

On Lattice Path Enumerations

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March 16, 2004

A New Method on Lattice Path Enumerations

Based on the connection between

Algebra: The theory of iterated Laurent series:

Unique factorization lemma.

and Combinatorics: The Concept of Gessel pair:

Unique factorization lemma of paths.

Catalan numbers are the most frequently used numbers in combinatorics other than the binomial coefficients.

It has more than 66 interpretations. (EC2).

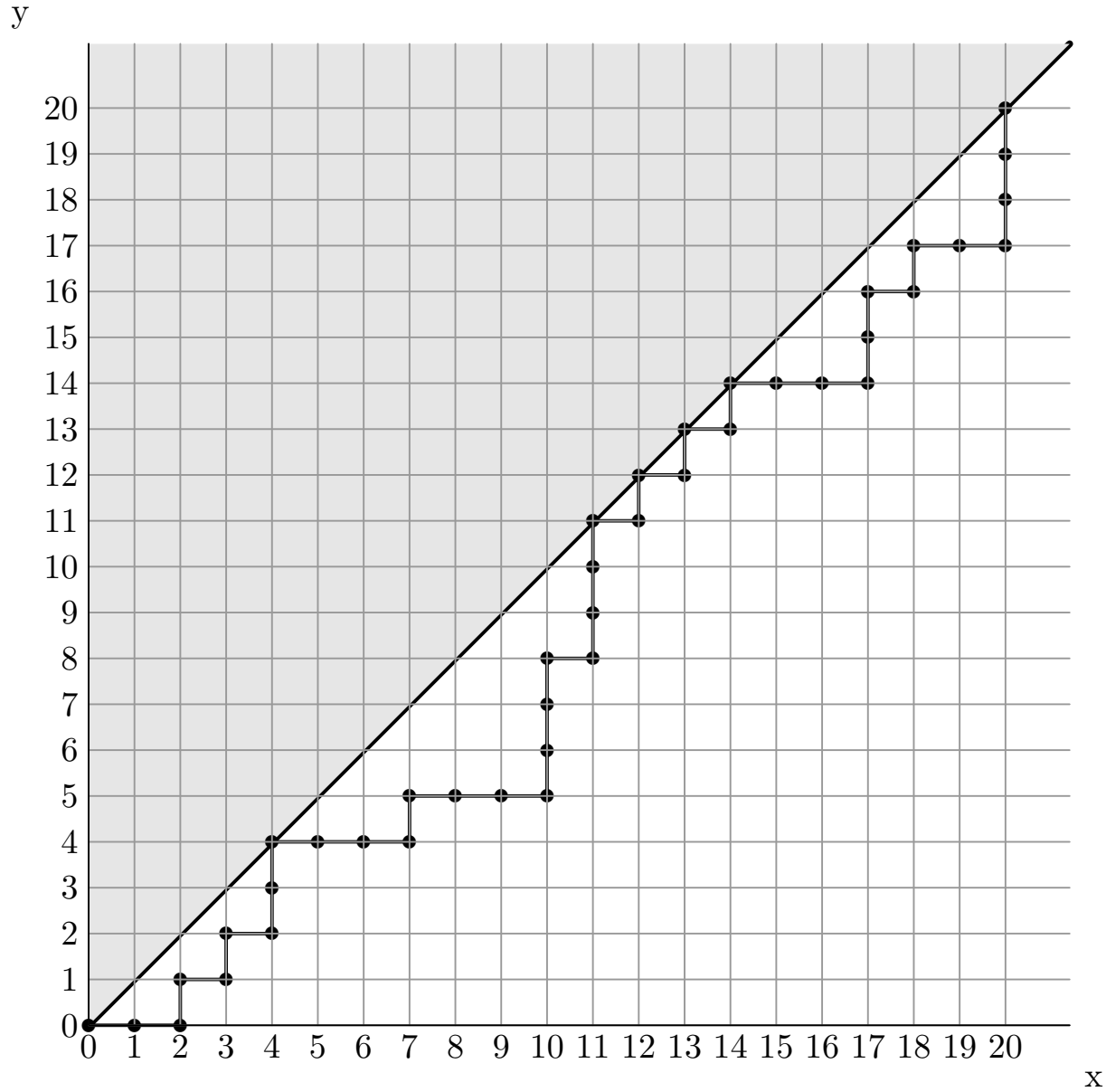
Theorem 1 (Classical Result). *The number of paths from $(0,0)$ to (n,n) with either an up step or a right step and never go above the diagonal is the n th Catalan number C_n .*

