

**On MacMahon's Partition Analysis And
q-Dyson's Conjecture**

Guoce Xin

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University of Florida

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The Outline of This Talk

- MacMahon's partition analysis

Previous work of Andrews et al.: the Omega package.

A new approach to MacMahon's partition analysis based on

- The theory of iterated Laurent series.

- A formal Laurent series proof of q -Dyson's Conjecture.

MacMahon's Partition Analysis

- **Goal:** to find solutions to a system of linear Diophantine equations and inequalities, and lattice points in convex polytopes, and
- **Idea:** to introduce new variables $\lambda_1, \lambda_2, \dots$ to replace the linear constraints, so that the problem is converted into constant term evaluations.
- **History:** It has been given a new life by Andrews et. al. (2001) in a series of papers.

MacMahon's Ω Operators

Definition .

$$\underset{=}{\overset{\Omega}{\geq}} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1, \dots, s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := \sum_{s_1=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} A_{s_1, \dots, s_r},$$

$$\underset{=}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1, \dots, s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := A_{0, \dots, 0}.$$

Everything is treated analytically. The method relies on the **unique** Laurent series representations of rational functions.

An Example

Find all nonnegative integral solutions to $a_1 + a_2 - a_3 = 0$.
Construct the generating function:

$$\sum_{\substack{a_1, a_2, a_3 \geq 0 \\ a_1 + a_2 - a_3 = 0}} x^{a_1} y^{a_2} z^{a_3}$$

• Introduce a new variable λ :

$$\begin{aligned} &= \sum_{a_1, a_2, a_3 \geq 0} \underline{\underline{\Omega}} \lambda^{a_1 + a_2 - a_3} x^{a_1} y^{a_2} z^{a_3} \\ &= \underline{\underline{\Omega}} \sum_{a_1, a_2, a_3 \geq 0} \lambda^{a_1 + a_2 - a_3} x^{a_1} y^{a_2} z^{a_3} \\ &= \underline{\underline{\Omega}} \sum_{a_1 \geq 0} \lambda^{a_1} x^{a_1} \cdot \sum_{a_2 \geq 0} \lambda^{a_2} y^{a_2} \cdot \sum_{a_3 \geq 0} \lambda^{-a_3} z^{a_3} \end{aligned}$$

An Example (continue)

- Apply the formula for the sum of a geometric series:

$$= \Omega \frac{1}{(1 - \lambda x)(1 - \lambda y)(1 - z/\lambda)},$$

- Eliminate λ using **Elliott's Reduction Procedure** to obtain

$$\frac{z(xyz - x - y)}{(1 - xz)(1 - yz)}$$

If we set $x = t, y = t, z = t$, we will get the generating function for the number of the above solutions such that $a_1 + a_2 + a_3 = n$:

$$\frac{t^2(t^2 - 2)}{(1 - t^2)^2}$$

(End of the example)

The Multivariate Case

- Replace r linear constraints with $\lambda_1, \lambda_2, \dots, \lambda_r$. Convert the counting problem to

$$\underline{\Omega} F(\lambda_1, \dots, \lambda_r, x_1, \dots, x_n),$$

where F will be an Elliott-rational function:

$$F = \frac{\text{polynomial in } \lambda_1, \dots, \lambda_r}{\prod_i (a_i - b_i)},$$

where a_i and b_i are monomials (might be 0).

- Reduce r -variate case to 1-variable case by iteration, using

Theorem (Elliott). *If F is Elliott-rational, then the constant terms of F in a field of iterated Laurent series are still Elliott-rational.*

Andrews et. al.'s Work

- **George Andrews** observed that MacMahon's idea can be implemented by computer: the **Omega package**.
- He and his coauthors, **Peter Paule**, **Axel Riese**, and **Volker Strehl**, have written 9 papers with many applications of **MacMahon's partition analysis**.
- Two technical difficulties arise in Omega Calculus: **run-time explosion**, **roots of unity** and **fractional powers**.
- It is desired to find an approach to solve **all** of the above technical problems.

How I started on this subject ...

