+j < 0$, then we have

$$\begin{aligned} W_d((i,j) \to (i,j+1)) &= x_i - y_{-(i+j)}, \qquad (3.10)\\ W_d(\varphi((i,j+1)) \to \varphi((i,j))) &= x_{n+1-i} - y_{-(i+j)}. \qquad (3.11) \end{aligned}$$

Let $A'_k = \varphi(B_{n+1-k}) = (1, -(k+J_k))$ and $B'_k = \varphi(A_{n+1-k}) = (k, -k)$. Note that the weights of horizontal steps are always 1. If w is a Grassmannian permutation in a symmetric group of order m satisfying

$$w_1 < \cdots < w_r > w_{r+1} < \cdots < w_m,$$

then $\mathfrak{S}_w$ is symmetric in $\{x_1, \ldots, x_r\}$ (see [23]). Therefore, for a Grassmannian permutation w_J, the Schubert polynomial $\mathfrak{S}_{w_J}(X_m, Y_m)$ is symmetric in $\{x_1, \ldots, x_n\}$. In this case, $\mathfrak{S}_{w_J}(X_m, Y_m)$ becomes the double Schur function. By the lattice path construction in [5], one has

$$\mathfrak{S}_{w_J}(X_m, Y_m) = GF(A,B). \qquad (3.12)$$

Since the involution φ only changes the indices of the x's in the evaluation of weights, from (3.12) it follows that $\mathfrak{S}_{w_J}(X_m, Y_m) = GF(A', B')$. By successively applying Lemma 3.1, we obtain

Theorem 3.2 *Let $A'_k = (1, -(k+J_k))$ and $B''_k = (k+I_k, -k-I_k)$. Let W_d be the weight function defined above. Then we have*

$$\mathbb{G}_{\langle J/I \rangle}(X_m, Y_m) = GF(A', B''), \qquad (3.13)$$

Applying the Gessel-Viennot argument, we can recover the determinantal formula (2.2).

Next we describe a bijection between the n-tuples of non-intersecting paths from A' to B'' and the flagged skew tableaux T of shape λ/μ with flag $\widehat{\phi}$. The i-th row T_i of T corresponds to the i-th path P_i, and the entries of T_i are just the indices of the x's of the weights of vertical steps from left to right. It is clear that the entries of T_i are smaller than or equal to $\widehat{\phi}_i$, when T is taken as a skew tableau with $n - \ell(\lambda)$ empty rows. Also, the column strictness of T follows from the non-intersecting property of the paths. Conversely, given a flagged tableau T such that the entries of T_i are smaller than or equal to $\widehat{\phi}_i$, we can construct an n-tuple $(P_1, P_2, \ldots, P_n)$ of non-intersecting paths by reversing the above procedure. Figure 3.1 is an illustration for the skew Schubert polynomial $G_{\langle[2,3,4]/[1,1,2]\rangle}$.

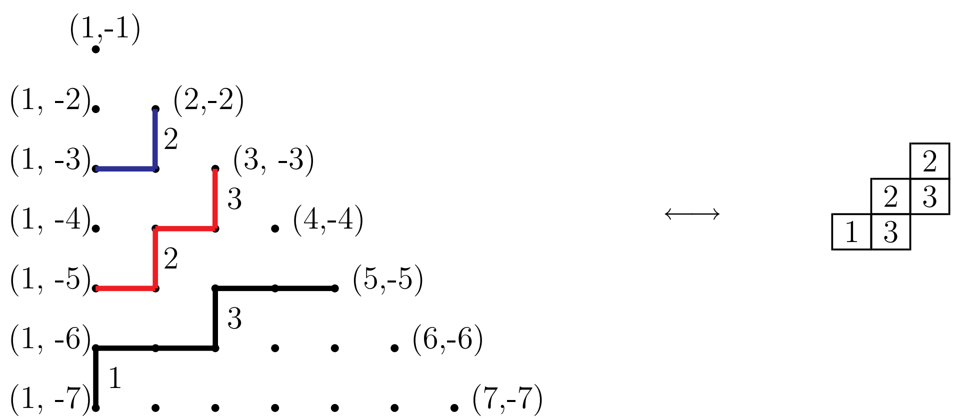


Figure 3.1 Lattice paths and skew tableaux.

