On MacMahon's Partition Analysis

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The Outline of This Talk

- What is MacMahon's partition analysis?

  Previous work of Andrews et al.: the Omega package.

- A new approach to MacMahon's partition analysis.

  The field of iterated Laurent series.

- Magic squares (if we have time).

MacMahon's Partition Analysis

· Goal: to find solutions to a system of linear Diophantine equations and inequalities, and lattice points in convex polytopes, and ....

· Idea: to introduce new variables $\lambda_1, \lambda_2, \ldots$ to replace the linear constraints, so that the problem is converted into constant term evaluations.

· Example: How many magic squares of order 3 are there?

· History: It has been given a new life by Andrews et. al. (2001) in a series of papers.

MacMahon's Ω Operators

Definition .

$$\underset{\geq}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\ldots,s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := \sum_{s_1=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} A_{s_1,\ldots,s_r},$$

$$\underset{=}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\ldots,s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := A_{0,\ldots,0}.$$

Everything is treated analytically. The method relies on the unique Laurent series representations of rational functions.

Guo-niu Han (2003) used a formal treatment: everything is in the ring $K[\Lambda, \Lambda^{-1}][[\mathbf{x}]]$, where Λ^{-1} is short for $(\lambda_1^{-1}, \ldots, \lambda_r^{-1})$, and $\mathbf{x}$ is short for $(x_1, \ldots, x_n)$.

An Example

Find all nonnegative integral solutions to $a_1 + a_2 - a_3 \geq 0$.
Construct the generating function:

$$\sum_{\substack{a_1, a_2, a_3 \geq 0 \\ a_1 + a_2 - a_3 \geq 0}} x^{a_1} y^{a_2} z^{a_3}$$

· Introduce a new variable λ:

$$= \sum_{a_1, a_2, a_3 \geq 0} \Omega_{\geq} \lambda^{a_1 + a_2 - a_3} x^{a_1} y^{a_2} z^{a_3}$$

$$= \Omega_{\geq} \sum_{a_1, a_2, a_3 \geq 0} \lambda^{a_1 + a_2 - a_3} x^{a_1} y^{a_2} z^{a_3}$$

$$= \Omega_{\geq} \sum_{a_1 \geq 0} \lambda^{a_1} x^{a_1} \cdot \sum_{a_2 \geq 0} \lambda^{a_2} y^{a_2} \cdot \sum_{a_3 \geq 0} \lambda^{-a_3} z^{a_3}$$

An Example (continue)

· Apply the formula for the sum of a geometric series:

$$= \underset{\geq}{\Omega} \frac{1}{(1-\lambda x)(1-\lambda y)(1-z/\lambda)},$$

· Eliminate λ by MacMahon's many formulas (will be explained later) to obtain

$$= \frac{1-xyz}{(1-x)(1-y)(1-xz)(1-yz)}$$

If we set $x = t, y = t, z = t$, we will get the generating function for the number of the above solutions such that $a_1 + a_2 + a_3 = n$: $\frac{1-t^3}{(1-t)^2(1+t)^2}$

(To be continued)

Elliott's reduction procedure

Base case: If $f(\lambda)$ contains only nonnegative powers in λ, then $\Omega_{\geq} f(\lambda) = f(1)$.

General case: Reduce to the base case by using Elliott Reduction Identity:

$$\frac{1}{(1-x\lambda^j)(1-y\lambda^{-k})} = \frac{1}{1-xy\lambda^{j-k}}\left(\frac{1}{1-x\lambda^j} + \frac{1}{1-y\lambda^{-k}} - 1\right).$$

In particular, we are going to use

$$\frac{1}{(1-x\lambda)(1-y\lambda^{-1})} = \frac{1}{1-xy}\left(\frac{1}{1-x\lambda} + \frac{1}{1-y\lambda^{-1}} - 1\right).$$

An Example

$$F = \frac{1}{(1 - x\lambda)(1 - y\lambda)(1 - z/\lambda)}$$

$$= \frac{1}{(1 - xz)(1 - y\lambda)} \left(\frac{1}{1 - x\lambda} + \frac{1}{1 - z/\lambda} - 1 \right);$$

$$\frac{1}{1 - y\lambda} \cdot \frac{1}{1 - z/\lambda} = \frac{1}{(1 - yz)} \left(\frac{1}{1 - y\lambda} + \frac{1}{1 - z/\lambda} - 1 \right)$$