Ira Gessel told me the following technique of finding $\text{CT}_\lambda F$:

$$\begin{aligned} F &= \frac{\lambda}{(1-x\lambda)(1-y\lambda)(\lambda-z)} \\ &= \frac{1/x}{(1-x\lambda)(1-y/x)(1-zx)} + \frac{1/y}{(1-y\lambda)(1-x/y)(1-zy)} \\ &\quad + \frac{z}{(\lambda-z)(1-xy)(1-xz)} \end{aligned}$$

$$\text{where } \text{CT}_\lambda \frac{1}{1-x\lambda} = 1, \text{CT}_\lambda \frac{1}{1-y\lambda} = 1$$

$$\text{and } \text{CT}_\lambda \frac{1}{\lambda-z} = \text{CT}_\lambda \frac{1}{\lambda} \frac{1}{1-z/\lambda} = \text{CT}_\lambda \sum_{n \geq 1} z^{n-1} / \lambda^n = 0.$$

When can we apply this technique?

This technique gives the constant term immediately, while the problems exist for multivariate case,

1. Does the PFD always exist? — this requires the coefficients to **lie in a field**.
2. The series expansion of a rational function is not always clear. E.g., what is the series expansion of $\frac{1}{\lambda x - y}$? — my treatment to this is to define a **total ordering** on the variables.
 - These motivated me to define the field of **iterated Laurent series**.

The Definition of Iterated Laurent Series

- ◇ The field $K\langle\langle x_1, x_2, \dots, x_n \rangle\rangle$ of **iterated Laurent series** is inductively defined as $K\langle\langle x_1, \dots, x_{n-1} \rangle\rangle((x_n))$, with $K\langle\langle x_1 \rangle\rangle = K((x_1))$.
- ◇ An element in $K\langle\langle x_1, x_2, \dots, x_n \rangle\rangle$ is first treated as a Laurent series in x_n , then a Laurent series in x_{n-1} , and so on.
- ◇ This means: for any positive integers $i < j$ and k , $x_j = o(x_i^k)$. In other words, x_j is **infinitely smaller** than x_i , whenever $i < j$.
- ◇ The expansion of $\frac{1}{x_1 x_2 - x_4^3}$ is clear.

Iterated Laurent Series Was Mentioned by ...

※ Physicists, e.g., [Wilson](#), 1962, used the notation

$$1 \gg x_1 \gg \cdots \gg x_n$$

to do integration in complex analysis.

※ [Stanley](#), 1974, used this approach in proving his [Combinatorial Reciprocity Theorems](#).

The Constant Term and the Fundamental Structure

Definition .

$$\begin{aligned} \text{CT}_{x_j} \sum_{(i_1, \dots, i_n) \in \mathbb{Z}^n} a_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n} \\ := \sum_{(i_1, \dots, i_n) \in \mathbb{Z}^n, i_j = 0} a_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n}, \end{aligned}$$

where a_{i_1, \dots, i_n} belongs to K .

Proposition (Fundamental structure). *A formal series in \mathbf{x} belongs to $K \langle\langle x_1, \dots, x_n \rangle\rangle$ if and only if it has a well-ordered support.*

New Approach to MacMahon's Partition Analysis

I apply the theory of iterated Laurent series to MacMahon's Partition Analysis.

- **Fix** a working field $K \langle\langle \Lambda, \mathbf{x} \rangle\rangle := K \langle\langle \lambda_1, \dots, \lambda_r, x_1, \dots, x_n \rangle\rangle$.
- **Define** a new operator PT_λ to replace **both** Ω_{\geq} and $\Omega_{=}$:

$$\text{PT}_\lambda \sum_{m=-\infty}^{\infty} a_m \lambda^m := \sum_{m=0}^{\infty} a_m \lambda^m.$$

MacMahon's operators can be realized as:

$$\begin{aligned} \Omega_{\geq} F(\Lambda, \mathbf{x}) &= \text{PT}_\Lambda F(\Lambda, \mathbf{x}) \Big|_{\Lambda=(1,\dots,1)}, \\ \Omega_{=} F(\Lambda, \mathbf{x}) &= \text{CT}_\Lambda F(\Lambda, \mathbf{x}) = \text{PT}_\Lambda F(\Lambda, \mathbf{x}) \Big|_{\Lambda=(0,\dots,0)}. \end{aligned}$$

The One Variable Case

- We reduce the problem into evaluating $\text{PT}_\lambda F(\lambda)$ with

$$F(\lambda) = \frac{\text{polynomial}(\lambda)}{\prod_{1 \leq i \leq n} (\lambda^{j_i} - z_i)},$$

where $j_i \in \mathbb{P}$, and z_i 's are independent of λ . Note that z_i 's are allowed to be 0.