Thus we are led to a tableau representation of skew Schubert polynomials:

Theorem 3.3 *Let σ be a 321-avoiding permutation and $\langle J/I \rangle$ its code. Let $\lambda_k = J_n - I_k, \mu_k = J_n - J_k$. Then we have*

$$\mathbb{G}_{\langle J/I \rangle}(X_m, Y_m) = \sum_T \prod_{(i,j) \in T} (x_{T_{i,j}} - y_{J_n+1-(T_{i,j}+C_{i,j})}), \tag{3.14}$$

where T ranges over all semistandard tableaux of shape λ/μ on $\{1, 2, \ldots, m\}$ in which all the entries in row i are bounded by $\widehat{\phi}_i$, and $(i,j) \in T$ means that (i,j) is a cell of T.

Proof. We have constructed a bijection between n-tuples $(P_1, \ldots, P_n)$ of non-intersecting paths from A' to B'' and flagged skew tableaux T of shape λ/μ with flag $\widehat{\phi}$. Notice that the kth vertical step of P_i corresponds to the $(k+\mu_i)$th cell of the i-th row of T. Recall that the element of the (i,j)cell of T

is $T_{i,j}$. The corresponding vertical step is from the point $(T_{i,j}, -(i+I_i)-(\lambda_i-j+1))$ to the point $(T_{i,j}, -(i+I_i)-(\lambda_i-j+1)+1)$ since $B''_i = (i+I_i, -i-I_i)$. From the flag conditions

$$\widehat{\phi}_i = I_i + i, \quad T_{i,j} \le \widehat{\phi}_i, \quad \text{and} \quad j \le \lambda_i,$$

it follows that

$$T_{i,j} - (i + I_i) - (\lambda_i - j + 1) < 0,$$

which implies that all the lattice paths in consideration lie below the diagonal line $y = -x$. Therefore, the weight of this vertical step is

$$x_{T_{i,j}} - y_{-(T_{i,j}-(i+I_i)-(\lambda_i-j+1))} = x_{T_{i,j}} - y_{(I_i+i)+(\lambda_i-j+1)-T_{i,j}}. \tag{3.15}$$

Since $I_i + i = J_n - \lambda_i + i$ and $C_{i,j} = j - i$, we may rewrite (3.15) as

$$x_{T_{i,j}} - y_{J_n+1-(T_{i,j}+C_{i,j})}.$$

Applying the above bijection, we may translate the lattice path interpretation into the desired tableau definition. ∎

The flagged double Schur function has been defined in [5]:

$$s_{\lambda,b}(X, Y) = \det(h_{\lambda_i-i+j}(X_{b_i} - Y_{b_i+\lambda_i-i}))_{t\times t}, \tag{3.16}$$

where the flag $b = (b_1, b_2, \cdots, b_t)$ is a sequence of a weakly increasing positive integers. Chen, Li, and Louck obtained a lattice path interpretation and a tableau representation of the flagged double Schur functions. As a consequence, one may get a lattice path interpretation and a tableau definition of the double Schubert polynomials indexed by vexillary permutations. Similarly, the skew Schubert polynomials also have a flag condition, where the flag is related to the code of the indexing permutation. From Eq. (3.13), we can naturally define the *flagged skew Schubert polynomials*

$$\mathbb{G}^{\phi}_{\langle J/I\rangle}(X_m, Y_m) = \det(h_{\lambda_i-\mu_j-i+j}(\{x_{\phi_j}, \ldots, x_{\widehat{\phi}_i}\} - Y_{\widehat{\phi}_i-\phi_j+\lambda_i-\mu_j-i+j}))_{n\times n}, \tag{3.17}$$

where ϕ is a weakly increasing flag sequence such that $\phi_i \le \widehat{\phi}_i$. Now using the lattice points $A''_i = (\phi_i, -(J_i+i))$ instead of $A'_i = (1, -(J_i+i))$, we obtain

$$\mathbb{G}^{\phi}_{\langle J/I\rangle}(X_m, Y_m) = GF(A'', B''). \tag{3.18}$$

Here is a more general result:

