The q-Log-convexity of the Generating Functions of the Squares of Binomial Coefficients

William Y. C. Chen[1], Robert L. Tang[2],
Larry X. W. Wang[3] and Arthur L. B. Yang[4]

Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P. R. China

[1]chen@nankai.edu.cn, [2]tangling@cfc.nankai.edu.cn,
[3]wxw@cfc.nankai.edu.cn, [4]yang@nankai.edu.cn

Abstract. We prove a conjecture of Liu and Wang on the q-log-convexity of the polynomial sequence $\{\sum_{k=0}^{n}\binom{n}{k}^2 q^k\}_{n\geq 0}$. By using Pieri's rule and the Jacobi-Trudi identity for Schur functions, we obtain an expansion of a sum of products of elementary symmetric functions in terms of Schur functions with nonnegative coefficients. Then the principal specialization leads to q-log-convexity. We also prove that a technical condition of Liu and Wang holds for the squares of the binomial coefficients. Hence we deduce that the linear transformation with respect to the triangular array $\{\binom{n}{k}^2\}_{0\leq k\leq n}$ is log-convexity preserving.

Keywords: q-log-convexity, Schur positivity, Pieri's rule, the Jacobi-Trudi identity, principal specialization.

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1 Introduction

The objective of this paper is to prove a conjecture of Liu and Wang [15] on the q-log-convexity of the following polynomials

$$W_n(q) = \sum_{k=0}^{n} \binom{n}{k}^2 q^k. \tag{1.1}$$

The polynomial $W_n(q)$ has appeared as the rank generating function of the lattice of noncrossing partitions of type B on $[n]$, see Reiner [10]. For the type A case, the rank generating function of the lattice of noncrossing partitions on $[n]$ is equal to the Narayana polynomial

$$N_n(q) = \sum_{k=0}^{n} \frac{1}{n}\binom{n}{k}\binom{n}{k+1} q^k. \tag{1.2}$$

Liu and Wang [15] also conjectured that $N_n(q)$ are q-log-convex. This conjecture was proved by Chen, Wang and Yang [5].

The polynomials $W_n(q)$ also arise in the theory of growth series of the root lattice. Recall that the classical root lattice A_n is generated by $\mathcal{M} = \{\mathbf{e}_i - \mathbf{e}_j : 0 \leq i, j \leq n+1 \,\text{with}\, i \neq j\}$. Then the growth series is defined to be the generating function

$$G(q) = \sum_{k\geq 0} S(k) q^k,$$

where $S(k)$ is the number of elements $\mathbf{u} \in A_n$ with length k. It is known that $G(q)$ is a rational function of the form

$$G(q) = \frac{h(q)}{(1-q)^d},$$

where d is the rank of A_n and $h(q)$ is a polynomial of degree less than or equal to d. The polynomial $h(q)$ is defined to be the coordinator polynomial of the growth series, see [2]. Recently, Ardila et al. [1] have shown that the above coordinator polynomial $h(q)$ of A_n equals the polynomial $W_n(q)$.

Recall that a sequence $\{a_k\}_{k\geq 0}$ of nonnegative numbers is said to be log-convex if $a_{k-1}a_{k+1} \geq a_k^2$ for any $k \geq 1$. The q-log-convexity is a property defined in [15] for sequences of polynomials over the field of real numbers, in contrast with the concept of q-log-concavity introduced by Stanley and studied by Butler [4], Krattenthaler [8], Leroux [9] and Sagan [11]. Given a sequence $\{f_n(q)\}_{n\geq 0}$, we say that it is *q-log-convex* if for any $k \geq 1$ the difference

$$f_{k+1}(q) f_{k-1}(q) - f_k(q)^2$$

has nonnegative coefficients as a polynomial of q. It has been shown that many combinatorial polynomials are q-log-convex, such as the Bell polynomials, the Eulerian polynomials, the Bessel polynomials, the Ramanujan polynomials and the Dowling polynomials, see Liu and Wang [15], and Chen, Wang and Yang [6].

Clearly, if a sequence $\{f_n(q)\}_{n\geq 0}$ is q-log-convex, then for any fixed positive number q the sequence $\{f_n(q)\}_{n\geq 0}$ is log-convex. It can be readily verified that the sequence of the central binomial coefficients $\{b_n\}_{n\geq 0}$ is log-convex, where

$$b_n = \binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Došlić and Veljan [7] obtained the log-convexity of the sequence of the central Delannoy numbers $\{d_n\}_{n\geq 0}$, where

$$d_n = \sum_{k=0}^{n} \binom{n}{k}^2 2^k.$$

These two examples are suggestive of the conjecture concerning the q-log-convexity of $W_n(q)$, see [15, Conjecture 5.3]. The first result of this paper is to give an affirmative answer to this conjecture.

Theorem 1.1 *The polynomials $W_n(q)$ form a q-log-convex sequence.*

By using the principal specialization, the q-log-convexity of $W_n(q)$ follows from the Schur positivity of a sum of products of elementary symmetric functions. We will establish an identity on symmetric functions by showing that both sides satisfy the same recurrence relation. To derive the required recurrence relation, we employ the Jacobi-Trudi identity and Pieri's rule for Schur functions. We would like to note that in general the polynomials

$$\sum_{k=0}^{n} \binom{n}{k}^m q^k, \quad n \geq 0$$

are not q-log-convex for $m \geq 3$.

The second result of this paper is concerned with a condition on linear transformations that preserve log-convexity. Let $\{a(n,k)\}_{0\leq k\leq n}$ be a triangular array of numbers. The linear transformation on a sequence $\{x_k\}_{k\geq 0}$ with respect to a triangular array $\{a(n,k)\}_{0\leq k\leq n}$ is defined by

$$y_n = \sum_{k=0}^{n} a(n,k)x_k.$$

Such a transformation is called *log-convexity preserving* if $\{y_n\}_{n\geq 0}$ is log-convex whenever $\{x_k\}_{k\geq 0}$ is log-convex. Liu and Wang [15] obtained a sufficient condition on a triangular array which ensures the corresponding transformation is log-convexity preserving.

Given a triangular array $\{a(n,k)\}_{0\leq k\leq n}$, define $\alpha(n,r,k)$ by

$$\alpha(n,r,k) = a(n+1,k)a(n-1,r-k)+a(n+1,r-k)a(n-1,k)-2a(n,r-k)a(n,k),$$

where $n \geq 1$, $0 \leq r \leq 2n$ and $0 \leq k \leq \lfloor \frac{r}{2} \rfloor$. The sufficient condition of Liu and Wang is stated as follows.

Theorem 1.2 ([15, Theorem 4.8]) *Assume that the polynomials*

$$A_n(q) = \sum_{k=0}^{n} a(n,k)q^k$$

form a q-log-convex sequence. For any given n and r, if there exists an integer $k' = k'(n,r)$ such that $\alpha(n,r,k) \geq 0$ for $k \leq k'$ and $\alpha(n,r,k) \leq 0$ for $k > k'$, then the linear transformation with respect to the triangular array $\{a(n,k)\}_{0\leq k\leq n}$ is log-convexity preserving.

Liu and Wang conjectured that the above sufficient condition holds for the coefficients of $W_n(q)$. We will also prove this conjecture.

Theorem 1.3 *The linear transformation with respect to the triangular array* $\{\binom{n}{k}^2\}_{0\le k\le n}$ *is log-convexity preserving.*

This paper is organized as follows. We recall some definitions and known results on symmetric functions in Section 2. In Section 3, we will give an inductive proof of the identity which implies the desired Schur positivity. In Section 4, we will present the proofs of Theorem 1.1 and Theorem 1.3.

2 Background on symmetric functions

Throughout this paper, we will adopt the notation and terminology on partitions and symmetric functions in Stanley [13]. Recall that a *partition* λ of a nonnegative integer n is a weakly decreasing sequence $(\lambda_1, \lambda_2, \ldots)$ of nonnegative integers satisfying $\sum_i \lambda_i = n$, denoted $\lambda \vdash n$. We usually omit the parts $\lambda_i = 0$. We also denote a partition $\lambda \vdash n$ by $(n^{m_n}, \ldots, 2^{m_2}, 1^{m_1})$ if λ has m_i i's for $1 \le i \le n$. Let $\mathrm{Par}(n)$ denote the set of partitions of n.

If $\lambda \vdash n$, we draw a left-justified array of n squares with λ_i squares in the i-th row. This array is called the *Young diagram* of λ. By transposing the diagram of λ, we get the *conjugate partition* of λ, denoted λ'. We use $\mu \subseteq \lambda$ to denote that the Young diagram of μ is contained in the diagram of λ.

A semistandard Young tableau of shape λ is an array $T = (T_{ij})$ of positive integers of shape λ such that it is weakly increasing in each row and strictly increasing in each column. The type of T is defined as the composition $\alpha = (\alpha_1, \alpha_2, \ldots)$, where α_i is the number of i's in T. Let $x = (x_1, x_2, \ldots)$ be a sequence of indeterminates. If $\mathrm{type}(T) = \alpha$, then we write

$$x^T = x_1^{\alpha_1} x_2^{\alpha_2} \cdots .$$

The *Schur function* $s_\lambda(x)$ is defined as the generating function

$$s_\lambda(x) = \sum_T x^T,$$

summed over semistandard Young tableaux T of shape λ. When $\lambda = \emptyset$, we set $s_\emptyset(x) = 1$.

It is well known that Schur functions $s_\lambda(x)$ form a basis for the ring of symmetric functions. A symmetric function $f(x)$ is called *Schur positive* if the coefficients a_λ are all nonnegative in the Schur expansion $f(x) = \sum_\lambda a_\lambda s_\lambda(x)$.