After simplifying, we get

$$\underset{\geq}{\Omega}\, F = \frac{1 - xyz}{(1 - x)(1 - y)(1 - xz)(1 - yz)}$$

(End of the example)

Multivariate Case 1/2

- Replace r linear constraints with $\lambda_1, \lambda_2, \ldots, \lambda_r$.
- Convert the counting problem to

$$\underset{\geq}{\Omega} F(\lambda_1, \ldots, \lambda_r, x_1, \ldots, x_n),$$

where F will be an Elliott-rational function:

$$F = \frac{\text{polynomial in } \lambda_1, \ldots, \lambda_r}{\prod_i (a_i - b_i)},$$

where a_i and b_i are monomials (might be 0).

Multivariate Case 2/2

· Reduce r-variate case to 1-variable case by iteration, using:

Theorem (Elliott). *If F is Elliott-rational, then the constant terms of F in a field of iterated Laurent series are still Elliott-rational.*

Recall Elliott reduction identity

$$\frac{1}{(1-x\lambda^{j})(1-y\lambda^{-k})}=\frac{1}{1-xy\lambda^{j-k}}\left(\frac{1}{1-x\lambda^{j}}+\frac{1}{1-y\lambda^{-k}}-1\right).$$

Andrews et. al.'s Work on 1-variable Case

George Andrews observed that MacMahon's idea can be implemented by computer: the Omega package.

He and his coauthors, Peter Paule, Axel Riese, and Volker Strehl, have written 9 papers with many applications of MacMahon's partition analysis.

One approach:

$$\underset{\geq}{\Omega} \frac{\lambda^a}{\prod_{1\leq i\leq n}(1-\lambda x_i)\prod_{1\leq i\leq m}(1-y_i/\lambda)} = \frac{P_{n,m,a}(x_1,\ldots,x_n;y_1,\ldots,y_n)}{\prod_{i=1}^n(1-x_i)\cdot\prod_{i=1}^n\prod_{j=1}^m(1-x_iy_j)},$$

where $P_{n,m,a}(x_1,\ldots,x_n;y_1,\ldots,y_n)$ is determined by a recurrence.

Andrews et. al.'s Work on r-variable Case

Two technique difficulties about the above approach: The problem about *run-time explosion*, and the problem about the roots of unity.

Another approach uses reduction by the formula:

$$\frac{1}{(1-x\lambda^j)(1-y\lambda^k)} = \frac{1}{y^j - x^k}\left(\frac{\sum_{i=0}^{j-1}\alpha_i\lambda^i}{1-x\lambda^j} + \frac{\sum_{i=0}^{k-1}\beta_i\lambda^i}{1-y\lambda^k}\right),$$

where

$$\alpha_i = \begin{cases} -x^k y^{i/k}, & \text{if } k|i \text{ or } i=0 \\ -y^{\cdots}x^{\cdots}, & \text{otherwise} \end{cases},$$

and β_i is similar as α_i. (End of Introduction)

How I started on this subject ⋯

My advisor Ira Gessel taught me the following technique:

$$F = \frac{\lambda}{(1-x\lambda)(1-y\lambda)(\lambda-z)}$$

$$= \frac{1/x}{(1-x\lambda)(1-y/x)(1-zx)} + \frac{1/y}{(1-y\lambda)(1-x/y)(1-zy)} + \frac{z}{(\lambda-z)(1-xy)(1-xz)}$$

$$\frac{1}{\lambda-z} = \frac{1}{\lambda}\frac{1}{1-z/\lambda} = \sum_{n\geq 1} z^{n-1}/\lambda^n.$$

When can we apply this technique?

This technique gives the constant term immediately, while the problems are,

1. Does the PFD always exist? — this requires the coefficients to lie in a field.

2. The series expansion of a rational function is not always clear. E.g., what is the series expansion of $\frac{1}{\lambda x - y}$? — my treatment to this problem is to define a total ordering on the variables.

These motivated me to define the field of **iterated Laurent series**.