Theorem 3.4 *Let σ be a 321-avoiding permutation and $\langle J/I\rangle$ its code. Let $\lambda_k = J_n - I_k, \mu_k = J_n - J_k$. Then*

$$\mathbb{G}^{\phi}_{\langle J/I\rangle}(X_m, Y_m) = \sum_T \prod_{(i,j)\in T} (x_{T_{i,j}} - y_{J_n+1-(T_{i,j}+C_{i,j})}), \tag{3.19}$$

where T ranges over all semistandard tableaux of shape λ/μ such that $\phi_i \le T_{i,j} \le \widehat{\phi}_i$, and $\widehat{\phi}_i = I_i + i$.

In the above setting, the flagged double Schur functions $s_{\lambda,b}(X,Y)$ can be viewed as specialized flagged skew Schubert polynomials. Setting $I=0$ and letting ϕ be a flag such that $\phi_i = n+1-b_{n+1-i}$ (for $k>t$, setting $b_k=n$), we obtain

$$\eta(\mathbb{G}^{\phi}_{\langle J/I\rangle}(X_m,Y_m)) = s_{\lambda,b}(X,Y), \tag{3.20}$$

where $\eta(x_i)=x_{n+1-i}$ for each i.

Remark. The flagged skew Schubert polynomials have a similar tableau representation to the flagged skew supersymmetric Schur functions studied by Hamel and Goulden [13]. They coincide with each other for the special case of double Schur functions.

4. Isobaric divided differences and flagged Schur functions

Similar to the divided difference, we can define the isobaric divided difference π_i. Let $g(x_1,\ldots,x_n)$ be a function over n variables, and we define

$$\pi_i g = \frac{x_i g - \sigma_i(x_i g)}{x_i - x_{i+1}}.$$

Lascoux has studied the action of isobaric divided difference on crystal graphs [16]. In this section, we present a lattice interpretation of isobaric divided difference. From the definition of π_m, we have

$$\pi_m(x_m^n) = \sum_{k=0}^{n} x_m^k x_{m+1}^{n-k}. \tag{4.21}$$

As usual, a lattice path in the plane consists of steps from (i,j) to $(i,j+1)$ or from (i,j) to $(i+1,j)$. The weight function W_s assigned to the lattice paths is defined as follows: for a vertical step from (i,j) to $(i,j+1)$, the weight is x_i; for a horizontal step from (i,j) to $(i+1,j)$, the weight is 1. The relation (4.21) can be easily rewritten in terms of lattice paths:

Lemma 4.1 *Let P be the vertical segment from (m,k) to (m,p) and $p>k$. Then the action of π_m on the weight of P yields the sum of weights of all lattice paths from (m,k) to $(m+1,p)$.*

An immediate consequence of the above lemma is the following result similar to Lemma 3.1:

Lemma 4.2 *Given the above weight function W_s, let $A = (A_1, A_2, \ldots, A_n)$ be a sequence of lattice points with $A_i = (m, k_i)$, and let $B = (B_1, B_2, \ldots, B_n)$ be a sequence of lattice points with $B_1 = (m, p)$ and $B_i = (m+1, t_i)$ for $i \geq 2$. Suppose $p > k_1 > \cdots > k_n$, $p > t_2 > \cdots > t_n$, and $k_i \leq t_i$ for $i \geq 2$. Then we have*

$$\pi_m GF(A, B) = GF(A, B'),$$

where B' is obtained from B by replacing B_1 with $(m+1, p)$.

