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STANLEY'S ZRANK CONJECTURE ON SKEW PARTITIONS

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ABSTRACT. We present an affirmative answer to Stanley's zrank conjecture, namely, the zrank and rank are equal for any skew partition. We show that certain classes of restricted Cauchy matrices are nonsingular and furthermore, the signs are determined by the number of zero entries. We also give a characterization of the rank in terms of the Giambelli type matrices of the corresponding skew Schur functions. Our approach also applies to the factorial Cauchy matrices and the inverse binomial coefficient matrices.

1. INTRODUCTION

In the study of tensor products of Yangian modules, Nazarov and Tarasov [12] give a generalization of the rank of an ordinary partition to a skew partition. Stanley [15] obtains several characterizations of the rank of a skew partition in terms of the reduced partition code, the Jacobi-Trudi matrix, and the minimal border strip decomposition. Stanley also introduces the notion of the zrank of a skew partition in terms of the specialization of the skew Schur function, and has proposed the problem of whether the zrank and the rank are always equal.

Yan, Yang and Zhou [16] find an equivalent characterization of Stanley's zrank conjecture in terms of the restricted Cauchy matrix based on two integer sequences. In this paper, we extend the definition of a restricted Cauchy matrix to two sequences of real numbers subject to certain conditions. We show that every restricted Cauchy matrix is nonsingular, and thus give an affirmative answer to Stanley's zrank conjecture.

In the spirit of Stanley's notion of the jrank of a skew partition which is defined as the number of rows in the Jacobi-Trudi matrix that do not contain entries equal to one, we introduce the notion of *grank* in terms of the Giambelli type matrix defined by Hamel and Goulden for a skew Schur function [9]. Given any outside decomposition of a skew partition, the grank is defined as the number of rows in which there are no entries equal to one. It can be seen that the grank is well-defined, namely, it does not depend on

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the specific outside decomposition of the skew partition. We show that the grank is always equal to the rank for any skew partition.

This paper is also concerned with the nonsingularity of the factorial Cauchy matrices and the inverse binomial coefficient matrices. Given a sequence A of real numbers and a sequence B of integers, we define the factorial Cauchy matrix as a matrix with each entry being either the inverse of the falling factorial or zero. We show that the determinant of the factorial Cauchy matrix has the same nonsingularity property as the restricted Cauchy matrix. A special case of the factorial Cauchy matrix arises in the calculation of some determinants involving the s-shifted factorial by Normand [13] in the study of probability density of the determinant of random matrices [7, 11].

The double Schur functions serve as a tool for proving the nonsingularity of the factorial Cauchy matrix without zero entries. The double Schur functions are an extension of the factorial Schur functions introduced by Biedenharn and Louck [3], and further studied by Chen and Louck [4], Goulden and Greene [8], Macdonald [10], Chen, Li and Louck [5].

As a direct application of the nonsingularity of factorial Cauchy matrices, we prove the nonsingularity of inverse binomial coefficient matrices whose entries are either zeros or the inverse of the binomial coefficients.

2. The restricted Cauchy matrices

Let $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$ be two sequences of real numbers. Suppose that A is strictly decreasing, B is strictly increasing, and $a_i > b_{n+1-i}$ and $a_i \neq b_j$ for any i, j. The *restricted real Cauchy matrix* $M = \mathbf{C}(A, B) = (c_{ij})_{i,j=1}^n$ is defined by

$$c_{ij} = \begin{cases} \dfrac{1}{a_i - b_j}, & \text{if } a_i > b_j, \\ 0, & \text{if } a_i < b_j. \end{cases} \tag{2.1}$$

If A and B are integer sequences, we call M a *restricted integer Cauchy matrix.*

Let $\omega(M)$ be the number of zero entries in M. The following theorem implies Stanley's zrank conjecture.