The n -th Catalan number is:

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

The generating function of Catalan number is

$$c(x) = \sum_{n \geq 0} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x}.$$



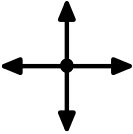
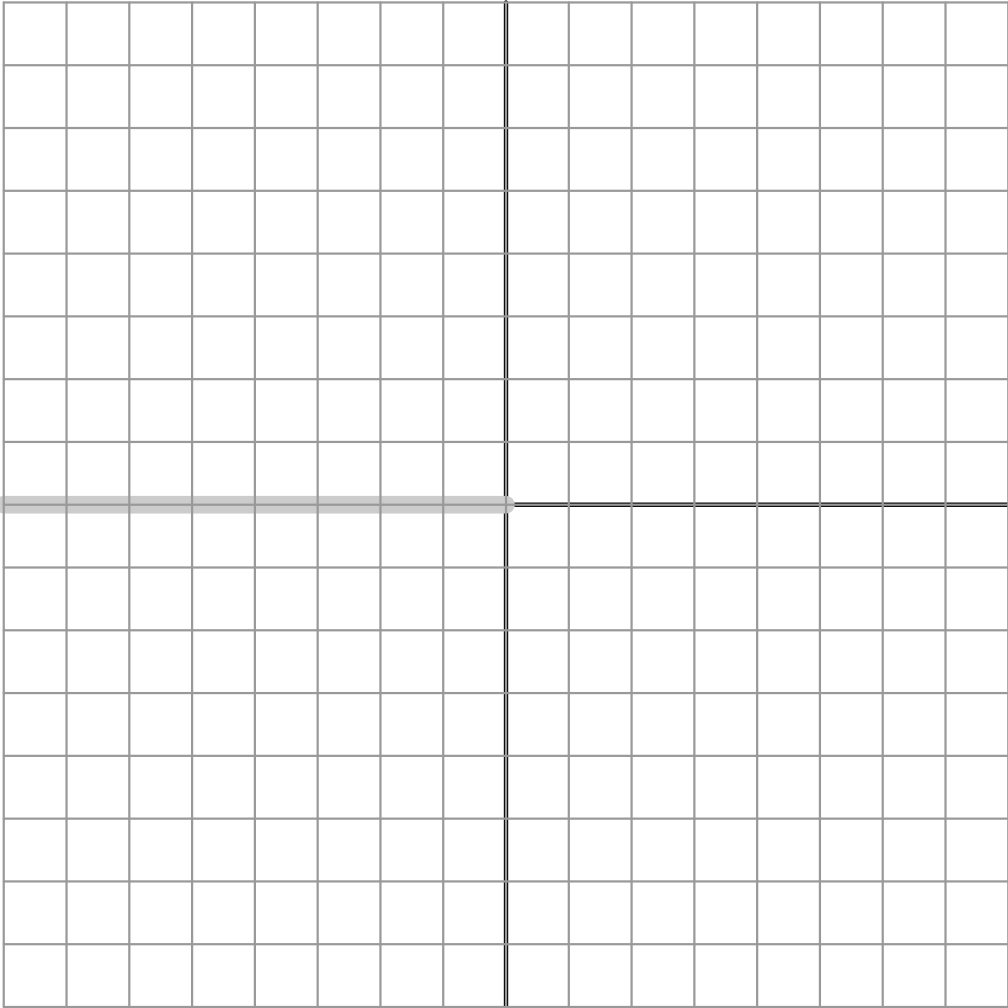
An example of Catalan path

A Surprising Result

Denote by \mathcal{H} the half line $\{(-k, 0); k \in \mathbb{N}\}$. Given a finite subset \mathcal{S} of \mathbb{Z}^2 , *walks on the slit plane* are paths that start at $(0, 0)$ with steps in \mathcal{S} and never hit the half line \mathcal{H} after the starting point.