Proof. From the definition of the isobaric divided difference we see that

$$\pi_m(g_1 g_2) = g_1 \pi_m(g_2), \text{ if } g_1(x_m, x_{m+1}) = g_1(x_{m+1}, x_m). \tag{4.22}$$

We proceed to show that what really matters for π_m is the segment of the path from A_1 to B_1 that is above the horizontal line $y = t_2 + 1$. The polynomial $GF(A, B)$ can be computed by the following procedure. Suppose $t_2 + 1 > k_1$. Then every path from A_2 to B_2 must have the segment from $(m+1, k_1 - 1)$ to $(m+1, t_2)$, and $GF(A, B)$ must contain the factor $(x_m x_{m+1})^{t_2 - k_1 + 1}$. If $k_2 > t_3$, then no path from A_3 to B_3 intersects any path from A_2 to B_2. By Lemma 4.1, the weights of non-intersecting paths from $(A_3, \ldots, A_n)$ to $(B_3, \ldots, B_n)$ contribute a symmetric factor in x_m and x_{m+1} to $GF(A, B)$. If $k_2 < t_3 + 1$, we may repeat the above procedure to get a factor $(x_m x_{m+1})^{t_3 - k_2 + 1}$. Throughout this process, we get factors symmetric in x_m and x_{m+1}. For the case $t_2 + 1 \leq k_1$, we first take out the factor $GF(A_1, B_1)$; then the remaining factors of $GF(A, B)$ are symmetric in x_m and x_{m+1}. In either case, we may apply Lemma 4.1 to reach the desired conclusion. ∎

Notice that the isobaric divided differences π_i also satisfy the braid relations,

$$\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}, \quad \pi_i \pi_j = \pi_j \pi_i,$$

where $|i - j| > 1$. Thus it is reasonable to define the operator π_w for $w = \sigma_{i_1} \cdots \sigma_{i_k}$ and $k = \ell(w)$,

$$\pi_w(g) = \pi_{i_k} \pi_{i_{k-1}} \cdots \pi_{i_1}(g), \tag{4.23}$$

where the operators are applied from right to left.

Theorem 4.3 *Every flagged Schur function $s_\lambda(b)$ is equal to $\pi_w(x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_m^{\lambda_m})$, where $w = (\sigma_m \sigma_{m+1} \cdots \sigma_{b_m - 1})(\sigma_{m-1} \sigma_m \cdots \sigma_{b_{m-1} - 1}) \cdots (\sigma_1 \sigma_2 \cdots \sigma_{b_1 - 1})$.*

Proof. Notice that $b_i \geq i$ for all i; otherwise $s_\lambda(b) = 0$. We begin with the m vertical lines $P_1, P_2, \ldots, P_m$, where P_i is from $A_i = (1, -i+1)$ to $B_i = (i, \lambda_i - i + 1)$. Fix the weight function W_s as above; then by Lemma

4.2 $\pi_m(GF(A,B))$ equals the generating function $GF(A,B')$, where B' is obtained from B by replacing B_m with $(m+1, \lambda_m - m+1)$. We continue with the action of π_{m+1} on $GF(A,B')$. For any sequence of paths $(P_1, P_2, \ldots, P_m)$ from A to B', what really matters for π_{m+1} is the area between the lines $x = m+1$ and $x = m+2$. It is clear that the points of $(P_1, P_2, \ldots, P_m)$ on the lines $x = m+1$ and $x = m+2$ satisfy the conditions in Lemma 4.2. By iteration, it follows that $(\pi_{b_m-1} \cdots \pi_{m+1}\pi_m)GF(A,B) = GF(A,B'')$, where B'' is obtained from B by replacing B_m with $(b_m, \lambda_m - m+1)$. Iterating the same argument, we obtain that $\pi_w(GF(A,B))$ is equal to $GF(A,B^{(m)})$, where $B^{(m)}$ is obtained from B by replacing B_i with $(b_i, \lambda_i - i+1)$ for each i. Now applying the Gessel-Viennot argument, we reach the desired conclusion. ∎