Theorem 2.1. *Any restricted real Cauchy matrix $M = \boldsymbol{C}(A, B)$ is nonsingular. Furthermore, the determinant* $\det(M)$ *is positive if $\omega(M)$ is even; or negative if $\omega(M)$ is odd.*

Before giving the proof of the above theorem, let us recall some definitions involving matrices. A matrix $M = (m_{ij})_{i,j=1}^n$ is called *partly decomposable* if it contains an $s \times t$ submatrix which contains only zeroes, where $s + t = n$. Otherwise, we say that M is *fully indecomposable.* Clearly, a restricted real Cauchy matrix $M = \mathbf{C}(A, B)$ is fully indecomposable if and only if $a_i > b_{n+2-i}$ for $i = 2, 3, \ldots, n$. Given a square matrix $M = (m_{ij})_{i,j=1}^n$, let

M_{ij} denote the (i,j)-th *minor* of M which is the matrix obtained from M by deleting the i-th row and the j-th column. We have the following lemma:

Lemma 2.2. If $M = \mathbf{C}(A,B)$ is a fully indecomposable restricted Cauchy matrix, then each minor M_{ij} is a restricted Cauchy matrix.

Proof. Let

$$\begin{aligned} A' &= (a'_1,\ldots,a'_{n-1}) = (a_1,\ldots,\hat{a}_i,\ldots,a_n),\\ B' &= (b'_1,\ldots,b'_{n-1}) = (b_1,\ldots,\hat{b}_j,\ldots,b_n), \end{aligned}$$

where $\hat{}$ stands for a missing entry. It suffices to show that $M_{ij} = \mathbf{C}(A',B')$, or equivalently, $a'_k > b'_{n-k}$ for any $k : 1 \le k \le n-1$. There are four cases:

(a) If $k < i$ and $n-k < j$, then $a'_k = a_k > b_{n+1-k} > b_{n-k} = b'_{n-k}$ since B is strictly increasing.
(b) If $k < i$ and $n-k \ge j$, then $a'_k = a_k > b_{n+1-k} = b'_{n-k}$.
(c) If $k \ge i$ and $n-k < j$, then $a'_k = a_{k+1} > b_{n-k} = b'_{n-k}$.
(d) If $k \ge i$ and $n-k \ge j$, then $a'_k = a_{k+1} > b_{n-k+1} = b'_{n-k}$ since $M = \mathbf{C}(A,B)$ is fully indecomposable.

This completes the proof. □

The *adjoint matrix* of M is defined as the matrix $\left((-1)^{i+j}\det(M_{ji})\right)_{i,j=1}^{n}$, denoted M^*. The *rank* of a matrix M is the maximum number of linearly independent rows or columns of the matrix, denoted $r(M)$. For an $n \times n$ square matrix M, we have the following relationship between $r(M)$ and $r(M^*)$:

$$r(M^*) = \begin{cases} n, & \text{if } r(M) = n,\\ 1, & \text{if } r(M) = n-1,\\ 0, & \text{if } r(M) < n-1. \end{cases} \tag{2.2}$$

The restricted Cauchy matrix $M = \mathbf{C}(A,B)$ reduces to the classical Cauchy matrix when $a_n > b_n$. In this case, the determinant $\det(M)$ is given by the following well known formula

$$\det\left(\frac{1}{a_i - b_j}\right)_{i,j=1}^{n} = \prod_{i<j}(a_i - a_j)\prod_{i<j}(b_j - b_i)\prod_{i,j}\frac{1}{a_i - b_j}. \tag{2.3}$$

Since A is strictly decreasing and B is strictly increasing, the above determinant is positive.

We are now ready to give the proof of Theorem 2.1.

Proof of Theorem 2.1. We use induction on n. The cases for $n = 1, 2$ are obvious. Assume that the theorem holds for matrices of order less than n. Then we will show that it also holds for matrices of order n.

If M is partly decomposable, then there exists an integer k greater than or equal to 2 such that $a_k < b_{n+2-k}$. Let us consider the following block decomposition

$$\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix},$$

where M_1 is a $(k-1)\times(n-k+1)$ matrix, M_2 is the $(k-1)\times(k-1)$ restricted Cauchy matrix $\mathbf{C}((a_1,\ldots,a_k),(b_{n-k+1},\ldots,b_n))$, M_3 is the $(n-k+1)\times(n-k+1)$ restricted Cauchy matrix $\mathbf{C}((a_{k+1},\ldots,a_n),(b_1,\ldots,b_{n-k}))$, and M_4 is an $(n-k+1)\times(k-1)$ zero block. So we have