Definition of Iterated Laurent Series

◇ A Laurent series over a field K is of the form $\sum_{n\geq N_0} c_n x^n$, where N_0 is an integer, possibly negative, and $c_n \in K$.

◇ The field $K\langle\langle x_1, x_2, \ldots, x_n\rangle\rangle$ of ***iterated Laurent series*** is inductively defined as $K\langle\langle x_1, \ldots x_{n-1}\rangle\rangle((x_n))$, with $K\langle\langle x_1\rangle\rangle = K((x_1))$.

◇ An element in $K\langle\langle x_1, x_2, \ldots, x_n\rangle\rangle$ is first treated as a Laurent series in x_n, then a Laurent series in x_{n-1}, and so on.

◇ This means: for any positive integers $i < j$ and k, $x_j = o(x_i^k)$. In other words, x_j is infinitely smaller than x_i, whenever $i < j$.

◇ In our settings, the expansion of $\dfrac{1}{x_1 x_2 - x_4^3}$ is clear.

Iterated Laurent Series Was Mentioned by . . .

⋇ Physicists, e.g., Wilson, 1962, used the notation

$$1 \gg x_1 \gg \cdots \gg x_n$$

to do integration in complex analysis.

⋇ Stanley, 1974, used this approach in proving his well-known Combinatorial Reciprocity Theorems.

A Natural Definition of the Constant Term

Definition (Natural Definition).

$$\operatorname*{CT}_{x_j} \sum_{(i_1,\dots,i_n)\in\mathbb{Z}^n} a_{i_1,\dots,i_n} x_1^{i_1}\cdots x_n^{i_n} := \sum_{(i_1,\dots,i_n)\in\mathbb{Z}^n, i_j=0} a_{i_1,\dots,i_n} x_1^{i_1}\cdots x_n^{i_n},$$

where $a_{i_1,\dots,i_n}$ *belongs to* K*.*

The set of all **formal** series does not form a ring. We can not do multiplications in it, so that the application of the constant term is limited.

But as we shall see (in a minute), everything works fine when we are restricted to the field of iterated Laurent series.

The Fundamental Structure of $K\langle\langle x_1, \ldots, x_n \rangle\rangle$

The *support* of a formal series is the set of the powers in its nonzero terms, which is a subset of $\mathbb{Z}^n$. For instance, the support of $2x_1^2 + x_1x_2 + x_2^{-1}$ is $\{ (2,0), (1,1), (0,-1) \}$.

We use the **reverse** lexicographical order on $\mathbb{Z}^n$.

Proposition (Fundamental structure). *A formal series in* $\mathbf{x}$ *belongs to* $K\langle\langle x_1, \ldots, x_n \rangle\rangle$ *if and only if it has a well-ordered support.*

Fact: ★ *Any subset of a well-ordered set is well-ordered* ★

The Orders on the Variables

▼ The idea of iterated Laurent series is to define a total ordering on its variables.

▲ We can consider $K\langle\langle x_{\sigma(1)}, \ldots, x_{\sigma(n)}\rangle\rangle$, where σ is a permutation on $\{1, \ldots, n\}$.

⧫ Rational functions may have different expansions in $K\langle\langle x_{\sigma(1)}, \ldots, x_{\sigma(n)}\rangle\rangle$ for different σ. For instance: the expansion of $\dfrac{1}{x+y}$ in $K\langle\langle x, y\rangle\rangle$ is different from $\dfrac{1}{x+y}$ in $K\langle\langle y, x\rangle\rangle$.

New Approach to MacMahon's Partition Analysis

Here is how I apply the theory of iterated Laurent series to MacMahon's Partition Analysis.