When $\lambda = (1^k)$ with $k \geq 1$, the Schur function $s_\lambda(x)$ becomes the k-th elementary symmetric function $e_k(x)$, i.e.,

$$s_{(1^k)}(x) = e_k(x) = \sum_{1 \leq i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}. \tag{2.3}$$

The dual Jacobi-Trudi identity gives an expression of the Schur function $s_\lambda(x)$ in terms of the elementary symmetric functions.

Theorem 2.1 ([13, Corollary 7.16.2]) *Let λ be a partition with the largest part $\leq n$ and λ' its conjugate. Then*

$$s_\lambda(x) = \det(e_{\lambda_i{}' - i + j}(x))_{i,j=1}^n,$$

where $e_0 = 1$ and $e_k = 0$ for $k < 0$.

Given any symmetric function $f(x)$, we may omit the variable set x if no confusion arises in the context. Now let us review the definition of the principal specialization ps_n^1 of a symmetric function. For any symmetric function f, the action of ps_n^1 is defined as

$$\mathrm{ps}_n^1(f) = f(\underbrace{1, \ldots, 1}_{n\, 1's}, 0, 0, \ldots).$$

In particular, by (2.3), we have $\mathrm{ps}_n^1(e_k) = \binom{n}{k}$ and

$$\mathrm{ps}_n^1(e_k) = \mathrm{ps}_{n-1}^1(e_k + e_{k-1}), \tag{2.4}$$

which is a restatement of the relation

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

We will also need the dual version of Pieri's rule which expresses the product of a Schur function s_μ and an elementary symmetric function e_k in terms of Schur functions.

Theorem 2.2 ([13]) *We have*

$$s_\mu e_k = \sum_\lambda s_\lambda,$$

summed over partitions λ such that $\mu \subseteq \lambda$ and the difference of the Young diagrams of λ and μ contains no two squares in the same row.

3 A Schur Positivity Identity

In this section, we aim to establish the following Schur positivity theorem which will be used to prove the q-log-convexity of $W_n(q)$.

Theorem 3.1 *For any $r \geq 1$, we have*

$$\sum_{k=0}^{r}(e_{k-1}e_{k-1}e_{r-k}e_{r-k} + e_{k-2}e_k e_{r-k}e_{r-k} - 2e_{k-1}e_k e_{r-k-1}e_{r-k}) = \sum_{\lambda} s_{\lambda}, \tag{3.5}$$

where λ sums over all partitions of $2r-2$ of the form $(4^{i_4}, 3^{2i_3}, 2^{2i_2}, 1^{2i_1})$ with i_1, i_2, i_3, i_4 being nonnegative integers.

Before proving the above theorem, it is informative to give examples for $r = 3, 4, 5$. By using the Maple package ACE [16], or SF [14], we find that

$$\begin{aligned}
&\sum_{k=0}^{3}(e_{k-1}e_{k-1}e_{3-k}e_{3-k} + e_{k-2}e_k e_{3-k}e_{3-k} - 2e_{k-1}e_k e_{3-k-1}e_{3-k}) \\
&\quad = s_{(1^4)} + s_{(2^2)} + s_{(4)}, \\
&\sum_{k=0}^{4}(e_{k-1}e_{k-1}e_{4-k}e_{4-k} + e_{k-2}e_k e_{4-k}e_{4-k} - 2e_{k-1}e_k e_{4-k-1}e_{4-k}) \\
&\quad = s_{(1^6)} + s_{(2^2,1^2)} + s_{(4,1^2)} + s_{(3^2)}, \\
&\sum_{k=0}^{5}(e_{k-1}e_{k-1}e_{5-k}e_{5-k} + e_{k-2}e_k e_{5-k}e_{5-k} - 2e_{k-1}e_k e_{5-k-1}e_{5-k}) \\
&\quad = s_{(4,2^2)} + s_{(4^2)} + s_{(1^8)} + s_{(2^2,1^4)} + s_{(2^4)} + s_{(4,1^4)} + s_{(3^2,1^2)}.
\end{aligned}$$

Let $L(r)$ and $R(r)$ denote the left-hand side and the right-hand side of (3.5), respectively. We will proceed to prove Theorem 3.1 by showing that $L(r)$ and $R(r)$ satisfy the same recurrence relation. It is easy to find the recurrence relation of $R(r)$. To derive the recurrence relation of $L(r)$, we will need several lemmas. For the sake of presentation, for $t \geq 0$, we define

$$\begin{aligned}
A_1(t,k,i,j) &= e_k s_{(3^i,2^{k-i+j},1^{4t-3k-2j-i})}, \\
A_2(t,k,i,j) &= e_k s_{(3^i,2^{k-i+j},1^{4t-3k-2j-i})}, \\
A_3(t,k,i,j) &= e_{k-1} s_{(3^i,2^{k-i+j},1^{4t-3k-2j-i+1})}, \\
A_4(t,k,i,j) &= e_k s_{(3^i,2^{k-i+j-1},1^{4t-3k-2j-i+2})}, \\
B_1(t,k,i,j) &= e_k s_{(3^i,2^{k-i+j},1^{4t-3k-2j-i-2})}, \\
B_2(t,k,i,j) &= e_k s_{(3^i,2^{k-i+j},1^{4t-3k-2j-i-2})},
\end{aligned}$$

$$B_3(t,k,i,j) = e_{k-1} s_{(3^i, 2^{k-i+j}, 1^{4t-3k-2j-i-1})},$$
$$B_4(t,k,i,j) = e_k s_{(3^i, 2^{k-i+j-1}, 1^{4t-3k-2j-i})},$$

and

$$A_1(t) = \sum_{k=0}^{t} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k} A_1(t,k,i,j),$$

$$A_2(t) = \sum_{k=0}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-1} A_2(t,k,i,j),$$

$$A_3(t) = \sum_{k=1}^{t} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k} A_3(t,k,i,j),$$

$$A_4(t) = \sum_{k=1}^{t} \sum_{i=0}^{k-1} \sum_{j=0}^{2t-2k+1} A_4(t,k,i,j),$$

$$B_1(t) = \sum_{k=0}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-1} B_1(t,k,i,j),$$

$$B_2(t) = \sum_{k=0}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-2} B_2(t,k,i,j),$$

$$B_3(t) = \sum_{k=1}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-1} B_3(t,k,i,j),$$

$$B_4(t) = \sum_{k=1}^{t} \sum_{i=0}^{k-1} \sum_{j=0}^{2t-2k} B_4(t,k,i,j).$$

The following lemma gives explicit expressions for $L(r)$ according to the parity of r.

Lemma 3.2 *For $t \geq 0$ we have*

$$L(2t+1) = A_1(t) + A_2(t) - A_3(t) - A_4(t),$$
$$L(2t) = B_1(t) + B_2(t) - B_3(t) - B_4(t).$$

Proof. For any $r \geq 0$, we have

$$L(r) = \sum_{k=0}^{r} (e_{k-1}e_{k-1}e_{r-k}e_{r-k} + e_{k-2}e_k e_{r-k}e_{r-k} - 2e_{k-1}e_k e_{r-k-1}e_{r-k})$$

$$= \sum_{k=0}^{r} (e_{k-1}e_{k-1}e_{r-k}e_{r-k} - e_{k-1}e_k e_{r-k-1}e_{r-k})$$

$$+ \sum_{k=0}^{r} (e_{k-2}e_k e_{r-k}e_{r-k} - e_{k-1}e_k e_{r-k-1}e_{r-k})$$

$$= \sum_{k=0}^{r} e_{k-1}e_{r-k} \det \begin{pmatrix} e_{k-1} & e_k \\ e_{r-k-1} & e_{r-k} \end{pmatrix}$$

$$+ \sum_{k=0}^{r} e_k e_{r-k} \det \begin{pmatrix} e_{k-2} & e_{k-1} \\ e_{r-k-1} & e_{r-k} \end{pmatrix}.$$

By the dual Jacobi-Trudi identity, we find that

$$L(r) = \sum_{\substack{k=0 \\ k-1 \geq r-k}}^{r} e_{k-1}e_{r-k}s_{(2^{r-k},1^{2k-r-1})} - \sum_{\substack{k=0 \\ k-1 < r-k-1}}^{r} e_{k-1}e_{r-k}s_{(2^k,1^{r-2k-1})}$$

$$+ \sum_{\substack{k=0 \\ k-2 \geq r-k}}^{r} e_k e_{r-k}s_{(2^{r-k},1^{2k-r-2})} - \sum_{\substack{k=0 \\ k-2 < r-k-1}}^{r} e_k e_{r-k}s_{(2^{k-1},1^{r-2k})}.$$

Applying the dual version of Pieri's rule to the products $e_{k-1}s_{(2^{r-k},1^{2k-r-1})}$, $e_{r-k}s_{(2^k,1^{r-2k-1})}$, $e_k s_{(2^{r-k},1^{2k-r-2})}$ and $e_{r-k}s_{(2^{k-1},1^{r-2k})}$, we get

$$L(r) = \sum_{\substack{k=0 \\ k-1 \geq r-k}}^{r} \sum_{i=0}^{r-k} \sum_{j=0}^{2k-r-1} e_{r-k}s_{(3^i,2^{r-k-i+j},1^{2k-r-2-j+k-i-j})}$$

$$- \sum_{\substack{k=0 \\ k-1 < r-k-1}}^{r} \sum_{i=0}^{k} \sum_{j=0}^{r-2k-1} e_{k-1}s_{(3^i,2^{k-i+j},1^{r-2k-j+r-k-i-j-1})}$$

$$+ \sum_{\substack{k=0 \\ k-2 \geq r-k}}^{r} \sum_{i=0}^{r-k} \sum_{j=0}^{2k-r-2} e_{r-k}s_{(3^i,2^{r-k-i+j},1^{2k-r-2-j+k-i-j})}$$

$$- \sum_{\substack{k=0 \\ k-2 < r-k-1}}^{r} \sum_{i=0}^{k-1} \sum_{j=0}^{r-2k} e_k s_{(3^i,2^{k-i+j-1},1^{r-2k-j+r-k-i-j})}.$$