$$\det(M) = (-1)^{\omega(M_4)}\det(M_2)\det(M_3).$$

By induction the sign of $\det(M_2)$ is $(-1)^{\omega(M_2)}$, and the sign of $\det(M_3)$ is $(-1)^{\omega(M_3)}$. Since $\omega(M_1)=0$, the sign of $\det(M)$ equals

$$(-1)^{\omega(M_4)+\omega(M_3)+\omega(M_2)} = (-1)^{\omega(M)}.$$

It remains to consider the case that $M=(m_{ij})_{i,j=1}^n$ is fully indecomposable. If M has no zero entry, then the theorem is true because of (2.3). If $\omega(M)>0$, then we have the following block decomposition of M

$$\begin{pmatrix} M_1' & M_2' \\ M_3' & 0 \end{pmatrix},$$

where M_1' is an $(n-1)\times(n-1)$ restricted Cauchy matrix, M_2' is an $(n-1)\times 1$ column vector, M_3' is a $1\times(n-1)$ row vector. By Lemma 2.2, we see that the minors $M_{11}, M_{nn}, M_{1n}, M_{n1}$ are also restricted Cauchy matrices. Consider the submatrix

$$\begin{pmatrix} \det(M_{11}) & (-1)^{n+1}\det(M_{n1}) \\ (-1)^{n+1}\det(M_{1n}) & \det(M_{nn}) \end{pmatrix}$$

of the adjoint matrix M^*. Note that the signs of $\det(M_{11}), \det(M_{n1}), \det(M_{1n})$ and $\det(M_{nn})$ are respectively $(-1)^{\omega(M_1')+\omega(M_2')+\omega(M_3')+1}$, $(-1)^{\omega(M_1')+\omega(M_2')}$, $(-1)^{\omega(M_1')+\omega(M_3')}$ and $(-1)^{\omega(M_1')}$. It follows that

$$\det\begin{pmatrix} \det(M_{11}) & (-1)^{n+1}\det(M_{n1}) \\ (-1)^{n+1}\det(M_{1n}) & \det(M_{nn}) \end{pmatrix} \neq 0.$$

Thus we have $r(M^*)\geq 2$. From the relationship (2.2) between $r(M^*)$ and $r(M)$, we see that $r(M^*)=n$, that is, M is nonsingular.

We proceed to show that the sign of $\det(M)$ is determined by the number of zero entries in M. Without loss of generality, we may assume that M is fully indecomposable. If M does not contain any zero entry, then $\det(M)$ is positive. We now assume that M contains at least one zero entry. From the definition of the restricted Cauchy matrix, we see that for any row in $\mathbf{C}(A,B)$, if there is a zero in the j-th column, then the entry in any column k $(k>j)$ must be zero. The same property also holds for the columns of $\mathbf{C}(A,B)$. Therefore, the (n,n)-entry in $\mathbf{C}(A,B)$ must be zero. Since M is fully indecomposable, there exists an integer $j\colon 2\leq j\leq n-1$ such that $m_{nj}\neq 0$, but $m_{n,j+1}=m_{n,j+2}=\cdots=m_{n,n}=0$. Let $\alpha=b_j$ and $\beta=\min(a_{n-1},b_{j+1})$. Then the determinant $\det(M)$ can be regarded as a continuous function of a_n on the open interval (α,β). Note that when a_n

varies in the open interval (α, β), the restricted Cauchy matrix M keeps the same shape, which means that the positions of zero entries are fixed. If $a_n = \eta$ for some $\eta \in (\alpha, \beta)$, we use notation W_η to denote the corresponding matrix M. When a_n tends to b_j from above, m_{nj} tends to $+\infty$, and for $k < j$ the entry m_{nk} tends to $\frac{1}{b_j - b_k}$.