- Find the PFD of $F(\lambda)$. Note that the idea of using PFD in this context was first used by Stanley (1974), but was thought to be impractical without using a computer.

Application of the PFD

Constant term can be read off as soon as we find the PFD.

Theorem . *Suppose that the factors in the denominator of F are pairwise relatively prime, and that the PFD of F is*

$$F = f(\lambda) + \sum_{1 \leq s \leq n} \frac{p_s(\lambda)}{\lambda^{j_s} - z_s},$$

where $f(\lambda)$ is a polynomial in λ , and $p_s(\lambda)$ is a polynomial of degree less than j_s for each s . Then

$$\text{PT}_{\lambda} F = f(\lambda) + \sum_s \frac{p_s(\lambda)}{\lambda^{j_s} - z_s},$$

where the sum ranges over all s such that $\lambda^{j_s} = o(z_s)$.

Proof by an Example

Example: Consider

$$\begin{aligned} F(\lambda) &= \frac{\lambda^2}{(1 - \lambda^2 x_1)(1 - \lambda^{-3} x_2)} = \frac{\lambda^5}{(1 - \lambda^2 x_1)(\lambda^3 - x_2)} \\ &= -\frac{1}{x_1} + \frac{\frac{1+x_1^2 x_2 \lambda}{x_1(1-x_1^3 x_2^2)}}{(1 - \lambda^2 x_1)} + \frac{\frac{x_2}{1-x_1^3 x_2^2} (x_1^2 x_2^2 + x_1 x_2 \lambda + \lambda^2)}{\lambda^3 (1 - \frac{x_2}{\lambda^3})} \end{aligned}$$

$$\text{Thus } \text{CT}_{\lambda} F(\lambda) = -\frac{1}{x_1} + \frac{1}{x_1 (1 - x_1^3 x_2^2)} = \frac{x_1^2 x_2^2}{1 - x_1^3 x_2^2}$$

The Multivariate Case

The **run-time explosion** problem is partially solved.

Example: Finding all triples (a, b, c) in \mathbb{N}^3 that satisfy the triangle inequalities is reduced to finding $\Omega_{\geq} F(\Lambda)$, where

$$F(\Lambda) = \frac{1}{(1 - \lambda_1 \lambda_3 x_1 / \lambda_2)(1 - \lambda_1 \lambda_2 x_2 / \lambda_3)(1 - \lambda_2 \lambda_3 x_3 / \lambda_1)}.$$

Step 1. Fix the working field $\mathbb{C}\langle\Lambda, x_3, x_2, x_1\rangle$.

Step 2. Eliminate λ_3 , we get

$$\Omega_{\geq, \lambda_3} F(\Lambda) = \frac{\lambda_2 \lambda_1^2 x_1}{(\lambda_1^2 x_1 - \lambda_2^2 x_3) (1 - \lambda_1^2 x_2 x_1) (\lambda_1 x_1 - \lambda_2)} - \frac{\lambda_1 \lambda_2^2 x_3}{(\lambda_1^2 x_1 - \lambda_2^2 x_3) (1 - \lambda_2^2 x_2 x_3) (\lambda_1 - \lambda_2 x_3)}$$

..... Continue

Step 3. Eliminate λ_2 for the two summands separately (do NOT combine the two summands):

$$\Omega_{\geq, \lambda_3, \lambda_2} F(\Lambda) = \frac{\lambda_1 (\lambda_1 + x_3) x_2}{(-x_3 + \lambda_1^2 x_2) (-1 + \lambda_1^2 x_2 x_1) (x_2 x_3 - 1)} + \frac{\lambda_1 x_3}{(-x_3 + \lambda_1^2 x_2) (-1 + x_1 x_3) (-x_3 + \lambda_1)}$$