- Fix a working field $K\langle\langle \Lambda, \mathbf{x} \rangle\rangle := K\langle\langle \lambda_1, \ldots, \lambda_r, x_1, \ldots, x_n \rangle\rangle$.
- Define a new operator PT_λ to replace $\Omega_\geq$ and $\Omega_=$:

$$\mathop{\mathrm{PT}}_{\lambda} \sum_{m=-\infty}^{\infty} a_m \lambda^m := \sum_{m=0}^{\infty} a_m \lambda^m.$$

MacMahon's operators can be realized as:

$$\mathop{\Omega}_{\geq} F(\Lambda, \mathbf{x}) = \mathop{\mathrm{PT}}_{\Lambda} F(\Lambda, \mathbf{x}) \Big|_{\Lambda=(1,\ldots,1)},$$

$$\mathop{\Omega}_{=} F(\Lambda, \mathbf{x}) = \mathop{\mathrm{CT}}_{\Lambda} F(\Lambda, \mathbf{x}) = \mathop{\mathrm{PT}}_{\Lambda} F(\Lambda, \mathbf{x}) \Big|_{\Lambda=(0,\ldots,0)}.$$

The One Variable Case

- We reduce the problem into evaluating $\mathrm{PT}_\lambda\, F(\lambda)$ with

$$F(\lambda) = \frac{\text{polynomial}(\lambda)}{\prod_{1\le i\le n}(\lambda^{j_i} - z_i)},$$

where $j_i \in \mathbb{P}$, and z_i's are independent of λ. **N**ote that z_i's are allowed to be 0.

- Find the PFD of $F(\lambda)$. Note that the idea of using PFD in this context was first used by Stanley (1974), but was thought to be impractical without using a computer.

Application of PFD

Constant term can be read off as soon as we find the PFD.

Theorem . *Suppose that the factors in the denominator of F are pairwise relatively prime, and that the PFD of F is*

$$F = f(\lambda) + \sum_{1 \le s \le n} \frac{p_s(\lambda)}{\lambda^{j_s} - z_s},$$

where $f(\lambda)$ is a polynomial in λ, and $p_s(\lambda)$ is a polynomial of degree less than j_s for each s. Then

$$\operatorname*{PT}_{\lambda} F = f(\lambda) + \sum_{s} \frac{p_s(\lambda)}{\lambda^{j_s} - z_s},$$

where the sum ranges over all s such that $\lambda^{j_s} = o(z_s)$.

Proof by Example

Example: Consider

$$F(\lambda) = \frac{\lambda^2}{(1-\lambda^2 x_1)(1-\lambda^{-3}x_2)} = \frac{\lambda^5}{(1-\lambda^2 x_1)(\lambda^3 - x_2)}$$

$$= \frac{-1}{x_1} + \frac{\frac{1+{x_1}^2 x_2 \lambda}{x_1\left(1-{x_1}^3{x_2}^2\right)}}{(1-\lambda^2 x_1)} + \frac{\frac{x_2}{1-x_1^3x_2^2}\left({x_1}^2{x_2}^2 + x_1 x_2 \lambda + \lambda^2\right)}{\lambda^3\left(1-\frac{x_2}{\lambda^3}\right)}$$

Thus $$\operatorname*{CT}_{\lambda} F(\lambda) = -\frac{1}{x_1} + \frac{1}{x_1\left(1-{x_1}^3{x_2}^2\right)} = \frac{x_1^2x_2^2}{1-x_1^3x_2^2}$$

Finding the PFD: a Formula for $p_s(\lambda)$

Let

$$\mathcal{F}(\lambda^j - a, \lambda^k - b) = \frac{\sum_{i=0}^{j'-1} \lambda^{ik} b^{j'-1-i}}{a^{k'} - b^{j'}},$$

where $j' = j/\gcd(j,k)$ and $k' = k/\gcd(j,k)$.

Theorem . *The polynomial $p_s(\lambda)$, as in the PFD of F, equals the remainder of*

$$P(\lambda) \prod_{i=1, i\neq s}^{n} \mathcal{F}(\lambda^{j_s} - z_s, \lambda^{j_i} - z_i),$$

when divided by $\lambda^{j_s} - z_s$ as a polynomial in λ.

This result is a consequence of a new algorithm for PFD, which we will not discuss here.

Using the Previous Formula is a Fast Approach

· Polynomial multiplication is fast.

· The **remainder** of $p(\lambda)$ when divided by the special kind of polynomial, $(\lambda^j - a)$, is fast. It can be obtained by replacing λ^d with $\lambda^{d \bmod j} a^{\lfloor d/j \rfloor}$ in $p(\lambda)$ for all d.

Example. The remainder of λ^{10} divided by $\lambda^4 - a$ is $a^2\lambda^2$. In this case, $d = 10$ and $j = 4$.