Since the minor M_{nj} is a restricted Cauchy matrix of order $n-1$, by Lemma 2.2, the induction hypothesis implies that $\det(M_{nj}) \neq 0$. Therefore, the sign of $\det(M)$ coincides with the sign of $(-1)^{n+j}\det(M_{nj})$ when a_n tends to α from above. It follows that there exists $\xi \in (\alpha, \beta)$ such that the sign of $\det(W_\xi)$ coincides with the sign of $(-1)^{n+j}\det(M_{nj})$. By induction, the sign of $\det(M_{nj})$ equals $(-1)^{\omega(M_{nj})}$, thus the sign of $\det(W_\xi)$ equals

$$(-1)^{n+j+\omega(M_{nj})} = (-1)^{(n-j)+\omega(M_{nj})} = (-1)^{\omega(W_\xi)}.$$

For any $\eta \in (\alpha, \beta)$, the sign of $\det(W_\eta)$ coincides with the sign of $\det(W_\xi)$. Otherwise, there exists a number ζ between ξ and η such that $\det(W_\zeta) = 0$, which is a contradiction. Since $\omega(W_\xi) = \omega(W_\eta)$, the proof is complete. $\square$

3. The zrank conjecture

We assume that the reader is familiar with the notation and terminology on partitions and symmetric functions in [14]. Given a partition λ with decreasing components $\lambda_1, \lambda_2, \ldots$, the rank of λ, denoted $\operatorname{rank}(\lambda)$, is the number of i's such that $\lambda_i \geq i$. Clearly, $\operatorname{rank}(\lambda)$ counts the number of diagonal boxes in the Young diagram of λ, where the Young diagram is an array of squares in the plane justified from the top and left corner with $\ell(\lambda)$ rows and λ_i squares in row i. A square (i, j) in the diagram is the square in row i from the top and column j from the left. The content of (i, j), denoted $\tau(i, j)$, is given by $j - i$.

Given two partitions λ and μ, if for each i we have $\lambda_i \geq \mu_i$, then the skew partition λ/μ is defined to be the diagram obtained from the diagram of λ by removing the diagram of μ at the top-left corner. A border strip is a connected skew partition without 2×2 squares. Nazarov and Tarasov [12] introduced a generalization of the rank of ordinary partitions to skew partitions: a square (i, j) is called an *inner corner* of λ/μ, if $(i, j), (i, j-1), (i-1, j) \in \lambda/\mu$ but $(i-1, j-1) \notin \lambda/\mu$; a square (i, j) is called an *outer corner* of λ/μ, if $(i, j) \in \lambda/\mu$ but $(i-1, j-1), (i, j-1), (i-1, j) \notin \lambda/\mu$; the *inner diagonal* is composed of the boxes $(i+p, j+p) \in \lambda/\mu$ if (i, j) is an inner corner; the *outer diagonal* is composed of the boxes $(i+p, j+p) \in \lambda/\mu$ if (i, j) is an outer corner; let d^+ be the number of boxes on all outer diagonals, and let d^- be the number of boxes on all inner diagonals; then the *rank* of λ/μ, denoted $\operatorname{rank}(\lambda/\mu)$, is the difference $d^+ - d^-$. For example, $\operatorname{rank}((6, 5, 5, 3)/(2, 1, 1)) = 3$, as illustrated in Figure 1.

Stanley [15] gave several characterizations of $\operatorname{rank}(\lambda/\mu)$. The first characterization is based on the border strip decomposition of the skew diagram. He proved that $\operatorname{rank}(\lambda/\mu)$ equals the smallest number k such that

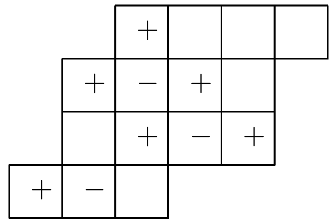

FIGURE 1. Outside and inside diagonals of $(6,5,5,3)/(2,1,1)$

λ/μ is a disjoint union of k border strips. As we see from Figure 2, $\mathrm{rank}((5,4,3,2)/(2,1,1)) = 3$.

