A short proof of the Zeilberger-Bressou q-Dyson theorem

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The Outline of This Talk

This is a joint work with Ira, M. Gessel (2004)

- The history of the Zeilberger-Bressoud q-Dyson theorem
- About the Dyson's conjecture (the $q = 1$ case)
- The proof of Zeilberger-Bressoud q-Dyson theorem
  - · The outline of the proof
  - · A little about iterated Laurent series
  - · Proof by an example.
- Summarize.

Andrews' q-Dyson conjecture (1975)

Theorem (Zeilberger-Bressoud (1985)). *For any nonnegative integers* $a_0, a_1, \ldots, a_n$,

$$\mathop{\mathrm{CT}}_{\mathbf{x}} \prod_{0\le i<j\le n} \left(\frac{x_i}{x_j}\right)_{a_i} \left(\frac{qx_j}{x_i}\right)_{a_j} = \frac{(q)_{a_0+\cdots+a_n}}{(q)_{a_0}\cdots(q)_{a_n}}, \tag{1}$$

where $(z)_m = (1-z)(1-qz)\cdots(1-q^{m-1}z)$.

Andrews's q-Dyson conjecture attracted much interest, but was not proved until 1985, combinatorially by Zeilberger and Bressoud. Other related work are done by Stanley (1984), Kadell (1985), Bressoud and Goulden (1985), Stembridge (1988), Cherednik (1995).

About Dyson's Conjecture

Dyson's Conjecture(1962) For nonnegative integers $a_0, a_1, \ldots, a_n$,

$$\operatorname*{CT}_{\mathbf{x}} \prod_{0 \le i \ne j \le n} \left(1 - \frac{x_i}{x_j}\right)^{a_j} = \frac{(a_0 + a_1 + \cdots a_n)!}{a_0!\, a_1! \cdots a_n!}. \tag{2}$$

Dyson's conjecture was quickly proved by Wilson and independently by Gunson. An elegant recursive proof was published by Good (1970). None of these proofs generalizes to Andrews q-Dyson conjecture.

Gessel's idea for Dyson's conjecture

Set $x_0 = 1$. For example, in the 3-variable case, we need to show

$$\operatorname*{CT}_{x,y} \frac{(1-x)^{a+b}(x-y)^{b+c}(1-y)^{a+c}}{x^{2b}y^{2c}} = \pm\binom{b+c}{b}\binom{a+b+c}{b+c}.$$

· Fix b and c. Both sides define a polynomial in a of degree $b+c$.

· L.H.S. equals the R.H.S. when $c = 0$ by induction.

· The R.H.S. varnishes for $c = -1, -2, \dots, -a-b$.

· It suffices to show the L.H.S. varnishes for $c = -1, -2, \dots, -a-b$.

Continue of Dyson's conjecture

Write $h = -a$. We need to show for $h = 1, 2, \ldots, b+c$

$$\operatorname*{CT}_{x,y} \frac{(x-y)^{b+c}}{x^{2b}y^{2c}(1-x)^{h-b}(1-y)^{h-c}} = 0. \text{ Equivalently}$$

$$[x^{2b}y^{2c}] \sum_{i,j\geq 0, i+j=b+c} \pm(1-x)^{i+b-h}(1-y)^{j+c-h} = 0$$

The contribution of the displayed summand is 0 if

i) $0 \leq i+b-h < 2b$ or ii) $0 \leq j+c-h < 2c$ or both hold.

The contribution of every summand is 0. Otherwise, if we have a pair (i,j) such that (for instance) $i \geq h+b$ and $j < h-c \Leftrightarrow -j \geq c-h+1$, then $i-j \geq b+c+1$. $\square$

The above idea works for all n. 1/2

$n+1$-variable case: $h=-a_0$, $x_0=1$. Show for $1 \le h \le \sum_{i=1}^{n} a_i$

$$[x_1^{na_1} \cdots x_n^{na_n}] \prod_{0 \le i < j \le n} \pm (x_i - x_j)^{a_i+a_j} = 0.$$

Using $(x_i - x_j)^{a_i+a_j} = ((1-x_j)-(1-x_i))^{a_i+a_j}$, we need to show:

There is no nonnegative integral matrix

$$\begin{pmatrix} a_1 & \alpha_{1,2} & \cdots & \alpha_{1,n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n,1} & \alpha_{n,2} & \cdots & a_n \end{pmatrix}$$

such that $\alpha_{i,j} + \alpha_{j,i} = a_i + a_j$, and there is no i with $0 \le \sum_j \alpha_{i,j} - h < na_i$.