Setting $r = 2t$ or $r = 2t + 1$, we obtain the required relations. ∎

To find a recurrence relation for $L(r)$, let us recall an operator Δ^{μ} associated with a partition μ, which acts on symmetric functions. This operator was introduced by Chen, Wang and Yang [5]. Given two partitions λ and μ, let $\lambda \cup \mu$ be the partition whose parts are obtained by taking the union of the parts of λ and μ. For a symmetric function f with the expansion

$$f = \sum_{\lambda} a_{\lambda} s_{\lambda},$$

the action of Δ^μ on f is defined by

$$\Delta^\mu(f) = \sum_\lambda a_\lambda s_{\lambda\cup\mu}.$$

In order to compute the difference $L(2t+1) - \Delta^{(1,1)}L(2t)$, we need to evaluate $A_m(t,k,i,j) - \Delta^{(1,1)}(B_m(t,k,i,j))$ for $1 \le m \le 4$. In fact, we will be able to express these differences as double sums of Schur functions. For $t \ge 0$, let

$$T_1(t,i,j,k) = \sum_{a=0}^{\beta_1+1} \sum_{b=0}^{\min(\beta_2,\beta_3+1)} P(t,k+1,i,j-1,a,b),$$

$$T_2(t,i,j,k) = \sum_{a=0}^{\beta_1} \sum_{b=0}^{\min(\beta_2,\beta_3)} P(t,k,i,j,a,b),$$

$$T_3(t,i,j,k) = \sum_{a=0}^{\beta_1+1} \sum_{b=0}^{\min(\beta_2,\beta_3+1)} P(t,k+1,i,j-1,a,b),$$

$$T_4(t,i,j,k) = \sum_{a=0}^{\beta_1} \sum_{b=0}^{\min(\beta_2,\beta_3)} P(t,k,i,j,a,b),$$

$$T_5(t,i,j,k) = \sum_{a=0}^{\beta_1-1} \sum_{b=0}^{\min(\beta_2,\beta_3-1)} P(t,k,i,j,a,b),$$

$$T_6(t,i,j,k) = \sum_{a=0}^{\beta_1-2} \sum_{b=0}^{\min(\beta_2,\beta_3-2)} P(t,k-1,i,j+1,a,b),$$

$$T_7(t,i,j,k) = \sum_{a=0}^{\beta_1-1} \sum_{b=0}^{\min(\beta_2-1,\beta_3-1)} P(t,k,i,j,a,b),$$

$$T_8(t,i,j,k) = \sum_{a=0}^{\beta_1-2} \sum_{b=0}^{\min(\beta_2-1,\beta_3-2)} P(t,k-1,i,j+1,a,b),$$

where

$$\begin{aligned}
P(t,k,i,j,a,b) &= s_{(4^a,3^{i-a+b},2^{4t-2k-2i-j-b},1^{4k+i+2j-a-b-4t})},\\
\beta_1 &= 4k+i+2j-4t,\\
\beta_2 &= k-i+j,\\
\beta_3 &= 4k+i+2j-a-4t.
\end{aligned}$$

Then we have the following result.

Lemma 3.3 *Suppose that $t \geq 0$.*

(i) If $0 \leq k \leq t-1, 0 \leq i \leq k, 0 \leq j \leq 2t-2k-1$, then

$$A_1(t,k,i,j) - \Delta^{(1,1)}(B_1(t,k,i,j)) = T_1(t,i,j,k) + T_2(t,i,j,k).$$

(ii) If $0 \leq k \leq t-1, 0 \leq i \leq k, 0 \leq j \leq 2t-2k-2$, then

$$A_2(t,k,i,j) - \Delta^{(1,1)}(B_2(t,k,i,j)) = T_3(t,i,j,k) + T_4(t,i,j,k).$$

(iii) If $1 \leq k \leq t-1, 0 \leq i \leq k, 0 \leq j \leq 2t-2k-1$, then

$$A_3(t,k,i,j) - \Delta^{(1,1)}(B_3(t,k,i,j)) = T_5(t,i,j,k) + T_6(t,i,j,k).$$

(iv) If $1 \leq k \leq t, 0 \leq i \leq k-1, 0 \leq j \leq 2t-2k$, then

$$A_4(t,k,i,j) - \Delta^{(1,1)}(B_4(t,k,i,j)) = T_7(t,i,j,k) + T_8(t,i,j,k).$$

Proof. We only give the proof of (i). The proofs of (ii),(iii) and (iv) are analogous to that of (i). Recall that

$$A_1(t,k,i,j) = e_k s_{(3^i,2^{k-i+j},1^{4t-3k-2j-i})},\ B_1(t,k,i,j) = e_k s_{(3^i,2^{k-i+j},1^{4t-3k-2j-i-2})}.$$

Applying the dual version of Pieri's rule, we see that

$$A_1(t,k,i,j) = \sum_{(a,b,c,d)} s_{(4^a,3^{i-a+b},2^{k-i+j-b+c},1^{4t-3k-2j-i-c+d})},$$

summed over nonnegative integer sequences (a,b,c,d) satisfying

$$a+b+c+d=k, \quad a \leq i, \quad b \leq k-i+j, \quad c \leq 4t-3k-2j-i.$$

Note that the shape $(4^a, 3^{i-a+b}, 2^{k-i+j-b+c}, 1^{4t-3k-2j-i-c+d})$ is obtained from the Young diagram of $(3^i, 2^{k-i+j}, 1^{4t-3k-2j-i})$ by adding a squares in the fourth column, b squares in the third column, c squares in the second column, and d squares in the first column. Similarly, we see that

$$B_1(t,k,i,j) = \sum_{(a,b,c,d)} s_{(4^a,3^{i-a+b},2^{k-i+j-b+c},1^{4t-3k-2j-i-2-c+d})},$$

summed over nonnegative integer sequences (a,b,c,d) satisfying

$$a+b+c+d=k, \quad a \leq i, \quad b \leq k-i+j, \quad c \leq 4t-3k-2j-i-2.$$

Therefore,

$$A_1(t,k,i,j) - \Delta^{(1,1)}(B_1(t,k,i,j))$$

$$= \sum_{\substack{(a,b,c,d)\\ c=4t-3k-2j-i}} s_{(4^a,3^{i-a+b},2^{k-i+j-b+c},1^{4t-3k-2j-i-c+d})}$$

$$+ \sum_{\substack{(a,b,c,d)\\ c=4t-3k-2j-i-1}} s_{(4^a,3^{i-a+b},2^{k-i+j-b+c},1^{4t-3k-2j-i-c+d})},$$

where both sums range over nonnegative integers a, b, d satisfying

$$a \le i, \quad b \le k-i+j, \quad a+b+c+d = k.$$

Since $0 \le j \le 2t-2k-1$, we have

$$\begin{aligned} k-(4t-3k-2j-i) &= 4k+2j+i-4t \\ &\le 4k+2(2t-2k-1)+i-4t \le i, \end{aligned}$$

and

$$\begin{aligned} k-(4t-3k-2j-i-1) &= 4k+2j+i-4t+1 \\ &\le 4k+2(2t-2k-1)+i-4t+1 \le i. \end{aligned}$$

Thus

$$A_1(t,k,i,j) - \Delta^{(1,1)}(B_1(t,k,i,j)) = T_1(t,i,j,k) + T_2(t,i,j,k).$$

This completes the proof of (i). ∎

In light of the above lemma, we will show that $L(2t+1) - \Delta^{(1,1)}L(2t)$ can be expressed in terms of ten sums $T_1(t), T_2(t), \ldots, T_{10}(t)$, as defined below,

$$T_1(t) = \sum_{k=0}^{t-1}\sum_{i=0}^{k}\sum_{j=0}^{2t-2k-1} T_1(t,i,j,k), \quad T_2(t) = \sum_{k=0}^{t-1}\sum_{i=0}^{k}\sum_{j=0}^{2t-2k-1} T_2(t,i,j,k),$$

$$T_3(t) = \sum_{k=0}^{t-1}\sum_{i=0}^{k}\sum_{j=0}^{2t-2k-2} T_3(t,i,j,k), \quad T_4(t) = \sum_{k=0}^{t-1}\sum_{i=0}^{k}\sum_{j=0}^{2t-2k-2} T_4(t,i,j,k),$$

$$T_5(t) = \sum_{k=1}^{t-1}\sum_{i=0}^{k}\sum_{j=0}^{2t-2k-1} T_5(t,i,j,k), \quad T_6(t) = \sum_{k=1}^{t-1}\sum_{i=0}^{k}\sum_{j=0}^{2t-2k-1} T_6(t,i,j,k),$$

$$T_7(t) = \sum_{k=1}^{t}\sum_{i=0}^{k-1}\sum_{j=0}^{2t-2k} T_7(t,i,j,k), \quad T_8(t) = \sum_{k=1}^{t}\sum_{i=0}^{k-1}\sum_{j=0}^{2t-2k} T_8(t,i,j,k),$$

$$T_9(t) = \sum_{k=0}^{t} e_k s_{(3^k,2^{2t-2k})}, \quad T_{10}(t) = \sum_{k=0}^{t-1} e_k s_{(3^{k+1},2^{2t-2k-2},1)}.$$

Lemma 3.4 *For $t \ge 0$, we have*

$$\begin{aligned} L(2t+1) - \Delta^{(1,1)}L(2t) &= (T_1(t)+T_2(t)+T_3(t)+T_4(t)) \\ &\quad - (T_5(t)+T_6(t)+T_7(t)+T_8(t)) + (T_9(t)-T_{10}(t)). \end{aligned}$$