Step 4. Eliminate λ_1 as in step 3, and simplify:

$$\Omega_{\geq} F(\Lambda) = \frac{1 + x_3 x_2 x_1}{(1 - x_1 x_3) (1 - x_2 x_1) (1 - x_2 x_3)}.$$

The problem is **divided** into simple pieces, and each piece is treated **separately**.
(End of MacMahon's Partition Analysis)

Andrews' q -Dyson conjecture (1975)

Also known as Zeilberger-Bressoud theorem:

Theorem (Zeilberger-Bressoud (1985)). *For any nonnegative integers a_0, a_1, \dots, a_n ,*

$$\text{CT}_{\mathbf{x}} \prod_{0 \leq i < j \leq n} \left(\frac{x_i}{x_j} \right)_{a_i} \left(\frac{qx_j}{x_i} \right)_{a_j} = \frac{(q)_{a_0 + \dots + a_n}}{(q)_{a_0} \cdots (q)_{a_n}}, \quad (1)$$

where $(z)_m = (1 - z)(1 - qz) \cdots (1 - q^{m-1}z)$.

Andrews's q -Dyson conjecture attracted much interest, but was not proved until 1985, combinatorially by Zeilberger and Bressoud.

Other related work are done by Stanley (1984), Kadell (1985), Bressoud and Goulden (1985), Stembridge (1988), ETC.

A Formal Laurent Series Proof of q-Dyson's Conjecture

This is a joint work with [Ira Gessel](#). The idea goes:

- Two polynomials of degree at most d are equal *iff* they are equal at $d + 1$ points.
- Fix nonnegative integers a_1, \dots, a_n , and let $d = a_1 + \dots + a_n$. The **R.H.S.** of (1)

$$\frac{(q)_{a_0 + \dots + a_n}}{(q)_{a_0} \cdots (q)_{a_n}} = \frac{(1 - q^{a_0+1})(1 - q^{a_0+2}) \cdots (1 - q^{a_0+d})}{(q)_{a_1} \cdots (q)_{a_n}}$$

is in fact a polynomial in q^{a_0} of degree at most d vanishing at d points: $a_0 = -1, -2, \dots, -d$.

- With the following [two lemmas](#), the [Z.B. theorem](#) follows straightforwardly using induction.

Main Lemmas

Fix $A = (a_1, \dots, a_n)$, $d = a_1 + \dots + a_n$, let

$$Q_A(q^b) = \text{CT}_{\mathbf{x}} \prod_{j=1}^n \binom{x_0}{x_j}_b \binom{x_j q}{x_0}_{a_j} \prod_{1 \leq i < j \leq n} \binom{x_i}{x_j}_{a_i} \binom{x_j q}{x_i}_{a_j}. \quad (2)$$

· Note that $Q_A(q^b)$ is the **L.H.S.** of (1) when evaluated at $b = a_0$, and is well defined for all b when regarded as a power series in x_0 .

Lemma . For fixed $A \in \mathbb{N}^n$, $Q_A(q^b)$ is a polynomial in q^b of degree at most d .

Lemma (Main Lemma). For any fixed $A \in \mathbb{N}^n$, $Q_A(q^b) = 0$ for $q^b = q^{-1}, q^{-2}, \dots, q^{-d}$.

The Proof of the Main Lemma

- We apply the theory of **iterated Laurent series** to prove the main lemma, working in the field $\mathbb{C}(q)\langle\langle x_n, \dots, x_0 \rangle\rangle$.
- We extensively use the commutativity

$$\text{CT}_{x_i} \text{CT}_{x_j} F(\mathbf{x}) = \text{CT}_{x_j} \text{CT}_{x_i} F(\mathbf{x})$$

★ We successively apply to $Q_A(q^b)$, $-d \leq b \leq -1$, the operators CT_{x_0} , followed by $\text{CT}_{x_{r_1}}$, followed by $\text{CT}_{x_{r_2}}$, and so on. This is being done in a certain order r_1, r_2, \dots , so that we are always taking the constant term of a **proper** rational function, using partial fraction decomposition.