The above idea works for all n. 2/2

Proof by contradiction: we can assume $\sum_j \alpha_{i,j} < h$ for $i = 1, 2, \ldots, k$ and $\sum_j \alpha_{i,j} \geq na_i + h$ for $i = k+1, \ldots, n$. Partition the matrix as $\begin{pmatrix} A^{k \times k} & B^{k \times (n-k)} \\ C^{(n-k) \times k} & D^{(n-k) \times (n-k)} \end{pmatrix}$. We have

$$sum(A) + sum(B) < kh \Leftrightarrow sum(B) < k(h - (a_1 + \cdots + a_k))$$

and

$$sum(C) + sum(D) \geq n(a_{k+1} + \cdots + a_n) + (n-k)h \Leftrightarrow$$
$$-sum(B) \geq (n-k)(h - (a_1 + \cdots + a_k)).$$

□

Laurent series proof of Zeilberger-Bressoud q-Dyson's theorem

· Two polynomials of degree at most d are equal *iff* they are equal at $d+1$ points.

· Fix nonnegative integers $a_1, \ldots, a_n$, and let $d = a_1 + \cdots + a_n$. The R.H.S. of (1)

$$\frac{(q)_{a_0+\cdots+a_n}}{(q)_{a_0}\cdots(q)_{a_n}} = \frac{(1-q^{a_0+1})(1-q^{a_0+2})\cdots(1-q^{a_0+d})}{(q)_{a_1}\cdots(q)_{a_n}}$$

is in fact a polynomial in q^{a_0} of degree at most d vanishing at d points: $a_0 = -1, -2, \ldots, -d$.

· With the following two lemmas, the Z.B. theorem follows straightforwardly using induction.

Main Lemmas

Fix $\mathbf{a} = (a_1, \ldots, a_n)$, $d = a_1 + \cdots + a_n$, let

$$Q_{\mathbf{a}}(q^b) = \operatorname*{CT}_{\mathbf{x}} \prod_{j=1}^{n} \left(\frac{x_0}{x_j}\right)_b \left(\frac{x_j}{x_0} q\right)_{a_j} \prod_{1 \le i < j \le n} \left(\frac{x_i}{x_j}\right)_{a_i} \left(\frac{x_j}{x_i} q\right)_{a_j}. \qquad (3)$$

· Note that $Q_{\mathbf{a}}(q^b)$ is the L.H.S. of (1) when evaluated at $b = a_0$, and is well defined for all b when regarded as a power series in x_0.

Lemma . *For fixed* $\mathbf{a} \in \mathbb{N}^n$, $Q_{\mathbf{a}}(q^b)$ *is a polynomial in* q^b *of degree at most* d.

Lemma (Main Lemma)**.** *For any fixed* $\mathbf{a} \in \mathbb{N}^n$, $Q_{\mathbf{a}}(q^b) = 0$ *for* $q^b = q^{-1}, q^{-2}, \ldots, q^{-d}$.

Basics of q-series

$$(z)_p = (1-z)(1-zq)\cdots(1-zq^{p-1}), \forall p \in \mathbb{N}$$

$$(z)_{-p} = \frac{1}{(1-zq^{-1})(1-zq^{-2})\cdots(1-zq^{-p})}.$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-m+1})}{(1-q)(1-q^2)\cdots(1-q^m)}, \forall n \in \mathbb{Z}, m \in \mathbb{N}$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{(q)_n}{(q)_m (q)_{n-m}}, \forall 0 \le m \le n.$$

The q-binomial theorem: $(u)_n = \sum_{k\ge 0} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} (-u)^k.$

An identity: $\left(\frac{x_i}{x_j}\right)_l \left(\frac{x_j}{x_i} q\right)_m = q^{\binom{m+1}{2}} \left(-\frac{x_j}{x_i}\right)^m \left(\frac{x_i}{x_j} q^{-m}\right)_{l+m}.$

The Proof of the Main Lemma

· We apply the theory of iterated Laurent series to prove the main lemma, working in the field
$\mathbb{C}(q)\langle\langle x_n, \ldots, x_0 \rangle\rangle \equiv \mathbb{C}(q)((x_n))\cdots((x_0))$.