Proof. By Lemma 3.2, $L(2t+1) - \Delta^{(1,1)}L(2t)$ equals

$$
\begin{aligned}
&(A_1(t) - \Delta^{(1,1)}B_1(t)) + (A_2(t) - \Delta^{(1,1)}B_2(t)) \\
&\quad - (A_3(t) - \Delta^{(1,1)}B_3(t)) - (A_4(t) - \Delta^{(1,1)}B_4(t)) \\
&= \left(\sum_{k=0}^{t} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k} A_1(t,k,i,j) - \sum_{k=0}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-1} \Delta^{(1,1)}(B_1(t,k,i,j)) \right) \\
&\quad + \left(\sum_{k=0}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-1} A_2(t,k,i,j) - \sum_{k=0}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-2} \Delta^{(1,1)}(B_2(t,k,i,j)) \right) \\
&\quad - \left(\sum_{k=1}^{t} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k} A_3(t,k,i,j) - \sum_{k=1}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-1} \Delta^{(1,1)}(B_3(t,k,i,j)) \right) \\
&\quad - \left(\sum_{k=1}^{t} \sum_{i=0}^{k-1} \sum_{j=0}^{2t-2k+1} A_4(t,k,i,j) - \sum_{k=1}^{t} \sum_{i=0}^{k-1} \sum_{j=0}^{2t-2k} \Delta^{(1,1)}(B_4(t,k,i,j)) \right) \\
&= \sum_{k=0}^{t} \sum_{i=0}^{k} A_1(t,k,i,2t-2k) + \sum_{k=0}^{t-1} \sum_{i=0}^{k} A_2(t,k,i,2t-2k-1) \\
&\quad + \sum_{k=0}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-1} \left(A_1(t,k,i,j) - \Delta^{(1,1)}(B_1(t,k,i,j)) \right) \\
&\quad + \sum_{k=0}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-2} \left(A_2(t,k,i,j) - \Delta^{(1,1)}(B_2(t,k,i,j)) \right) \\
&\quad - \sum_{k=1}^{t} \sum_{i=0}^{k} A_3(t,k,i,2t-2k) - \sum_{k=1}^{t} \sum_{i=0}^{k-1} A_4(t,k,i,2t-2k+1) \\
&\quad - \sum_{k=1}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-1} \left(A_3(t,k,i,j) - \Delta^{(1,1)}(B_3(t,k,i,j)) \right) \\
&\quad - \sum_{k=1}^{t} \sum_{i=0}^{k-1} \sum_{j=0}^{2t-2k} \left(A_4(t,k,i,j) - \Delta^{(1,1)}(B_4(t,k,i,j)) \right).
\end{aligned}
$$

Applying Lemma 3.3, we get the desired relation. ∎

In order to derive a recurrence relation of $L(r)$, we still need to compute

$$(L(2(t+1)+1) - \Delta^{(1,1)}L(2(t+1))) - \Delta^{(2,2)}(L(2t+1) - \Delta^{(1,1)}L(2t)).$$

By Lemma 3.4, the above expression equals

$$\sum_{i=1}^{4}(T_i(t+1) - \Delta^{(2,2)}T_i(t)) - \sum_{i=5}^{8}(T_i(t+1) - \Delta^{(2,2)}T_i(t))$$

$$+ (T_9(t+1) - \Delta^{(2,2)}T_9(t)) + (T_{10}(t+1) - \Delta^{(2,2)}T_{10}(t)). \tag{3.6}$$

Set

$$Q_1(t,k,i,j,a,b) = s_{(4^a,3^{k-a+b+1},2^{4t-4k-j-b-1},1^{5k+2j-a-b-4t+3})}, \tag{3.7}$$

$$Q_2(t,k,i,j,a,b) = s_{(4^a,3^{k-a+j+1},2^{4t-3k-i-2j},1^{3k+2i+j-a-4t+1})}, \tag{3.8}$$

$$\gamma_1 = 5k + 2j + 2 - 4t, \tag{3.9}$$

$$\gamma_2 = 5k + 2j - a + 2 - 4t, \tag{3.10}$$

$$\gamma_3 = 3k + 2i + j - 4t, \tag{3.11}$$

and let

$$T_{11}(t) = \sum_{k=0}^{t-1} \sum_{j=0}^{2t-2k-1} \sum_{a=0}^{\gamma_1} \sum_{b=0}^{\min(j,\gamma_2)} Q_1(t,k,i,j,a,b),$$

$$T_{12}(t) = \sum_{k=0}^{t} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-1} \sum_{a=0}^{\gamma_3} Q_2(t,k,i,j,a,b),$$

$$T_{21}(t) = \sum_{k=0}^{t-1} \sum_{j=-1}^{2t-2k-2} \sum_{a=0}^{\gamma_1+1} \sum_{b=0}^{\min(j+1,\gamma_2+1)} Q_1(t,k,i,j,a,b),$$

$$T_{22}(t) = \sum_{k=0}^{t-1} \sum_{i=-1}^{k-1} \sum_{j=0}^{2t-2k-1} \sum_{a=0}^{\gamma_3+1} Q_2(t,k,i,j,a,b),$$

$$T_{31}(t) = \sum_{k=0}^{t} \sum_{j=0}^{2t-2k-2} \sum_{a=0}^{\gamma_1} \sum_{b=0}^{\min(j,\gamma_2)} Q_1(t,k,i,j,a,b),$$

$$T_{32}(t) = \sum_{k=0}^{t} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-2} \sum_{a=0}^{\gamma_3} Q_2(t,k,i,j,a,b),$$

$$T_{41}(t) = \sum_{k=0}^{t-1} \sum_{j=-1}^{2t-2k-3} \sum_{a=0}^{\gamma_1+1} \sum_{b=0}^{\min(j+1,\gamma_2+1)} Q_1(t,k,i,j,a,b),$$

$$T_{42}(t) = \sum_{k=0}^{t-1} \sum_{i=-1}^{k-1} \sum_{j=0}^{2t-2k-2} \sum_{a=0}^{\gamma_3+1} Q_2(t,k,i,j,a,b),$$

$$T_{51}(t) = \sum_{k=0}^{t-1} \sum_{j=-1}^{2t-2k-2} \sum_{a=0}^{\gamma_1} \sum_{b=0}^{\min(j+1,\gamma_2)} Q_1(t,k,i,j,a,b),$$

$$T_{52}(t) = \sum_{k=0}^{t-1} \sum_{i=-1}^{k-1} \sum_{j=0}^{2t-2k-1} \sum_{a=0}^{\gamma_3} Q_2(t,k,i,j,a,b),$$

$$T_{61}(t) = \sum_{k=0}^{t-1} \sum_{j=-2}^{2t-2k-3} \sum_{a=0}^{\gamma_1+1} \sum_{b=0}^{\min(j+2,\gamma_2+1)} Q_1(t,k,i,j,a,b),$$

$$T_{62}(t) = \sum_{k=-2}^{t-3} \sum_{i=0}^{k+2} \sum_{j=2}^{2t-2k-3} \sum_{a=0}^{\gamma_3+1} Q_2(t,k,i,j,a,b),$$

$$T_{71}(t) = \sum_{k=0}^{t} \sum_{j=-2}^{2t-2k-2} \sum_{a=0}^{\gamma_1+1} \sum_{b=-1}^{\min(j+1,\gamma_2)} Q_1(t,k,i,j,a,b),$$

$$T_{72}(t) = \sum_{k=0}^{t} \sum_{i=0}^{k-1} \sum_{j=-1}^{2t-2k-1} \sum_{a=0}^{\gamma_3} Q_2(t,k,i,j,a,b),$$

$$T_{81}(t) = \sum_{k=0}^{t-1} \sum_{j=0}^{2t-2k-2} \sum_{a=0}^{\gamma_1+1} \sum_{b=0}^{\min(j,\gamma_2+1)} Q_1(t,k,i,j,a,b),$$

$$T_{82}(t) = \sum_{k=-1}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-2} \sum_{a=0}^{\gamma_3+1} Q_2(t,k,i,j,a,b).$$

The following lemma shows that each $T_i(t+1) - \Delta^{(2,2)} T_i(t)$ can be expressed in terms of two sums for $1 \leq i \leq 8$, where each sum is a quadruple sum of Schur functions.

Lemma 3.5 *For $t \geq 0$ and $1 \leq i \leq 8$, we have*

$$T_i(t+1) - \Delta^{(2,2)} T_i(t) = T_{i1}(t) + T_{i2}(t).$$

Proof. We will give only the proof of the identity for $T_1(t)$, since the other cases can be verified by the same argument. Observe that

$$T_1(t) = \sum_{k=0}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-1} \sum_{a=0}^{4k+i+2j-4t+1} \sum_{b=0}^{\min(k-i+j,4k+i+2j-a-4t+1)} \left(s_{(4^a,3^{i-a+b},2^{4t-2k-2i-j-b-1},1^{4k+i+2j-a-b-4t+2})} \right).$$