The key polynomials are investigated in [21];they are called standard bases by Lascoux and Schützenberger. Unlike the Schubert polynomials that are indexed by permutations, the key polynomials are indexed by compositions, which are integer sequences γ with non-negative components. There is also a recursive definition for the key polynomials:

$$\kappa_\gamma = \begin{cases} x_1^{\gamma_1} x_2^{\gamma_2} \cdots, & \text{if } \gamma \text{ is weakly decreasing,} \\ \pi_i \kappa_{\sigma_i \gamma}, & \text{if } \gamma_i < \gamma_{i+1}. \end{cases}$$

Now we easily have the following result from the definition of key polynomials , which is a consequence of the characterization theorem in [25]. For this special case, our argument does not involve the flagged Littlewood-Richardson rule.

Corollary 4.4 *Every flagged Schur function $s_\lambda(b)$ is some key polynomial κ_γ; moreover λ is a weakly decreasing reordering of γ.*

Theorem 4.3 is analogous to the following theorem of Wachs.

Theorem 4.5 ([28, Theorem 2.4]) *Every flagged Schur function $s_\lambda(b)$ is equal to $\partial_w(x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m})$, where $a_i = \lambda_i + b_i - i$ and $w = (\sigma_m \sigma_{m+1} \cdots \sigma_{b_m-1})$ $(\sigma_{m-1}\sigma_m \cdots \sigma_{b_{m-1}-1}) \cdots (\sigma_1 \sigma_2 \cdots \sigma_{b_1-1})$.*

Comparing the above two theorems, we see that they have similar forms. But it is not generally true that $\pi_w = \partial_w(x_1^{b_1-1} x_2^{b_2-2} \cdots x_m^{b_m-m})$. It's worth mentioning the special case of $\pi_{w_0} = \partial_{w_0}(x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1)$, where w_0 is the maximal permutation $[n, n-1, \ldots, 1]$; see [20].

5. The super-Giambelli identity

The Giambelli identity for classical Schur functions $s_\lambda(X)$ first appeared in [8]. The first bijective proof was due to Eğecioğlu and Remmel [6], and the lattice path approach was first given by Stembridge [27] and later by Fulmek and Krattenthaler in a different form [7]. In this section, we present a lattice path construction for the super-Giambelli identity (2.4).

Again, we consider paths in the integer lattice consisting of unit horizontal and vertical steps. By a horizontal step we still mean a directed edge from (i, j) to $(i+1, j)$, but for a vertical step we mean a directed edge from (i, j) to $(i, j-1)$ if $i \leq 0$; or from (i, j) to $(i, j+1)$ if $i > 0$. With the Frobenius notation $(\alpha|\beta)$ of partition λ defined above, we choose the origin vertices $A_i = (-\alpha_i, \infty)$ and the destination vertices $B_i = (\beta_i + 1, \infty)$, $i = 1, 2, \ldots, r$, where r is the rank of λ. The weight function W_g is defined as follows: the weight of a vertical step is always 1; for a horizontal step from (i, j) to $(i+1, j)$ strictly to the left of the y-axis, it is given weight $x_j - y_{j-i}$; for a horizontal step strictly to the right of the y-axis, it is given weight $x_{i+j} - y_j$. Since every path from A_i to B_j is determined by a hook with shape $(\alpha_i|\beta_j)$, we have

Lemma 5.1 *Given the above weight function W_g as above , we have*

$$GF(A_i, B_j) = S_{(\alpha_i|\beta_j)}(X, Y). \tag{5.24}$$

The bijection between tableaux and tuples of non-intersecting paths from A to B is illustrated in Figure 5.1. Applying the Gessel-Viennot method, we obtain

$$S_\lambda(X, Y) = GF(A, B) = \det(S_{(\alpha_i|\beta_j)}(X, Y))_{r\times r}. \tag{5.25}$$