· We extensively use the commutativity

$$\operatorname*{CT}_{x_i} \operatorname*{CT}_{x_j} F(\mathbf{x}) = \operatorname*{CT}_{x_j} \operatorname*{CT}_{x_i} F(\mathbf{x})$$

★ We successively apply to $Q_A(q^b)$, $-d \le b \le -1$, the operators CT_{x_0}, followed by $\mathrm{CT}_{x_{r_1}}$, followed by $\mathrm{CT}_{x_{r_2}}$, and so on. This is being done in a certain order $r_1, r_2, \ldots$, so that we are always taking the constant term of a proper rational function, using partial fraction decomposition.

Lemma from partial fraction decomposition

· We call x_i/x_jq^k small if $i < j$, and large if $i > j$.

· $\underset{x_i}{\mathrm{CT}} \dfrac{1}{1 - x_i/x_jq^k} = 1$ if x_i/x_jq^k small, and 0 if x_i/x_jq^k large.

Lemma(Partial fraction decomposition) Let

$$R = \frac{p(x_k)}{x_k^d \prod_{i=1}^m (1 - x_k/\alpha_i)},$$

be a proper rational function of x_k and the α_i are distinct monomials, each of the form x_tq^s. Then

$$\underset{x_k}{\mathrm{CT}}\, R = \sum_j \big(R\,(1 - x_k/\alpha_j)\big)\Big|_{x_k=\alpha_j},$$

where the sum ranges over all j such that x_k/α_j is small.

An Example For The Main Lemma: Settings

Consider the simplest nontrivial case

$n = 2, a_1 = 2, a_2 = 3, b = -4, d = a_1 + a_2 = 5,$

$$Q = \frac{\left(\frac{x_1}{x_0} q\right)_2 \left(\frac{x_2}{x_0} q\right)_3 \left(\frac{x_1}{x_2}\right)_2 \left(\frac{x_2}{x_1} q\right)_3}{\left(1 - \frac{x_0}{x_1 q}\right) \cdots \left(1 - \frac{x_0}{x_1 q^4}\right) \cdot \left(1 - \frac{x_0}{x_2 q}\right) \cdots \left(1 - \frac{x_0}{x_2 q^4}\right)}$$

Want to show: $\mathrm{CT}_{x_0,x_1,x_2}\, Q = 0.$

· We apply the partial fraction decomposition lemma to Q with respect to x_0 to obtain a sum of 8 terms.

An Example For The Main Lemma: 1

· Step 1: take CT_{x_0} on Q:

$$\mathop{\mathrm{CT}}_{x_0} Q = \sum_{i=1}^{2} \sum_{k=1}^{4} \left[Q \cdot \left(1 - \frac{x_0}{x_i q^k} \right) \right] \Bigg|_{x_0 = x_i q^k}$$

· For each of the 8 terms, we continue to take constant term in another carefully chosen variables.

An Example For The Main Lemma: 2

· Step 2: we look at the first and the last term and another typical term in details.

The first term

$$Q_1 = \frac{\left(\frac{x_1}{x_1 q} q\right)_2 \left(\frac{x_2}{x_1 q} q\right)_3 \left(\frac{x_1}{x_2}\right)_2 \left(\frac{x_2}{x_1} q\right)_3}{\left(1 - \frac{x_1 q}{x_1 q^2}\right)\left(1 - \frac{x_1 q}{x_1 q^3}\right)\left(1 - \frac{x_1 q}{x_1 q^4}\right) \cdot \left(1 - \frac{x_1 q}{x_2 q}\right) \cdots \left(1 - \frac{x_1 q}{x_2 q^4}\right)}$$

$$\left(\frac{x_1}{x_1 q} q\right)_2 = (1-1)(1-q) = 0 \Rightarrow Q_1 = 0$$

An Example For The Main Lemma: 3

The last term

$$Q_8 = \frac{\left(\frac{x_1 q}{x_2 q^4}\right)_2 \left(\frac{x_2 q}{x_2 q^4}\right)_3 \left(\frac{x_1}{x_2}\right)_2 \left(\frac{x_2}{x_1} q\right)_3}{\left(1 - \frac{x_2 q^4}{x_1 q}\right) \cdots \left(1 - \frac{x_2 q^4}{x_1\, q^4}\right) \cdot \left(1 - \frac{x_2 q^4}{x_2 q}\right)\left(1 - \frac{x_2 q^4}{x_2 q^2}\right)\left(1 - \frac{x_2 q^4}{x_2 q^3}\right)}$$