It follows that

$$T_1(t+1) = \sum_{k=0}^{t} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k+1} \sum_{a=0}^{4k+i+2j-4t-3} \sum_{b=0}^{\min(k-i+j,4k+i+2j-a-4t-3)}$$

$$\left(s_{(4^a,3^{i-a+b},2^{4t-2k-2i-j-b+3},1^{4k+i+2j-a-b-4t-2})}\right)$$

$$= \sum_{k=-1}^{t-1}\sum_{i=0}^{k+1}\sum_{j=0}^{2t-2k-1}\sum_{a=0}^{4k+i+2j-4t+1}\sum_{b=0}^{\min(k-i+j+1,4k+i+2j-a-4t+1)}\left(s_{(4^a,3^{i-a+b},2^{4t-2k-2i-j-b+1},1^{4k+i+2j-a-b-4t+2})}\right)$$

$$= \sum_{k=0}^{t-1}\sum_{i=0}^{k+1}\sum_{j=0}^{2t-2k-1}\sum_{a=0}^{4k+i+2j-4t+1}\sum_{b=0}^{\min(k-i+j+1,4k+i+2j-a-4t+1)}\left(s_{(4^a,3^{i-a+b},2^{4t-2k-2i-j-b+1},1^{4k+i+2j-a-b-4t+2})}\right),$$

where the last equality holds because the upper bound $4k+i+2j-4t+1$ of a is negative for $k=-1$. Hence we deduce that

$$T_1(t+1) - \Delta^{(2,2)}T_1(t)$$

$$= \sum_{k=0}^{t-1}\sum_{i=k+1}^{k+1}\sum_{j=0}^{2t-2k-1}\sum_{a=0}^{4k+i+2j-4t+1}\sum_{b=0}^{\min(k-i+j+1,4k+i+2j-a-4t+1)}\left(s_{(4^a,3^{i-a+b},2^{4t-2k-2i-j-b+1},1^{4k+i+2j-a-b-4t+2})}\right)$$

$$+ \sum_{k=0}^{t-1}\sum_{i=0}^{k}\sum_{j=0}^{2t-2k-1}\sum_{a=0}^{4k+i+2j-4t+1}\sum_{b=0}^{\min(k-i+j+1,4k+i+2j-a-4t+1)}\left(s_{(4^a,3^{i-a+b},2^{4t-2k-2i-j-b+1},1^{4k+i+2j-a-b-4t+2})}\right)$$

$$- \sum_{k=0}^{t-1}\sum_{i=0}^{k}\sum_{j=0}^{2t-2k-1}\sum_{a=0}^{4k+i+2j-4t+1}\sum_{b=0}^{\min(k-i+j,4k+i+2j-a-4t+1)}\left(s_{(4^a,3^{i-a+b},2^{4t-2k-2i-j-b+1},1^{4k+i+2j-a-b-4t+2})}\right)$$

$$= \sum_{k=0}^{t-1}\sum_{i=k+1}^{k+1}\sum_{j=0}^{2t-2k-1}\sum_{a=0}^{4k+i+2j-4t+1}\sum_{b=0}^{\min(k-i+j+1,4k+i+2j-a-4t+1)}\left(s_{(4^a,3^{i-a+b},2^{4t-2k-2i-j-b+1},1^{4k+i+2j-a-b-4t+2})}\right)$$

$$+ \sum_{k=0}^{t-1}\sum_{i=0}^{k}\sum_{j=0}^{2t-2k-1}\sum_{\substack{a=0\\ k-i+j+1\le 4k+i+2j-a-4t+1}}^{4k+i+2j-4t+1}\sum_{b=0}^{k-i+j+1}\left(s_{(4^a,3^{i-a+b},2^{4t-2k-2i-j-b+1},1^{4k+i+2j-a-b-4t+2})}\right)$$

$$-\sum_{k=0}^{t-1}\sum_{i=0}^{k}\sum_{j=0}^{2t-2k-1}\sum_{\substack{a=0\\ k-i+j+1\le 4k+i+2j-a-4t+1}}^{4k+i+2j-4t+1}\sum_{b=0}^{k-i+j}$$

$$\left(s_{(4^a,3^{i-a+b},2^{4t-2k-2i-j-b+1},1^{4k+i+2j-a-b-4t+2})}\right)$$

$$=\sum_{k=0}^{t-1}\sum_{i=k+1}^{k+1}\sum_{j=0}^{2t-2k-1}\sum_{a=0}^{4k+i+2j-4t+1}\sum_{b=0}^{\min(k-i+j+1,4k+i+2j-a-4t+1)}$$

$$\left(s_{(4^a,3^{i-a+b},2^{4t-2k-2i-j-b+1},1^{4k+i+2j-a-b-4t+2})}\right)$$

$$+\sum_{k=0}^{t-1}\sum_{i=0}^{k}\sum_{j=0}^{2t-2k-1}\sum_{\substack{a=0\\ k-i+j+1\le 4k+i+2j-a-4t+1}}^{4k+i+2j-4t+1}\sum_{b=k-i+j+1}^{k-i+j+1}$$

$$\left(s_{(4^a,3^{i-a+b},2^{4t-2k-2i-j-b+1},1^{4k+i+2j-a-b-4t+2})}\right)$$

$$=\sum_{k=0}^{t-1}\sum_{j=0}^{2t-2k-1}\sum_{a=0}^{5k+2j+2-4t}\sum_{b=0}^{\min(j,5k+2j-a+2-4t)}$$

$$\left(s_{(4^a,3^{k-a+b+1},2^{4t-4k-j-b-1},1^{5k+2j-a-b-4t+3})}\right)$$

$$+\sum_{k=0}^{t-1}\sum_{i=0}^{k}\sum_{j=0}^{2t-2k-1}\sum_{a=0}^{3k+2i+j-4t}$$

$$\left(s_{(4^a,3^{k-a+j+1},2^{4t-3k-i-2j},1^{3k+2i+j-a-4t+1})}\right),$$

as desired. This completes the proof. ∎

To compute (3.6), it is necessary to simplify

$$\sum_{i=1}^{4}(T_i(t+1)-\Delta^{(2,2)}T_i(t))-\sum_{i=5}^{8}(T_i(t+1)-\Delta^{(2,2)}T_i(t)),$$

by the above lemma, which equals

$$\sum_{i=1}^{4}(T_{i1}(t)+T_{i2}(t))-\sum_{i=5}^{8}(T_{i1}(t)+T_{i2}(t)).$$

Moreover, we need to rearrange the terms into groups in order to reduce the relevant quadruple sums to triple sums, and then to double sums. For $t\ge 0$, let

$$N_1(t)=\sum_{k=0}^{t-1}\sum_{j=0}^{2t-2k-1}\sum_{a=0}^{5k+j-4t-1}s_{(4^a,3^{k-a+j+1},2^{4t-4k-2j+1},1^{5k+j-1-4t-a})},$$

$$N_2(t) = \sum_{k=0}^{t-1} \sum_{j=0}^{2t-2k-2} \sum_{a=0}^{5k+j-4t+2} s_{(4^a,3^{k-a+j+1},2^{4t-4k-2j-1},1^{5k+j+3-4t-a})},$$

$$N_3(t) = \sum_{k=0}^{t-1} \sum_{a=0}^{k} \sum_{b=0}^{\min(2t-2k-2,k-a)} s_{(4^a,3^{k-a+b+1},2^{2t-2k-b},1^{k-a-b+1})},$$

$$N_4(t) = \sum_{k=0}^{t-1} \sum_{a=0}^{k} \sum_{b=0}^{\min(2t-2k-1,k-a)} s_{(4^a,3^{k+1-a+b},2^{2t-2k-b},1^{k+1-a-b})},$$

$$N_5(t) = \sum_{k=0}^{t-1} \sum_{j=0}^{2t-2k-2} \sum_{a=\max(0,5k+j-4t+3)}^{5k+2j-4t+3} s_{(4^a,3^{6k-4t+2j-2a+4},2^{8t-9k-3j+a-4})},$$

$$N_6(t) = \sum_{k=0}^{t-1} \sum_{j=0}^{2t-2k-1} \sum_{a=\max(0,5k+j-4t+1)}^{5k+2j-4t+1} s_{(4^a,3^{6k-4t+2j-2a+2},2^{8t-9k-3j+a-1})},$$

$$M_1(t) = \sum_{k=0}^{t-1} \sum_{j=0}^{2t-2k-2} \sum_{a=0}^{5k+j+2-4t} s_{(4^a,3^{k+1-a+j},2^{4t-2j-1-4k},1^{5k+j-4t+3-a})},$$

$$M_2(t) = \sum_{k=0}^{t-1} \sum_{j=0}^{2t-2k-2} \sum_{a=0}^{5k+j-4t-1} s_{(4^a,3^{k+1-a+j},2^{4t+1-4k-2j},1^{5k+j-4t-1-a})},$$

$$M_3(t) = \sum_{k=0}^{t-1} \sum_{i=1}^{k} \sum_{a=0}^{k-2t+2i-4} s_{(4^a,3^{2t-k-a},2^{k+4-i},1^{k+2i-2t-4-a})},$$

$$M_4(t) = \sum_{k=0}^{t-1} \sum_{i=0}^{k} \sum_{a=0}^{k+2i-2t-1} s_{(4^a,3^{2t-a-k},2^{k-i+2},1^{k+2i-2t-a})},$$

$$M_5(t) = \sum_{k=0}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-2} \sum_{a=\max(0,3k+j+2i-4t+1)}^{3k+j+2i-4t+1} s_{(4^a,3^{k+1-a+j},2^{4t-2j-i-3k})},$$

$$M_6(t) = \sum_{k=1}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-1} \sum_{a=\max(0,3k+j+2i-4t-1)}^{3k+j+2i-4t-1} s_{(4^a,3^{k+1-a+j},2^{4t-2j-i-3k+1})}.$$

The following lemma gives a strategy to group the terms of $T_{i1}(t)$ and $T_{i2}(t)$, which leads to the reduction from quadruple sums to triple sums.