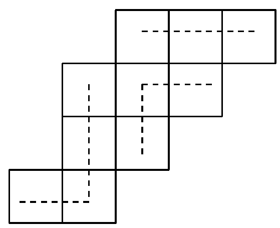

FIGURE 2. A minimal border strip decomposition of $(5,4,3,2)/(2,1,1)$

The Jacobi-Trudi identity for the skew Schur function is given by

$$s_{\lambda/\mu} = \det\left(h_{\lambda_i-\mu_j-i+j}\right)_{i,j=1}^{\ell(\lambda)}, \tag{3.1}$$

where h_k denotes the k-th complete symmetric function, $h_0 = 1$ and $h_k = 0$ for $k < 0$. Let $J_{\lambda/\mu}$ be the matrix in (3.1). Stanley defined the *jrank* of λ/μ, denoted $\mathrm{jrank}(\lambda/\mu)$, as the number of rows of $J_{\lambda/\mu}$ without entries equal to one, and obtained the relation $\mathrm{jrank}(\lambda/\mu) = \mathrm{rank}(\lambda/\mu)$. For example,

$$J_{(6,5,5,3)/(2,1,1)} = \begin{pmatrix} h_4 & h_6 & h_7 & h_9 \\ h_2 & h_4 & h_5 & h_7 \\ h_1 & h_3 & h_4 & h_6 \\ 0 & 1 & h_1 & h_3 \end{pmatrix}.$$

The third characterization of $\mathrm{rank}(\lambda/\mu)$ involves the reduced code of λ/μ, denoted $\mathrm{c}(\lambda/\mu)$. The reduced code $\mathrm{c}(\lambda/\mu)$ is also known as the *partition sequence* of λ/μ [1, 2]. Consider the two boundary lattice paths of the diagram of λ/μ with steps $(0,1)$ or $(1,0)$ from the bottom-leftmost point to the top-rightmost point. Replacing each step $(0,1)$ by 1 and each step $(1,0)$ by 0, we obtain two binary sequences by reading the lattice paths from the bottom-left corner to the top-right corner. Denote the top-left binary sequence by $f_1, f_2, \ldots, f_k$, and the bottom-right binary sequence by $g_1, g_2, \ldots, g_k$. The reduced code $\mathrm{c}(\lambda/\mu)$ is defined by the two-row array

$$\begin{array}{cccc} f_1 & f_2 & \cdots & f_k \\ g_1 & g_2 & \cdots & g_k \end{array}.$$

For example, the reduced code of the skew partition $(5,4,3,2)/(2,1,1)$ in Figure 3 is

$$\begin{array}{ccccccccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}.$$

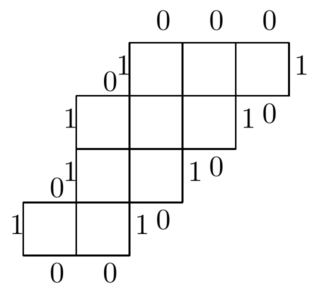

FIGURE 3. The reduced code of $(5,4,3,2)/(2,1,1)$

Stanley proved that the rank of a skew partition also equals the number of columns $\substack{1\\0}$ in $\mathrm{c}(\lambda/\mu)$, as we see from Figure 3.

In [15] Stanley introduced the notion of *zrank* of a skew partition, and conjectured that the zrank is always equal to the rank for any skew partition. Let $s_{\lambda/\mu}(1^t)$ denote the skew Schur function $s_{\lambda/\mu}$ evaluated at $x_1 = \cdots = x_t = 1$, $x_i = 0$ for $i > t$. The zrank of λ/μ, denoted $\mathrm{zrank}(\lambda/\mu)$, is defined as the exponent of the largest power of t dividing $s_{\lambda/\mu}(1^t)$.

The following equivalence was established by Yan, Yang and Zhou [16]:

Theorem 3.1. *The following two statements are equivalent:*
(i) The zrank and rank are equal for any skew partition.
(ii) Any restricted integer Cauchy matrix is nonsingular.

As an immediate consequence of Theorem 2.1 and Theorem 3.1, we have the following conclusion.

Theorem 3.2. *For all skew partitions λ/μ,* $\mathrm{zrank}(\lambda/\mu) = \mathrm{rank}(\lambda/\mu)$.

The above theorem allows us to give other characterizations of rank in terms of the Giambelli type matrix. Let us recall the Giambelli type determinant formulas of the skew Schur function. An *outside decomposition*, introduced by Hamel and Goulden [9], is a border strip decomposition of λ/μ for which every strip has an initial square on the left or bottom perimeter of the diagram and a terminal square on the right or top perimeter. Chen, Yan and Yang [6] introduced the notion of the cutting strip of an outside decomposition and obtained a transformation theorem on the Giambelli-type determinantal formulas for the skew Schur function.