The factors of $\left(1 - \frac{x_2 q^4}{x_1 q}\right) \cdots \left(1 - \frac{x_2 q^4}{x_1\, q^4}\right)$ in the denominator has no contribution, and Q_8 is proper in $x_2 \Rightarrow$

$$\operatorname*{CT}_{x_2} Q_8 = 0.$$

An Example For The Main Lemma: 3

The typical term

$$Q_4 = \frac{\left(\frac{x_1}{x_1 q^4} q\right)_2 \left(\frac{x_2}{x_1 q^4} q\right)_3 \left(\frac{x_1}{x_2}\right)_2 \left(\frac{x_2}{x_1} q\right)_3}{\left(1 - \frac{x_1 q^4}{x_1 q}\right)\left(1 - \frac{x_1 q^4}{x_1 q^2}\right)\left(1 - \frac{x_1 q^4}{x_1 q^3}\right) \cdot \left(1 - \frac{x_1 q^4}{x_2 q}\right) \cdots \left(1 - \frac{x_1 q^4}{x_2 q^4}\right)}$$

This is a proper rational function in x_1 and we take constant term in x_1. The result will be 4 terms, all varnishes. Our key argument is a tournament argument in the next slide.

The Tournament Argument

Lemma by tournament: Let $A_1, \ldots, A_s$ be nonnegative integers. Then for any positive integers $k_1, \ldots, k_s$ with $1 \le k_i \le A_1 + \cdots + A_s$ for all i, either $1 \le k_i \le A_i$ for some i or $-A_j \le k_i - k_j \le A_i - 1$ for some $i < j$.

The idea: Proof by contradiction. If the lemma is not true, then there is a $k_1, \ldots, k_s$ such that either $k_i - k_j \ge A_i$ or $k_i - k_j \le -A_j - 1$.

We then draw arcs for all $i < j$, if $k_i - k_j \ge A_i$ then draw $i \xleftarrow{A_i} j$; if $k_i - k_j \le -1 - A_j$ then draw $i \xrightarrow{A_j+1} j$.

Now it will be an easy exercise to show a contradiction. $\square$

Summarize the Proof of the ZB-Theorem

· A polynomial of degree d is uniquely determined by its $d+1$ values: constant of polynomial $\rightarrow$ constant term of rational function.

· The theory of iterated Laurent series: gives us a kind of freedom in evaluating constant terms. (Wait for the next slide).

· Partial fraction decomposition of proper rational functions. This applies to solve a statistical problem from Persi Diaconis reducing to extracting coefficient of a product of polynomials (joint with Persi Diaconis).

· Tournament argument.

Iterated Laurent Series Is Used To . . .

· Prove Andrews' q-Dyson's conjecture (joint work with Ira Gessel).

· Give a new approach on MacMahon's partition analysis, which is related to counting lattice points in convex polytopes.

  · Find a formula to generate all magic squares of order 3.

· Evaluate combinatorial sums (similar to Egorychev's work).

The generalization to Malcev-Neumann series is used to

· Prove a generalization of Stanley's Monster Reciprocity Theorem (joint work with Richard Stanley).

Food for Thought

Does this method works for Macdonald's constant term conjecture for other root systems? Probably not, but need to try.

Macdonald's constant term conjecture (1982): Define

$$\Delta(q,k) = \prod_{\alpha \in R(+)} \prod_{i=1}^{k} (1 - q^{i-1}\mathrm{e}^{-\alpha})(1 - q^{i}\mathrm{e}^{\alpha}).$$

Then

$$\mathrm{CT}(\Delta(q,k)) = \prod_{j=1}^{n} \begin{bmatrix} d_j k \\ k \end{bmatrix}$$

for all $k \in \mathbb{N}$. Here $d_1, d_2, \cdots, d_n$ are certain positive integers associated with the Weyl group W, the so-called primitive degrees.

Thanks!

Thanks!

Thanks!

Thanks!