Lemma 3.6 *For $t \geq 0$, we have*

$$T_{41}(t) - T_{61}(t) = -\, N_1(t),$$

$$
\begin{aligned}
T_{31}(t) - T_{71}(t) =& N_2(t) - N_3(t),\\
T_{11}(t) - T_{81}(t) =& N_4(t) - N_5(t),\\
T_{21}(t) - T_{51}(t) =& N_6(t),\\
T_{32}(t) - T_{72}(t) =& - M_1(t)\\
T_{42}(t) - T_{62}(t) =& M_2(t) - M_3(t),\\
T_{12}(t) - T_{82}(t) =& M_4(t) - M_5(t),\\
T_{22}(t) - T_{52}(t) =& M_6(t).
\end{aligned}
$$

Proof. We will prove the first identity, since the others can be proved by the same argument. Using the notation $Q_1(t,k,i,j,a,b)$, γ_1 and γ_2, as given in (3.7)-(3.10), we find

$$
\begin{aligned}
T_{41}(t) - T_{61}(t) =& \sum_{k=0}^{t-1} \sum_{j=-1}^{2t-2k-3} \sum_{a=0}^{\gamma_1+1} \sum_{b=0}^{\min(j+1,\gamma_2+1)} Q_1(t,k,i,j,a,b)\\
&- \sum_{k=0}^{t-1} \sum_{j=-2}^{2t-2k-3} \sum_{a=0}^{\gamma_1+1} \sum_{b=0}^{\min(j+2,\gamma_2+1)} Q_1(t,k,i,j,a,b)\\
=& \sum_{k=0}^{t-1} \sum_{j=-2}^{2t-2k-3} \sum_{a=0}^{\gamma_1+1} \sum_{b=0}^{\min(j+1,\gamma_2+1)} Q_1(t,k,i,j,a,b)\\
&- \sum_{k=0}^{t-1} \sum_{j=-2}^{2t-2k-3} \sum_{a=0}^{\gamma_1+1} \sum_{b=0}^{\min(j+2,\gamma_2+1)} Q_1(t,k,i,j,a,b)\\
=& - \sum_{k=0}^{t-1} \sum_{j=-2}^{2t-2k-3} \sum_{\substack{a=0\\ j+2\le\gamma_2+1}}^{\gamma_1+1} \sum_{b=j+2}^{j+2} Q_1(t,k,i,j,a,b)\\
=& - \sum_{k=0}^{t-1} \sum_{j=-2}^{2t-2k-3} \sum_{a=0}^{5k+j-4t+1} Q_1(t,k,i,j,a,j+2).
\end{aligned}
$$

Substituting j with $j-2$ in the last summation, we are led to the required relation. This completes the proof. ∎

For $t \ge 0$, let

$$
C_1(t) = \sum_{k=1}^{t} \sum_{a=0}^{5k-4t-3} s_{(4^a,3^{k-a},2^{4t-4k+3},1^{5k-4t-a-2})},
$$

$$C_2(t) = \sum_{k=1}^{t-1} \sum_{j=0}^{2t-2k-1} \sum_{a=\max(0,5k+j-4t-1)}^{5k+j-4t-1} s_{(4^a,3^{k-a+j+1},2^{4t-4k-2j+1})},$$

$$C_3(t) = \sum_{k=0}^{t-1} \sum_{a=0}^{3k-2t+1} s_{(4^a,3^{2t-k-a},2,1^{3k-2t-a+2})},$$

$$C_4(t) = \sum_{k=0}^{t-1} \sum_{a=\max(0,5k-4t+1)}^{5k-4t+1} s_{(4^a,3^{k+1-a},2^{4t-4k})},$$

$$C_5(t) = \sum_{k=1}^{t-1} \sum_{j=0}^{2t-2k-2} \sum_{a=\max(0,5k+j-4t+2)}^{5k+j-4t+2} s_{(4^a,3^{k+j-a+2},2^{4t-4k-2j-2})},$$

$$D_1(t) = \sum_{k=0}^{t-2} \sum_{j=1}^{2t-2k-3} \sum_{a=\max(0,5k+j-4t+3)}^{5k+j-4t+3} s_{(4^a,3^{k+1-a+j},2^{4t-2j-4k-1})},$$

$$D_2(t) = \sum_{a=0}^{t-3} s_{(4^a,3^{t-a},2^3,1^{t-a-2})},$$

$$D_3(t) = \sum_{k=0}^{t-2} \sum_{a=0}^{5k-4t+2} s_{(4^a,3^{k+1-a},2^{4t-4k-1},1^{5k-4t-a+3})},$$

$$D_4(t) = \sum_{k=0}^{t-2} \sum_{a=0}^{3k-2t} s_{(4^a,3^{2t-k-a-1},2^3,1^{3k-2t-a+1})},$$

$$D_5(t) = \sum_{k=0}^{t-1} \sum_{a=0}^{3k-2t-1} s_{(4^a,3^{2t-a-k},2^2,1^{3k-2t-a})},$$

$$D_6(t) = \sum_{k=0}^{t-1} \sum_{a=0}^{3k-2t-3} s_{(4^a,3^{2t-a-k},2^3,1^{3k-2t-a-2})},$$

$$D_7(t) = \sum_{k=0}^{t-1} \sum_{i=0}^{k-2} \sum_{a=\max(0,k+2i-2t)}^{k+2i-2t} s_{(4^a,3^{2t-k-a},2^{k-i+2})},$$

$$D_8(t) = \sum_{k=1}^{t-1} \sum_{i=0}^{k-1} \sum_{a=\max(0,k-2t+2i)}^{k-2t+2i} s_{(4^a,3^{2t-a-k},2^{k-i+2})},$$

$$D_9(t) = \sum_{k=1}^{t-1} \sum_{j=0}^{2t-2k-2} \sum_{a=\max(0,5k+j-4t+1)}^{5k+j-4t+1} s_{(4^a,3^{k+1-a+j},2^{4t-4k-2j})}.$$

The following lemma shows that one can group the terms of $N_i(t)$ and $M_i(t)$ to reduce the involved triple sums to double sums.

Lemma 3.7 *For any $t \geq 0$, we have*

$$\begin{aligned}
N_2(t) - N_1(t) =& C_1(t) - C_2(t),\\
N_4(t) - N_3(t) =& C_3(t),\\
N_6(t) - N_5(t) =& C_4(t) + C_5(t),\\
M_2(t) - M_1(t) =& D_1(t) - D_2(t) - D_3(t) - D_4(t),\\
M_4(t) - M_3(t) =& D_5(t) + D_6(t) - D_7(t),\\
M_6(t) - M_5(t) =& D_8(t) - D_9(t).
\end{aligned}$$

Proof. We have

$$\begin{aligned}
N_2(t) - N_1(t) =& \sum_{k=0}^{t-1} \sum_{j=0}^{2t-2k-2} \sum_{a=0}^{5k+j-4t+2} s_{(4^a,3^{k-a+j+1},2^{4t-4k-2j-1},1^{5k+j+3-4t-a})}\\
&- \sum_{k=0}^{t-1} \sum_{j=0}^{2t-2k-1} \sum_{a=0}^{5k+j-4t-1} s_{(4^a,3^{k-a+j+1},2^{4t-4k-2j+1},1^{5k+j-1-4t-a})}\\
=& \sum_{k=1}^{t} \sum_{j=0}^{2t-2k} \sum_{a=0}^{5k+j-4t-3} s_{(4^a,3^{k-a+j},2^{4t-4k-2j+3},1^{5k+j-2-4t-a})}\\
&- \sum_{k=0}^{t-1} \sum_{j=0}^{2t-2k-1} \sum_{a=0}^{5k+j-4t-1} s_{(4^a,3^{k-a+j+1},2^{4t-4k-2j+1},1^{5k+j-1-4t-a})}\\
=& \sum_{k=1}^{t} \sum_{j=-1}^{2t-2k-1} \sum_{a=0}^{5k+j-4t-2} s_{(4^a,3^{k-a+j+1},2^{4t-4k-2j+1},1^{5k+j-1-4t-a})}\\
&- \sum_{k=0}^{t-1} \sum_{j=0}^{2t-2k-1} \sum_{a=0}^{5k+j-4t-1} s_{(4^a,3^{k-a+j+1},2^{4t-4k-2j+1},1^{5k+j-1-4t-a})}\\
=& \sum_{k=1}^{t} \sum_{j=-1}^{-1} \sum_{a=0}^{5k+j-4t-2} s_{(4^a,3^{k-a+j+1},2^{4t-4k-2j+1},1^{5k+j-1-4t-a})}\\
&+ \sum_{k=1}^{t} \sum_{j=0}^{2t-2k-1} \sum_{a=0}^{5k+j-4t-2} s_{(4^a,3^{k-a+j+1},2^{4t-4k-2j+1},1^{5k+j-1-4t-a})}
\end{aligned}$$

$$-\sum_{k=0}^{t-1}\sum_{j=0}^{2t-2k-1}\sum_{a=0}^{5k+j-4t-1} s_{(4^a,3^{k-a+j+1},2^{4t-4k-2j+1},1^{5k+j-1-4t-a})}.$$

It follows that

$$N_2(t)-N_1(t)=C_1(t)-C_2(t).$$

Similarly, one can check the other identities. This completes the proof. ∎

It remains to compute $T_9(t+1)-\Delta^{(2,2)}T_9(t)$ and $T_{10}(t+1)-\Delta^{(2,2)}T_{10}(t)$. Let

$$E_1(t)=\sum_{a=0}^{t+1} s_{(4^a,3^{t+1-a},1^{t+1-a})},$$

$$E_2(t)=\sum_{k=0}^{t}\sum_{a=0}^{3k-2t-1} s_{(4^a,3^{2t-k-a+1},2,1^{3k-2t-1-a})},$$

$$E_3(t)=\sum_{k=0}^{t}\sum_{a=0}^{3k-2t-2} s_{(4^a,3^{2t-k-a+2},1^{3k-2t-2-a})},$$

$$E_4(t)=\sum_{a=0}^{t} s_{(4^a,3^{t+1-a},1^{t+1-a})},$$

$$E_5(t)=\sum_{a=0}^{t-1} s_{(4^a,3^{t+1-a},2,1^{t-1-a})},$$

$$E_6(t)=\sum_{k=0}^{t-1}\sum_{a=0}^{3k-2t+1} s_{(4^a,3^{2t-k-a},2,1^{3k+2-2t-a})},$$

$$E_7(t)=\sum_{k=0}^{t-1}\sum_{a=0}^{3k-2t} s_{(4^a,3^{2t+1-k-a},1^{3k+1-2t-a})},$$

$$E_8(t)=\sum_{k=0}^{t-1}\sum_{a=0}^{3k-2t} s_{(4^a,3^{2t-k-a},2,2,1^{3k-2t-a})}$$

$$E_9(t)=\sum_{k=0}^{t-1}\sum_{a=0}^{3k-2t-1} s_{(4^a,3^{2t+1-k-a},2,1^{3k-1-2t-a})}.$$

The following lemma gives the Schur expansions of $T_9(t+1)-\Delta^{(2,2)}T_9(t)$ and $T_{10}(t+1)-\Delta^{(2,2)}T_{10}(t)$.