Suppose that λ/μ has k diagonals. Given an outside decomposition of λ/μ, we see that each square in the diagram can be assigned a direction in the following way: starting with the bottom-left corner of a strip, we say that a square of a strip has the up direction (resp. right direction) if the next square in the strip lies on its top (resp. to its right). Then the squares on the same diagonal of λ/μ have the same direction. Based on this property, the cutting strip ϕ of an outside decomposition $\mathbf{D}$ of λ/μ can be defined as follows: for $i = 1, 2, \ldots, k-1$ the i-th square in ϕ keeps the same direction as the i-th diagonal of λ/μ with respect to $\mathbf{D}$. Given a border strip θ of $\mathbf{D}$, let $p(\theta)$ denote the lower left-hand square of θ, and let $q(\theta)$ denote the upper right-hand square. Hamel and Goulden [9] derived the following determinantal formula.

Theorem 3.3 ([9, Theorem 3.1])**.** *For an outside decomposition* $\mathbf{D}$ *with* k *border strips* $\theta_1, \theta_2, \ldots, \theta_k$*, we have*

$$s_{\lambda/\mu} = \det\left(s_{[\tau(p(\theta_i)),\tau(q(\theta_j))]}\right)_{i,j=1}^{k}, \tag{3.2}$$

where for any two integers α, β*, a strip* $[\alpha, \beta]$ *is defined by the following rule: if* $\alpha \leq \beta$*, then let* $[\alpha, \beta]$ *be the segment of* ϕ *from the square with content* α *to the square with content* β*; if* $\alpha = \beta + 1$*, then let* $[\alpha, \beta]$ *be the empty strip and* $s_{[\alpha,\beta]} = 1$*; if* $\alpha > \beta + 1$*, then* $[\alpha, \beta]$ *is undefined and* $s_{[\alpha,\beta]} = 0$*. The content function* τ *is defined on the original skew diagram.*

Denote the matrix in (3.2) by $G^{\mathbf{D}}_{\lambda/\mu}$. Given an outside decomposition $\mathbf{D}$ of λ/μ, let $\mathrm{grank}_{\mathbf{D}}(\lambda/\mu)$ be the number of rows in $G^{\mathbf{D}}_{\lambda/\mu}$ that do not contain entries equal to one. Then we have the following theorem:

Theorem 3.4. *For any skew partition* λ/μ *and any outside decomposition* $\mathbf{D}$ *of* λ/μ*,* $\mathrm{grank}_{\mathbf{D}}(\lambda/\mu) = \mathrm{rank}(\lambda/\mu)$.

Proof. By Theorem 2.1, we have $\mathrm{rank}(\lambda/\mu) = \mathrm{zrank}(\lambda/\mu)$. So it suffices to show that $\mathrm{grank}_{\mathbf{D}}(\lambda/\mu) = \mathrm{zrank}(\lambda/\mu)$. According to the definition of $\mathrm{zrank}(\lambda/\mu)$, we need to consider the terms with the lowest degree in the expansion of $\det(G^{\mathbf{D}}_{\lambda/\mu}(1^t))$. Suppose that the square with content $\tau(p(\theta_i))$ lies in the p_i-th row of the cutting strip ϕ of $\mathbf{D}$, and the square with content $\tau(q(\theta_j))$ lies in the q_j-th row. If the border strip $[\tau(p(\theta_i)), \tau(q(\theta_j))]$ is nonempty, it is easy to show that

$$(t^{-1} s_{[\tau(p(\theta_i)),\tau(q(\theta_j))]}(1^t))_{t=0} = \frac{(-1)^{p_i - q_j}}{\tau(q(\theta_j)) + 1 - \tau(p(\theta_i))}. \tag{3.3}$$