An Example For The [Main Lemma](#): Settings

Consider the simplest nontrivial case

$$n = 2, a_1 = 2, a_2 = 3, b = -4, d = a_1 + a_2 = 5,$$

$$Q = \frac{\left(\frac{x_1}{x_0}q\right)_2 \left(\frac{x_2}{x_0}q\right)_3 \left(\frac{x_1}{x_2}\right)_2 \left(\frac{x_2}{x_1}q\right)_3}{\left(1 - \frac{x_0}{x_1q}\right) \cdots \left(1 - \frac{x_0}{x_1q^4}\right) \cdot \left(1 - \frac{x_0}{x_2q}\right) \cdots \left(1 - \frac{x_0}{x_2q^4}\right)}$$

Want to show: $\text{CT}_{x_0, x_1, x_2} Q = 0$.

· We do partial fraction decomposition to Q with respect to x_0 to obtain a sum of 8 terms, and then take the constant term in x_0 .

An Example For The Main Lemma: 1

- We call x_i/x_jq^k **small** if $i < j$, and **large** if $i > j$.
- $\text{CT}_{x_i} \frac{1}{1 - x_i/x_jq^k} = 1$ if x_i/x_jq^k small, and 0 if x_i/x_jq^k large.
- For each of the 8 factors of the denominator of Q , only those with x_i/x_jq^k **small** have contributions when taking constant terms in x_i . If all of the factors have no contribution, then we get 0 as the result.
- Step 1: take CT_{x_0} on Q :

$$\text{CT}_{x_0} Q = \sum_{i=1}^2 \sum_{k=1}^4 \left[Q \cdot \left(1 - \frac{x_0}{x_i q^k} \right) \right] \Big|_{x_0 = x_i q^k}$$

An Example For The Main Lemma: 2

- Step 2: we look at the **first** and the **last** term in details.

The **first** term

$$Q_1 = \frac{\left(\frac{x_1}{x_1 q} q\right)_2 \left(\frac{x_2}{x_1 q} q\right)_3 \left(\frac{x_1}{x_2}\right)_2 \left(\frac{x_2}{x_1} q\right)_3}{\left(1 - \frac{x_1 q}{x_1 q^2}\right) \left(1 - \frac{x_1 q}{x_1 q^3}\right) \left(1 - \frac{x_1 q}{x_1 q^4}\right) \cdot \left(1 - \frac{x_1 q}{x_2 q}\right) \cdots \left(1 - \frac{x_1 q}{x_2 q^4}\right)}$$

$$\left(\frac{x_1}{x_1 q} q\right)_2 = (1 - 1)(1 - q) = 0 \Rightarrow Q_1 = 0$$

An Example For The Main Lemma: 3

The last term

$$Q_8 = \frac{\left(\frac{x_1 q}{x_2 q^4}\right)_2 \left(\frac{x_2 q}{x_2 q^4}\right)_3 \left(\frac{x_1}{x_2}\right)_2 \left(\frac{x_2}{x_1} q\right)_3}{\left(1 - \frac{x_2 q^4}{x_1 q}\right) \cdots \left(1 - \frac{x_2 q^4}{x_1 q^4}\right) \cdot \left(1 - \frac{x_2 q^4}{x_2 q}\right) \left(1 - \frac{x_2 q^4}{x_2 q^2}\right) \left(1 - \frac{x_2 q^4}{x_2 q^3}\right)}$$

The factors of $\left(1 - \frac{x_2 q^4}{x_1 q}\right) \cdots \left(1 - \frac{x_2 q^4}{x_1 q^4}\right)$ in the denominator has no contribution, and Q_8 is proper in $x_2 \Rightarrow$

$$\text{CT}_{x_2} Q_8 = 0.$$

· Step 3 ...: We do similar things with the other parts of $\text{CT}_{x_0} Q$, and show that $\text{CT}_{\mathbf{x}} Q = 0$ eventually.

(End of Andrew's q -Dyson's Conjecture)

Iterated Laurent Series Is Used To ...

- Prove **Andrews'** q -Dyson's conjecture (joint work with **Ira Gessel**).
- Give a new approach on **MacMahon's** partition analysis.
- Prove **Bousquet-Mélou** and **Schaeffer's** conjecture on **Slit Plan Walks**.
- Evaluate combinatorial sums (similar to **Egorychev's** work).

The generalization to **Malcev-Neumann series** is used to

- Prove a generalization of **Stanley's** **Monster Reciprocity Theorem** (joint work with **Richard Stanley**).