Lemma 3.8 *For any $t\geq 0$, we have*

$$T_9(t+1)-\Delta^{(2,2)}T_9(t)=E_1(t)+E_2(t)+E_3(t), \tag{3.12}$$

$$T_{10}(t+1) - \Delta^{(2,2)} T_{10}(t) = E_4(t) + E_5(t) + E_6(t) + E_7(t) + E_8(t) + E_9(t). \tag{3.13}$$

Proof. We will present the proof of the identity (3.13), because it is easier to prove (3.12) by using Pieri's rule. Recall that

$$T_{10}(t) = \sum_{k=0}^{t-1} e_k s_{(3^{k+1},2^{2t-2k-2},1)}.$$

We have

$$\begin{aligned} T_{10}(t+1) - \Delta^{(2,2)} T_{10}(t) =& \sum_{k=0}^{t} e_k s_{(3^{k+1},2^{2t-2k},1)} - \Delta^{(2,2)} \left(\sum_{k=0}^{t-1} e_k s_{(3^{k+1},2^{2t-2k-2},1)} \right) \\ =& e_t s_{(3^{t+1},1)} + \sum_{k=0}^{t-1} e_k s_{(3^{k+1},2^{2t-2k},1)} \\ & - \sum_{k=0}^{t-1} \Delta^{(2,2)} \big(e_k s_{(3^{k+1},2^{2t-2k-2},1)} \big). \end{aligned}$$

Applying the dual version of Pieri's rule, we deduce that

$$e_t s_{(3^{t+1},1)} = \sum_{a=0}^{t} s_{(4^a,3^{t+1-a},1^{t+1-a})} + \sum_{a=0}^{t-1} s_{(4^a,3^{t+1-a},2,1^{t-1-a})},$$

and, for $0 \le k \le t-1$,

$$e_k s_{(3^{k+1},2^{2t-2k-2},1)} = \sum_{a,b,c,d} s_{(4^a,3^{k+1-a+b},2^{2t-2k-2-b+c},1^{1-c+d})},$$

where

$$0 \le a \le k+1,\ 0 \le b \le 2t-2k-2,\ 0 \le c \le 1,\ d \ge 0,\ a+b+c+d = k.$$

Therefore,

$$e_k s_{(3^{k+1},2^{2t-2k},1)} - \Delta^{(2,2)} \big(e_k s_{(3^{k+1},2^{2t-2k-2},1)} \big) = \sum_{a,b,c,d} s_{(4^a,3^{k+1-a+b},2^{2t-2k-b+c},1^{1-c+d})},$$

where

$$0 \le a \le k+1,\ 2t-2k-1 \le b \le 2t-2k,\ 0 \le c \le 1,\ d \ge 0,\ a+b+c+d = k.$$

In view of the ranges of b and c, the above sum is divided into four sums:

$$\begin{aligned} & \textstyle\sum_{a=0}^{3k-2t+1} s_{(4^a,3^{2t-k-a},2,1^{3k+2-2t-a})} + \sum_{a=0}^{3k-2t} s_{(4^a,3^{2t+1-k-a},1^{3k+1-2t-a})} \\ & + \textstyle\sum_{a=0}^{3k-2t} s_{(4^a,3^{2t-k-a},2,2,1^{3k-2t-a})} + \sum_{a=0}^{3k-2t-1} s_{(4^a,3^{2t+1-k-a},2,1^{3k-1-2t-a})}. \end{aligned}$$

This completes the proof. ∎

We are now in a position to give a recurrence relation for $L(r)$ by using the above lemmas. Note that it is easy to establish a recurrence relation for $R(r)$. For $t \geq 0$, let

$$\begin{aligned} R_{o,1}(t) &= R(2t+1) - \Delta^{(1,1)} R(2t), \\ R_{e,1}(t) &= R(2t+2) - \Delta^{(1,1)} R(2t+1), \\ L_{o,1}(t) &= L(2t+1) - \Delta^{(1,1)} L(2t), \\ L_{e,1}(t) &= L(2t+2) - \Delta^{(1,1)} L(2t+1), \end{aligned}$$

and

$$\begin{aligned} R_{o,2}(t) &= R_{o,1}(t+1) - \Delta^{(2,2)} R_{o,1}(t), \\ R_{e,2}(t) &= R_{e,1}(t+1) - \Delta^{(2,2)} R_{e,1}(t), \\ L_{o,2}(t) &= L_{o,1}(t+1) - \Delta^{(2,2)} L_{o,1}(t), \\ L_{e,2}(t) &= L_{e,1}(t+1) - \Delta^{(2,2)} L_{e,1}(t). \end{aligned}$$

The following lemma gives the recurrence relations of $L(r)$ and $R(r)$.

Lemma 3.9 *Let* $\mathrm{Par}_{\{3,4\}}(n)$ *denote the set of partitions of* n *with parts* 3 *and* 4*. Then for any* $t \geq 0$ *we have*

$$R_{o,2}(t) = \sum_{\lambda \in \mathrm{Par}_{\{3,4\}}(4t+4)} s_\lambda, \quad R_{e,2}(t) = \sum_{\lambda \in \mathrm{Par}_{\{3,4\}}(4t+6)} s_\lambda, \tag{3.14}$$

$$L_{o,2}(t) = \sum_{\lambda \in \mathrm{Par}_{\{3,4\}}(4t+4)} s_\lambda, \quad L_{e,2}(t) = \sum_{\lambda \in \mathrm{Par}_{\{3,4\}}(4t+6)} s_\lambda. \tag{3.15}$$

Proof. The identities in (3.14) are easy to check. It remains to prove the identities in (3.15). Here we will consider only the identity concerning $L_{o,2}(t)$, since the identity for $L_{e,2}(t)$ can be justified in the same manner.

By Lemma 3.4, we obtain

$$\begin{aligned} L_{o,2}(t) = &\sum_{m=1}^{4} (T_m(t+1) - \Delta^{(2,2)} T_m(t)) - \sum_{m=5}^{8} (T_m(t+1) - \Delta^{(2,2)} T_m(t)) \\ &+ (T_9(t+1) - \Delta^{(2,2)} T_9(t)) - (T_{10}(t+1) - \Delta^{(2,2)} T_{10}(t)). \end{aligned}$$

With the aid of Lemma 3.5, it follows that

$$\begin{aligned} L_{o,2}(t) = &(T_{41}(t) - T_{61}(t)) + (T_{31}(t) - T_{71}(t)) + (T_{11}(t) - T_{81}(t)) \\ &+ (T_{21}(t) - T_{51}(t)) + (T_{32} - T_{72}(t)) + (T_{42}(t) - T_{62}(t)) \end{aligned}$$

$$+ (T_{12}(t) - T_{82}(t)) + (T_{22}(t) - T_{52}(t))$$
$$+ (T_9(t+1) - \Delta^{(2,2)}T_9(t)) - (T_{10}(t+1) - \Delta^{(2,2)}T_{10}(t)).$$

Applying Lemmas 3.6-3.8, we obtain

$$\begin{aligned}
L_{o,2}(t) =&(N_2(t) - N_1(t)) + (N_4(t) - N_3(t)) + (N_6(t) - N_5(t)) \\
&+ (M_2(t) - M_1(t)) + (M_4(t) - M_3(t)) + (M_6(t) - M_5(t)) \\
&+ (E_1(t) + E_2(t) + E_3(t)) \\
&- (E_4(t) + E_5(t) + E_6(t) + E_7(t) + E_8(t) + E_9(t)) \\
=&(C_1(t) - C_2(t) + C_3(t) + C_4(t) + C_5(t)) \\
&+ (D_1(t) - D_2(t) - D_3(t) - D_4(t) + D_5(t) \\
&+ D_6(t) - D_7(t) + D_8(t) - D_9(t)) \\
&+ (E_1(t) + E_2(t) + E_3(t) - E_4(t) - E_5(t) \\
&- E_6(t) - E_7(t) - E_8(t) - E_9(t)) \\
=&[(C_3(t) - E_6(t)) + (E_2(t) - E_9(t)) - E_5(t)] \\
&+ [(C_5(t) - D_9(t)) + (D_5(t) - E_8(t)) + C_4(t)] \\
&+ [(D_6(t) - D_4(t)) + (C_1(t) - D_3(t)) - D_2(t)] \\
&+ [(D_8(t) - D_7(t)) + (D_1(t) - C_2(t))] \\
&+ [(E_1(t) - E_4(t)) + (E_3(t) - E_7(t))] \\
=&0 + 0 + 0 + 0 + \sum_{k=1}^{t+1} s_{(4^{3k-2t-2},3^{4t-4k+4})},
\end{aligned}$$

where the last equality comes from the following relations:

$$\begin{aligned}
C_3(t) - E_6(t) =&0, \\
E_2(t) - E_9(t) =&E_5(t), \\
D_6(t) - D_4(t) =&0, \\
E_1(t) - E_4(t) =&s_{(4^{t+1})}, \\
C_5(t) - D_9(t) =&\sum_{k=1}^{t-1} \sum_{a=\max(0,3k-2t)}^{3k-2t} s_{(4^a,3^{2t-k-a},2^2)} \\
&- \sum_{k=1}^{t-1} \sum_{a=\max(0,5k-4t+1)}^{5k-4t+1} s_{(4^a,3^{k+1-a},2^{4t-4k})},
\end{aligned}$$

$$C_1(t) - D_3(t) = \sum_{a=0}^{t-3} s_{(4^a,3^{t-a},2^3,1^{t-a-2})},$$

$$D_5(t) - E_8(t) = -\sum_{k=0}^{t-1} \sum_{a=\max(0,3k-2t)}^{3k-2t} s_{(4^a,3^{2t-k-a},2^2)},$$

$$D_8(t) - D_7(t) = \sum_{k=1}^{t-1} \sum_{a=\max(0,3k-2t-2)}^{3k-2t-2} s_{(4^a,3^{2t-a-k},2^3)},$$

$$D_1(t) - C_2(t) = -\sum_{k=0}^{t-2} \sum_{a=\max(0,3k-2t+1)}^{3k-2t+1} s_{(4^a,3^{2t-a-k-1},2^3)},$$

$$E_3(t) - E_7(t) = \sum_{k=1}^{t} \sum_{a=\max(0,3k-2t-2)}^{3k-2t-2} s_{(4^a,3^{2t-k-a+2})}.$$

So we have obtained the desired Schur expansion of $L_{o,2}(t)$. This completes the proof. ∎

Based on the above lemma, we obtain the following relations.