This completes the proof of the super-Giambelli identity (2.4). ∎

6. The super-Lascoux-Pragacz identity

The ribbon identity for the classical Schur functions is due to Lascoux and Pragacz [17]. Ueno [29] gave a lattice path interpretation of this identity based on the work of Stembridge [27]. The goal of this section is to extend Ueno's technique to super-Schur functions

Suppose that the rank of a partition λ is r, and $(\Theta_1, \Theta_2, \ldots, \Theta_r)$ is the ribbon decomposition of the Ferrers diagram of λ. Let u_i be the number

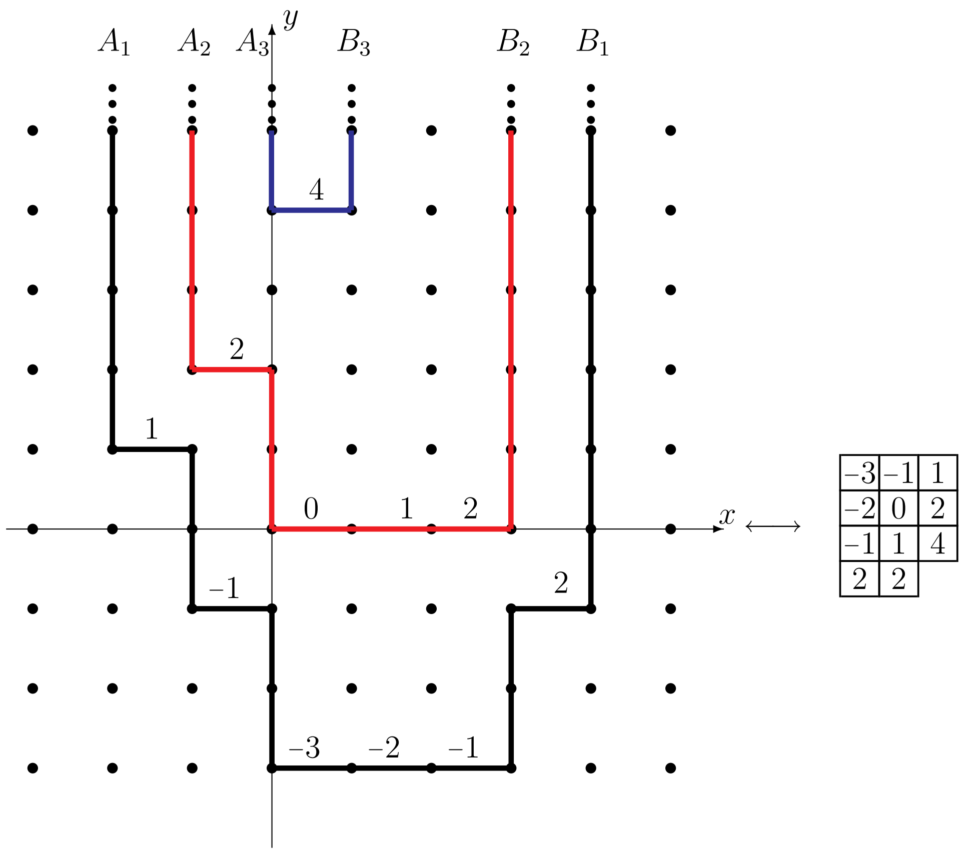


Figure 5.1 Non-intersecting lattice paths and tableau

of cells in Θ_i^+, and v_i the number of cells in Θ_i^-. For our lattice path construction, we choose the origin points $A_i = (-u_i, -\infty)$, and the destination points $B_i = (v_i + 1, -\infty)$. The use of points at infinity can be reformulated in finite terms. However, we find it convenient to use the points at infinity. For the shape $(5, 4, 3, 2)$ in Figure 6.2, we have $A_1 = (-4, -\infty)$, $A_2 = (-2, -\infty)$, $A_3 = (0, -\infty)$, $B_1 = (4, -\infty)$, $B_2 = (3, -\infty)$, $B_3 = (1, -\infty)$.