Theorem 2 (M. Bousquet-Mélou and G. Schaeffer, 2002). *The number of walks of length $2n + 1$ on the slit plane, with steps in $\{(\pm 1, \pm 1)\}$ and ending at $(1, 0)$ is the Catalan number C_{2n+1} .*

Slit Plane



The Field of Iterated Laurent Series

Let K be a field. Then $K\langle\langle x_1 \rangle\rangle$ is defined to be the field $K((x_1))$ of Laurent series in x_1 . Inductively we define $K\langle\langle x_1 \dots x_m \rangle\rangle = K\langle\langle x_1, x_2, \dots, x_{m-1} \rangle\rangle((x_m))$.

Proposition 3 (Fundamental structure). *A formal Laurent series in \mathbf{x} belongs to $K\langle\langle x_1 \dots x_m \rangle\rangle$ if and only if it has a well-ordered support.*

Definition 4.

$$\begin{aligned} & \text{CT}_{x_j} \sum_{(i_1, \dots, i_m) \in \mathbb{Z}^m} a_{i_1, \dots, i_m} x_1^{i_1} \cdots x_m^{i_m} \\ & := \sum_{(i_1, \dots, i_m) \in \mathbb{Z}^m, i_j = 0} a_{i_1, \dots, i_m} x_1^{i_1} \cdots x_m^{i_m}, \end{aligned}$$

where a_{i_1, \dots, i_m} belongs to K .

Unique Factorization Lemma

Lemma 5 (Unique Factorization Lemma, Gessel and Bousquet-Mélou). *Let $h(x, t)$ be an element in $K((x))[[t]]$, in which $h(x, 0) = 1$. Then h has a unique factorization in $K((x))[[t]]$ such that $h = h_- h_0 h_+$, where $h_- \in K[x^{-1}][[t]]$, $h_0 \in K[[t]]$, and $h_+ \in K[[x, t]]$, more over, h_- , h_0 , and h_+ are all 1 when setting $t = 0$.*