Lemma 3.10 *For $t \geq 0$, we have*

$$L_{o,1}(t) = R_{o,1}(t), \quad L_{e,1}(t) = R_{e,1}(t).$$

Proof. We conduct induction on t. We first consider that relation $L_{o,1}(t) = R_{o,1}(t)$. Clearly, the equality holds for $t = 0$ and $t = 1$. Assume that $L_{o,1}(t-1) = R_{o,1}(t-1)$. Note that

$$L_{o,1}(t) = L_{o,2}(t-1) + \Delta^{(2,2)} L_{o,1}(t-1),$$
$$R_{o,1}(t) = R_{o,2}(t-1) + \Delta^{(2,2)} R_{o,1}(t-1).$$

It follows from Lemma 3.9 that $L_{o,2}(t-1) = R_{o,2}(t-1)$. Hence by induction we have $L_{o,1}(t) = R_{o,1}(t)$. Similarly, it can be shown that $L_{e,1}(t) = R_{e,1}(t)$. ∎

We have established the recurrence relations for $L(r)$ and $R(r)$. We now proceed to prove Theorem 3.1 based on these recurrence relations.

Proof of Theorem 3.1. We conduct induction on r. It is easy to verify that the identity holds for $r = 1$. Assume that $L(r) = R(r)$. We proceed to prove that $L(r+1) = R(r+1)$. If $r = 2t$, then

$$L(r+1) = L(2t+1) = \Delta^{(1,1)} L(2t) + L_{o,1}(t),$$
$$R(r+1) = R(2t+1) = \Delta^{(1,1)} R(2t) + R_{o,1}(t).$$

From Lemma 3.10 it follows that $L(r+1) = R(r+1)$. If $r = 2t+1$, then

$$L(r+1) = L(2t+2) = \Delta^{(1,1)} L(2t+1) + L_{e,1}(t),$$
$$R(r+1) = R(2t+2) = \Delta^{(1,1)} R(2t+1) + R_{e,1}(t).$$

By the inductive hypothesis and Lemma 3.10, we also reach the conclusion $L(r+1) = R(r+1)$. This completes the proof. ∎

4 The q-log-convexity

In this section, we aim to prove Theorem 1.1 and Theorem 1.3. The proof of Theorem 1.1 is based on the identity (3.5).

Proof of Theorem 1.1. For any $n \geq 1$, we have $\mathrm{ps}_n^1(e_k) = \binom{n}{k}$. So we have

$$W_n(q) = \sum_{k=0}^{n} \binom{n}{k}^2 q^k = \sum_{k=0}^{n} (\mathrm{ps}_n^1(e_k))^2 q^k.$$

Thus, for any $r \geq 0$, the coefficient of q^r in $W_{n-1}(q)W_{n+1}(q) - (W_n(q))^2$ is given by

$$\sum_{k=0}^{r} \mathrm{ps}_{n-1}^1(e_k)^2 \mathrm{ps}_{n+1}^1(e_{r-k})^2 - \mathrm{ps}_n^1(e_k)^2 \mathrm{ps}_n^1(e_{r-k})^2.$$

By (2.4), the above sum equals

$$\mathrm{ps}_{n-1}^1 \left(\sum_{k=0}^{r} {e_k}^2 (e_{r-k} + 2e_{r-k-1} + e_{r-k-2})^2 - (e_k + e_{k-1})^2 (e_{r-k} + e_{r-k-1})^2 \right).$$

To evaluate the above sum, we first expand the squares, and then apply the following relations

$$\sum_{k=0}^{r} e_k^2 e_{r-k-2}^2 = \sum_{k=0}^{r} e_{k-1}^2 e_{r-k-1}^2,$$
$$\sum_{k=0}^{r} e_{k-1}^2 e_{r-k}^2 = \sum_{k=0}^{r} e_k^2 e_{r-k-1}^2,$$
$$\sum_{k=0}^{r} e_k^2 e_{r-k} e_{r-k-1} = \sum_{k=0}^{r} e_{r-k}^2 e_k e_{k-1},$$
$$\sum_{k=0}^{r} e_k^2 e_{r-k} e_{r-k-2} = \sum_{k=0}^{r} e_{r-k}^2 e_k e_{k-2},$$

$$\sum_{k=0}^{r} e_{k-1}^2 e_{r-k} e_{r-k-1} = \sum_{k=0}^{r} e_{r-k-1}^2 e_k e_{k-1} = \sum_{k=0}^{r} e_k^2 e_{r-k-1} e_{r-k-2}.$$

After simplification we obtain the following expression

$$2\,\mathrm{ps}_{n-1}^1 \left(\sum_{k=0}^{r} e_{k-1}^2 e_{r-k}^2 + e_{k-2} e_k e_{r-k}^2 - 2 e_{k-1} e_k e_{r-k-1} e_{r-k} \right).$$

By (3.5), we see that the sum in the above expression is Schur positive. This implies that the polynomials $W_n(q)$ form a q-log-convex sequence, completing the proof. ∎

To prove Theorem 1.3, we introduce the following polynomials. For any $n \geq 1$ and $0 \leq r \leq 2n$, define

$$\begin{aligned} f_1(x) &= (n+1)^2(n-x+1)^2(n-x)^2, \\ f_2(x) &= (n+1)^2(n-(r-x)+1)^2(n-(r-x))^2, \\ f_3(x) &= n^2(n-x+1)^2(n-(r-x)+1)^2. \end{aligned}$$

Set

$$f(x) = f_1(x) + f_2(x) - 2f_3(x).$$

Proof of Theorem 1.3. It suffices to show that the polynomials $W_n(q)$ satisfy the conditions in Theorem 1.2. Clearly, for any n and r, if $k \leq r-n-1$, then $n \leq (r-k)-1$ and $\alpha(n,r,k)=0$. We only need to determine the sign of $\alpha(n,r,k)$ for $r-n-1 < k \leq \lfloor \frac{r}{2} \rfloor$. It is easy to see that $\alpha(n,r,k)$ can be rewritten as

$$\alpha(n,r,k) = \frac{1}{n^2(n-k+1)^2(n-r+k+1)^2} \binom{n}{k}^2 \binom{n}{r-k}^2 f(k). \qquad (4.16)$$

Let us consider the value of $f(k)$ for given r. Taking the derivative of $f(x)$ with respect to x, we obtain that

$$f'(x) = 2(2x-r)g(x),$$

where

$$\begin{aligned} g(x) = 2x^2 + 4nx^2 - 4nrx - 2rx + 4nr^2 - 17n^2r + 2n^2r^2 - 12nr \\ - 8n^3r + 1 + 8n + 21n^2 + 8n^4 + 22n^3 + 2r^2 - 3r. \end{aligned}$$

Differentiating with respect to x, we find that

$$g'(x) = 2(2x-r)(1+2n).$$

This yields that $g'(x) \leq 0$ for $x \leq \frac{r}{2}$. Thus $g(x)$ is decreasing on the interval $(-\infty, \frac{r}{2}]$. Since $g(x) \to +\infty$ when $x \to -\infty$, the polynomial $g(x)$ has at most one real root on the interval $(-\infty, \frac{r}{2}]$, say x_0, if it exists. Consequently, $f'(x)$ either has a unique root $\frac{r}{2}$ on the interval $(-\infty, \frac{r}{2}]$, or has two real roots x_0 and $\frac{r}{2}$. In the former case, $f(x)$ is decreasing on the interval $(-\infty, \frac{r}{2}]$. In the latter case, $f(x)$ is decreasing on the interval $(-\infty, x_0]$ and increasing on the interval $[x_0, \frac{r}{2}]$. Combining the two cases, it suffices to show that $f(r/2) \leq 0$. Observe that

$$\begin{aligned} f(r/2) &= 2(n+1)^2(n-\frac{r}{2}+1)^2(n-\frac{r}{2})^2 - 2n^2(n-\frac{r}{2}+1)^4 \\ &= -\frac{r(2n(2n-r)+(2n-r)+2n)(2+2n-r)^2}{8}, \end{aligned}$$

which is nonpositive for $0 \leq r \leq 2n$. By (4.16), there exists an integer $k' = k'(n,r)$ such that $\alpha(n,r,k) \geq 0$ for $k \leq k'$ and $\alpha(n,r,k) \leq 0$ for $k > k'$. Therefore, by Theorem 1.1 and the conditions of Liu and Wang, we deduce that the linear transformation defined by the triangular array $\{\binom{n}{k}^2\}_{0 \leq k \leq n}$ preserves q-log-convexity. This completes the proof. ∎

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