In particular, when $j = 1$, we just replace λ by a.

An Example

$$F(\lambda) = \frac{\lambda^2}{(1-\lambda^2 x_1)(1-\lambda^{-3}x_2)} = \frac{\lambda^5}{-x_1(\lambda^2 - 1/x_1)(\lambda^3 - x_2)}$$

Let $a = 1/x_1$, $b = x_2$.

$$\frac{a^3-b^2}{(\lambda^2-a)(\lambda^3-b)} = \frac{\lambda^6-b^2}{(\lambda^2-a)(\lambda^3-b)} - \frac{\lambda^6-a^3}{(\lambda^2-a)(\lambda^3-b)}$$
$$= \frac{\lambda^3+b}{\lambda^2-a} + \frac{*}{\lambda^3-b}$$

$$\lambda^5 \frac{\lambda^3+b}{\lambda^2-a} = \frac{\lambda^8+b\lambda^5}{\lambda^2-a} = \text{poly} + \frac{a^4+a^2b\lambda}{\lambda^2-a}$$

The Run-Time Explosion Problem is Partially Solved

We show by an example of computing $\Omega_{\geq} F(\Lambda)$, where

$$F(\Lambda) = \frac{1}{((1-\lambda_1\lambda_3 x_1/\lambda_2)(1-\lambda_1\lambda_2 x_2/\lambda_3)(1-\lambda_2\lambda_3 x_3/\lambda_1))}.$$

This example comes from finding all triples (a, b, c) in $\mathbb{N}^3$ such that they satisfy the triangle inequalities.

Step 1. Fix the working field $\mathbb{C}\langle\langle \Lambda, x_3, x_2, x_1 \rangle\rangle$.

Step 2. Eliminate λ_3, we get

$$\Omega_{\geq,\lambda_3} F(\Lambda) = \frac{\lambda_2\lambda_1^2 x_1}{\left(\lambda_1{}^2 x_1 - \lambda_2^2 x_3\right)\left(1-\lambda_1{}^2 x_2 x_1\right)\left(\lambda_1 x_1 - \lambda_2\right)} - \frac{\lambda_1 \lambda_2{}^2 x_3}{\left(\lambda_1{}^2 x_1 - \lambda_2{}^2 x_3\right)\left(1-\lambda_2{}^2 x_2 x_3\right)\left(\lambda_1 - \lambda_2 x_3\right)}$$

······ Continue

Step 3. Eliminate λ_2 for the two summands separately (do NOT combine the two summands):

$$\Omega_{\geq,\lambda_3,\lambda_2} F(\Lambda) = \frac{\lambda_1 (\lambda_1 + x_3) x_2}{\left(-x_3 + {\lambda_1}^2 x_2\right) \left(-1 + {\lambda_1}^2 x_2 x_1\right) (x_2 x_3 - 1)} + \frac{\lambda_1 x_3}{\left(-x_3 + {\lambda_1}^2 x_2\right) (-1 + x_1 x_3) (-x_3 + \lambda_1)}$$

Step 4. Eliminate λ_1 as in step 3, and simplify:

$$\underset{\geq}{\Omega} F(\Lambda) = \frac{1 + x_3 x_2 x_1}{(1 - x_1 x_3)(1 - x_2 x_1)(1 - x_2 x_3)}.$$

(End of the New Approach)

Definition of Magic Squares

A *magic square* of *order* n is an n by n matrix with distinct positive integer entries such that every row sum, column sum, and (two) diagonal sums equals to the same number m, the *magic number*.

A magic square is *pure* if the entries are the consecutive numbers from 1 to n^2, and hence it has magic number $\frac{1}{n}\sum_{i=1}^{n^2} i = \frac{n(n^2+1)}{2}$.

The Yu magic square is given by $\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}$.

Generating All Magic Squares of Order Three

Theorem . *Every magic square of order three, up to rotation and reflection, can be written uniquely as* $iA + jC + kD$*, where* i*,* j*, and* k *are positive integers with* $j \neq k$ *and*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \; D = \begin{bmatrix} 3 & 0 & 3 \\ 2 & 2 & 2 \\ 1 & 4 & 1 \end{bmatrix}.$$