Note that for any $i \neq j$ we have

$$\tau(q(\theta_i)) \neq \tau(q(\theta_j)), \quad \tau(p(\theta_i)) \neq \tau(p(\theta_j))$$

subject to the definition of the outside decomposition $\mathbf{D}$. By removing the rows and columns with ones from $G^{\mathbf{D}}_{\lambda/\mu}$, extracting t from each row without ones, and putting $t = 0$, we obtain a restricted Cauchy matrix up to permutations of rows and columns. From Theorem 2.1, we get the desired equality $\mathrm{grank}_{\mathbf{D}}(\lambda/\mu) = \mathrm{zrank}(\lambda/\mu)$. $\square$

In fact, the above theorem can be proved in a different way. Given a border strip decomposition $\mathbf{D} = \{\theta_1, \theta_2, \ldots, \theta_m\}$ of λ/μ, let

$$P_{\mathbf{D}} = \{\tau(p(\theta_1)), \tau(p(\theta_2)), \ldots, \tau(p(\theta_m))\}$$

and

$$Q_{\mathbf{D}} = \{\tau(q(\theta_1)) + 1, \tau(q(\theta_2)) + 1, \ldots, \tau(q(\theta_m)) + 1\}.$$

The following theorem is implicit in [16]:

Theorem 3.5. *For any border strip decomposition* $\mathbf{D}$*, the two sets* $P_{\mathbf{D}} - Q_{\mathbf{D}}$ *and* $Q_{\mathbf{D}} - P_{\mathbf{D}}$ *are independent of the border strip decomposition* $\mathbf{D}$*, hence uniquely determined by the skew shape* λ/μ*.*

Remark 3.6. We omit the proof of the above theorem, since it is similar to that of [16, Proposition 3.1]. Note that a border strip decomposition $\mathbf{D}$ may not be an outside decomposition. As shown by Yan, Yang and Zhou [16], these two sets are related to the noncrossing interval sets of a given skew partition. If $\mathbf{D}$ is a minimal border strip decomposition, then $P_{\mathbf{D}}$ and $Q_{\mathbf{D}}$ are disjoint. Otherwise, the cardinality of the intersection $P_{\mathbf{D}} \cap Q_{\mathbf{D}}$ is equal to the number of rows containing ones in $G^{\mathbf{D}}_{\lambda/\mu}$. From this point of view, Theorem 3.5 is more general than Theorem 3.4.

4. The factorial Cauchy matrices

Before defining the factorial Cauchy matrix and the inverse binomial coefficient matrix, let us review some background on double Schur functions. Let $X = \{x_1, x_2, \ldots\}$ and $Y = \{y_1, y_2, \ldots\}$ be two sets of variables. For a positive integer k, we set

$$(x_i|Y)_k = \prod_{1\le j\le k} (x_i - y_j), \tag{4.1}$$

and define $(x_i|Y)_0 = 1$. Taking $y_i = i - 1$, we obtain the falling factorial $(x_i)_k = x_i(x_i-1)\cdots(x_i-k+1)$. Taking $y_i = 1-i$, we get the rising factorial $(x_i)^k = x_i(x_i+1)\cdots(x_i+k-1)$.

Now we review two equivalent definitions of the double Schur function $S_\lambda(X, Y)$. The first definition of $S_\lambda(X, Y)$ is a determinantal form. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. The determinantal definition is as follows:

$$S_\lambda(X, Y) = \frac{\det\Big((x_i|Y)_{\lambda_j+n-j} \Big)_{i,j=1}^{n}}{\Delta(X)}, \tag{4.2}$$

where $\Delta(X)$ is the Vandermonde determinant in $x_1, x_2, \ldots, x_n$:

$$\Delta(X) = \prod_{i<j} (x_i - x_j).$$

The second definition of $S_\lambda(X,Y)$ is obtained by Macdonald [10], and Goulden and Greene [8]. Recall that a *semistandard Young tableau* T of shape λ is a configuration of the Young diagram of λ with positive integers such that each row is weakly increasing and each column is strictly increasing. Given a Young tableau T and a cell α of T, let $T(\alpha)$ be the number filled in the cell α. The combinatorial definition of $S_\lambda(X,Y)$ is as follows.

Theorem 4.1. *Let λ be a partition of length n. Then*

$$S_\lambda(X,Y)=\sum_T\prod_{\alpha\in T}(x_{T(\alpha)}-y_{T(\alpha)+\tau(\alpha)}),$$

summing over all column strict tableaux T on $\{1,2,\ldots,n\}$ of shape λ.