We continue with our lattice path construction. There are three types of moves in the lattice: right move, up move, and down move. However, for each line parallel to the y-axis, there is a given direction, either up or down, which specifies the direction of possible moves along this line. So, at any point one may either make a right move, or a vertical move along the specified direction. Given the points A_i and B_i, we only need to consider the region between the line $x = -u_1$ and the line $x = v_1 + 1$. From the example in Figure 6.2, we see that there are $v_1 + u_1 + 2$ lines parallel to the y-axis, which is the number of cells in the rim of λ plus one. Equivalently, each cell in the rim of λ corresponds to a subdivision of the region formed by two adjacent lines parallel to the y-axis.

To determine the directions of the lines parallel to the y-axis, we need the notion of the *code* of a partition λ (See Stanley [26]), which is also called the *partition sequence* by Bessenrodt [2]. Along the borderline of a partition λ, i.e., the edges of the rim of λ, we put a 1 to the right of each vertical edge and a 0 underneath each horizontal edge. Then we read off the 0-1 labels from top to bottom, and the resulting binary sequence is the code of λ. For example, the code of $(5, 4, 3, 2)$ in Figure 6.1 is $(1, 0, 1, 0, 1, 0, 1, 0, 0)$.

For each line parallel to the y-axis, if it is the jth line between the line $x = -u_1$ and the line $x = v_1 + 1$, then it is given the up or down direction depending on whether the jth component of the code is 1 or 0. An example is given in Figure 6.2.

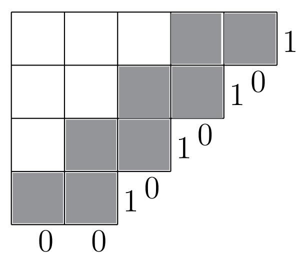


Figure 6.1 The code of the partition $(5, 4, 3, 2)$

We proceed to define the weight function W_r of a lattice path. First, all the vertical steps (either up move or down move) are given weight 1. For a horizontal step from (i, j) to $(i + 1, j)$, we will give a labelling as shown in Figure 6.2. Note that a bar over a number means the minus sign. The following is the procedure for giving the labelling of the lattice according to a shape.

Suppose that the step from (i, j) to $(i + 1, j)$ is labelled k; then the step from $(i, j+1)$ to $(i+1, j+1)$ is labelled $k+1$ and the step from $(i, j-1)$ to $(i+1, j-1)$ is labelled $k-1$. Therefore, we only need to label the horizontal steps on the x-axis. The first step is to label the leftmost horizontal step as $-r+1$, then label the next step (on the right) according to the following rule: if the right vertical line next to the current step has the down direction, then use the same label for the next step; otherwise, we increase the labelling by 1. This labelling rule ensures that the step from $(0, 0)$ to $(1, 0)$ is labelled 0, as shown in Figure 6.2. We assign the weight $x_k - y_{k-i}$ to the step from (i, j) to $(i, j + 1)$, where k is the labelling of this step.