The factorial Cauchy matrix and the inverse binomial coefficient matrix are defined as follows. Let $A=(a_1,\ldots,a_n)$ be a strictly decreasing sequence of real numbers, and let $B=(b_1,\ldots,b_n)$ be a strictly increasing sequence of positive integers. Suppose that for any i,j we have $a_i>b_{n+1-i}-1$ and $a_i\neq b_j-1$. Then the *factorial Cauchy matrix* $\mathbf{F}(A,B)$ is given by $(c_{ij})_{i,j=1}^n$, where

$$c_{ij}=\begin{cases}\dfrac{1}{(a_i)_{b_j}}, & \text{if } a_i>b_j-1,\\[2mm] 0, & \text{if } a_i<b_j-1.\end{cases}$$

When A is also a sequence of positive integers, the *inverse binomial coefficient matrix* $\mathbf{R}(A,B)=(d_{ij})_{i,j=1}^n$ is defined by

$$d_{ij}=\begin{cases}\dbinom{a_i}{b_j}^{-1}, & \text{if } a_i\geq b_j,\\[2mm] 0, & \text{if } a_i<b_j.\end{cases}$$

We now consider the evaluation of the factorial Cauchy matrix $\mathbf{F}(A,B)$ without zero entries, i.e., $a_i>b_j-1$ for any i,j.

Lemma 4.2. Let $\mathbf{F}(A,B)$ be the factorial Cauchy matrix with $a_i>b_j-1$ for any i,j. Then we have

$$\det(\mathbf{F}(A,B))=\frac{\Delta(X)S_\lambda(X,Y)}{\Pi_{k=1}^n(a_k)_{b_n}},\tag{4.3}$$

where $\lambda_j=b_n-b_j+j-n$, $x_i=a_i-b_n+1$, and $y_j=-j+1$. In particular, we have $\det(\mathbf{F}(A,B))>0$.

Proof. Since $a_i>b_j-1$ for any i,j, we get

$$\mathbf{F}(A,B)=\left(\frac{1}{(a_i)_{b_j}}\right)_{i,j=1}^n.$$

So we have

$$\begin{aligned}\det(\mathbf{F}(A,B)) &= \det\left(\frac{1}{(a_i)_{b_j}}\right)_{i,j=1}^{n}\\ &= \frac{\det\left((a_i-b_n+1)^{b_n-b_j}\right)_{i,j=1}^{n}}{\Pi_{k=1}^{n}(a_k)_{b_n}}\\ &= \frac{\Delta(X)S_\lambda(X,Y)}{\Pi_{k=1}^{n}(a_k)_{b_n}},\end{aligned}$$

where the last equality follows from the algebraic definition (4.2) of $S_\lambda(X,Y)$.

Applying Theorem 4.1, it follows that $\det(\mathbf{F}(A,B))>0$. □

We now present the main result of this section.

Theorem 4.3. *Any factorial Cauchy matrix $M = \boldsymbol{F}(A,B)$ is nonsingular. Furthermore, the determinant $\det(M)$ is positive if $\omega(M)$ is even; or negative if $\omega(M)$ is odd.*

Proof. We use induction on the order of M. The proof is analogous to that of Theorem 2.1. Notice that Lemma 4.2 will play the same role as Equation (2.3) in the proof of Theorem 2.1 when considering the case that M contains no zero entry. Moreover, to show that the sign of $\det(M)$ is determined by the number of zero entries in M, we need to change the assignments of α and β by setting $\alpha = b_j - 1$ and $\beta = \min(a_{n-1}, b_{j+1} - 1)$. □

From the relation

$$\det(\mathbf{R}(A,B)) = \det(\mathbf{F}(A,B))\prod_{i=1}^{n} b_j!,$$

we obtain the following

Corollary 4.4. Any inverse binomial coefficient matrix $M = \mathbf{R}(A,B)$ is nonsingular. Furthermore, the determinant $\det(M)$ is positive if $\omega(M)$ is even; or negative if $\omega(M)$ is odd.

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