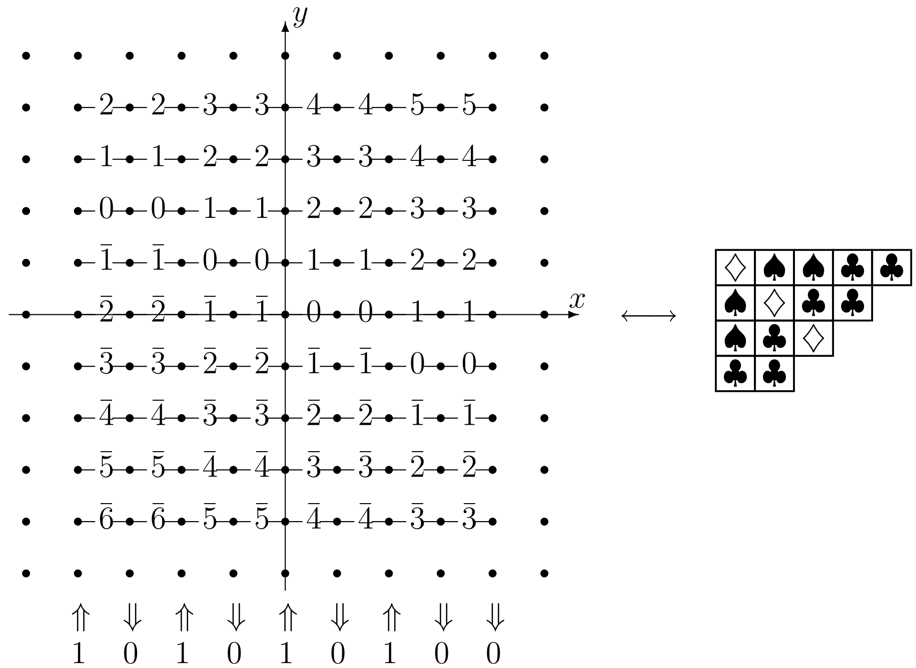


Figure 6.2 Labelling and ribbon decomposition of shape $(5,4,3,2)$, $\bar{i} = -i$

Theorem 6.1 *Let $A_i = (-u_i, -\infty)$ and $B_i = (v_i + 1, -\infty)$, and let the weight function W_r be defined as above. Then*

$$GF(A_i, B_j) = S_{\Theta_i^+ \& \Theta_j^-}(X, Y). \tag{6.26}$$

The following lemma describes the D-compatible conditions introduced by Stembridge [27]. Once we chose the directions of the edges as given before, then we have

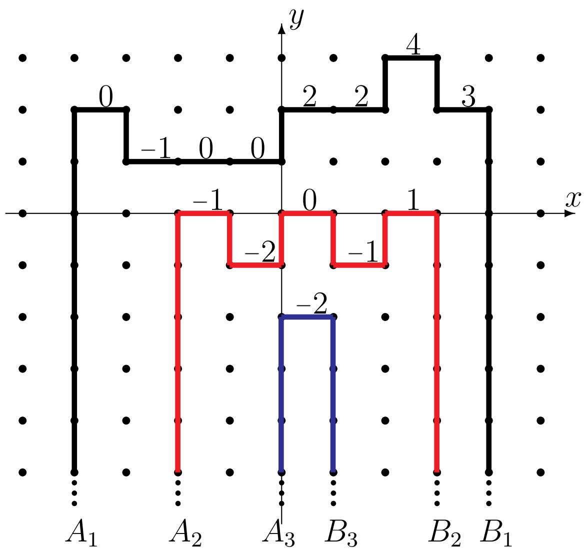
Lemma 6.2 *The vertical steps are given up directions on the line $x = -u_i$, and down directions on the line $x = v_i + 1$ for all $i = 1, 2, \ldots, r$. Thus every tuple of lattice paths from $(A_1, \ldots, A_r)$ to $(B_{\pi_1}, \ldots, B_{\pi_r})$ must intersect unless π is the identity permutation.*

Given an r-tuple of non-intersecting paths from $(A_1, \ldots, A_r)$ to $(B_1, \ldots, B_r)$, we may construct a tableau with shape λ. Given a lattice path from A_i to B_i, we may fill the ith rim from top to bottom with the labellings of the steps on the lattice path. Thus the non-intersecting property ensures that we get a tableau with shape λ. Conversely, we can construct the lattice path from the tableau. This bijection turns out to be weight preserving. From Lemma 6.2, Theorem 6.1, and the Gessel-Viennot argument, it follows that

$$GF(A, B) = \det(S_{\Theta_i^+ \& \Theta_j^-}(X, Y))_{r \times r}. \tag{6.27}$$

Hence we get the super-Lascoux-Pragacz identity (2.5). ∎

Figure 6.3 shows such a bijection between a tableau and the sequence of non-intersecting paths.



$\longleftrightarrow$

−2	−2	−1	−1	0
−1	0	0	0	
1	2	2		
3	4			

Figure 6.3 Non-intersecting lattice paths and tableau for ribbon identity

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