Proof. Let $\log h = \sum_{i,j} b_{ij} x^i t^j$. Then

$$h_- = \exp \left(\sum_{i < 0, j > 0} b_{ij} x^i t^j \right),$$

$$h_0 = \exp \left(\sum_{j > 0} b_{0j} t^j \right),$$

$$h_+ = \exp \left(\sum_{i > 0, j > 0} b_{ij} x^i t^j \right).$$

The uniqueness is obvious. □

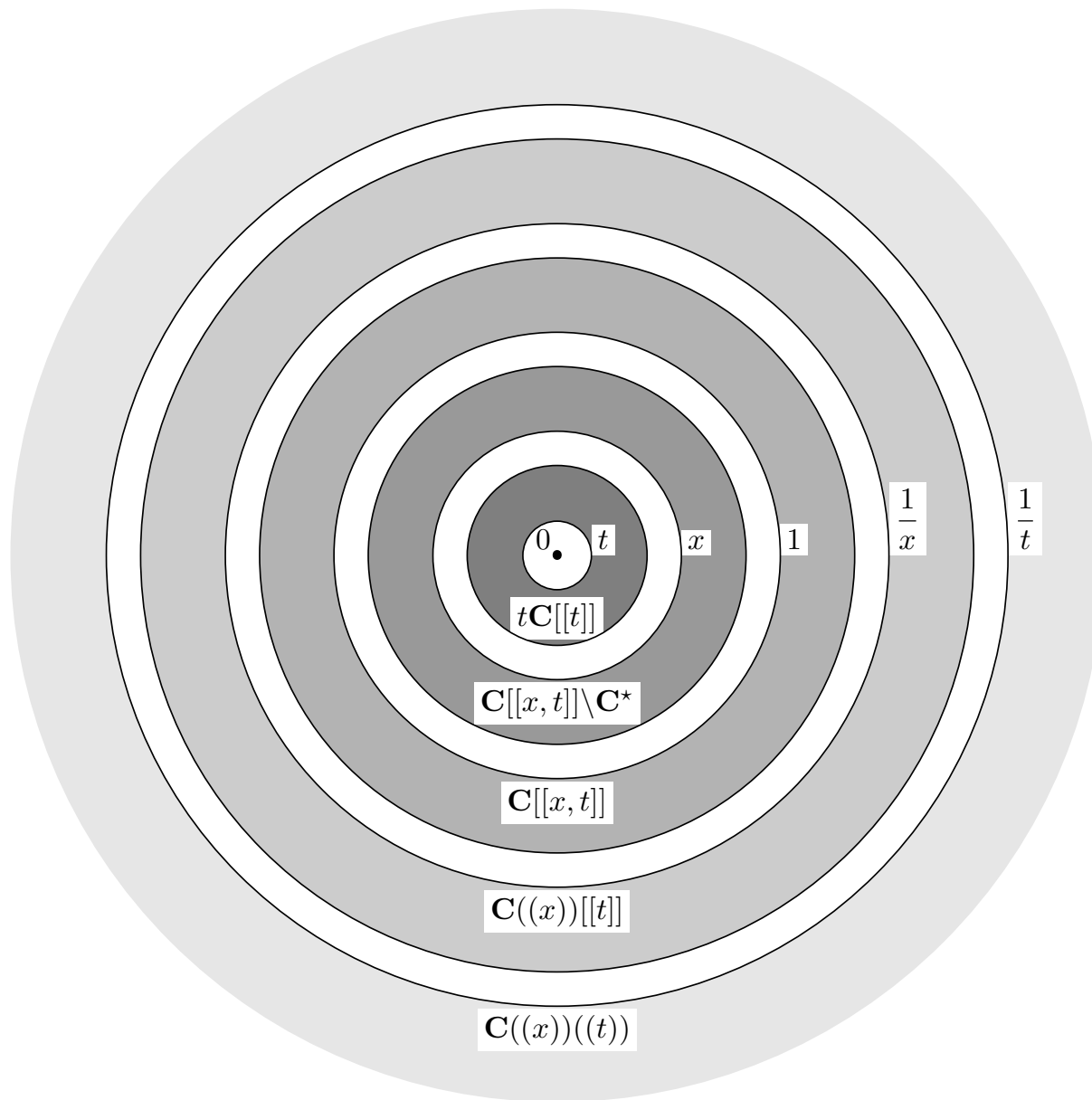
One Useful Tool in Lattice Path Enumeration.

Theorem 6. *Let $G(x, t), F(x, t) \in K[[x, t]]$. If $G(x, 0)$ can be written as $ax +$ higher terms with $a \neq 0$, then*

$$\text{CT}_x \frac{x}{G(x, t)} F(x, t) = \frac{F(x, t)}{\frac{\partial}{\partial x} G(x, t)} \Big|_{x=X}, \quad (1)$$

where $X = X(t)$ is the unique element in $tK[[t]]$ such that $G(X, t) = 0$.

Lemma 7. *If $G(x, t) \in R[[x, t]]$ and $G(x, 0)$ can be written as $ax +$ higher terms with $a \neq 0$, then $G(x, t)$ has a unique positive root $X(t)$ for x , and this $X = X(t)$ belongs to $tR[[t]]$.*



The plane of $\mathbf{C}((x))((t))$

Basic Concepts of Lattice Paths

A *path* σ in \mathbb{Z}^2 is a finite sequence of lattice points $(a_0, b_0), \dots, (a_n, b_n)$ in \mathbb{Z}^2 , in which we call (a_0, b_0) the starting point, (a_n, b_n) the ending point, $(a_i - a_{i-1}, b_i - b_{i-1})$ the steps of σ , and n the length of σ .

Given two paths σ_1 and σ_2 , we define their product $\sigma_1\sigma_2$ to be the path whose steps are those of σ_1 followed by those of σ_2 . If $\pi = \sigma_1\sigma_2$, then we call σ_1 a *head* of π , and σ_2 a *tail* of π .

Denote by S^* the set of all such paths. Then any $\sigma \in S^*$ can be uniquely factored as $\sigma = s_1 s_2 \cdots s_n$ for some $n \geq 0$, and $s_i \in S$ for all i . Note that the empty path belongs to S^* .

The weight of a step $(a, b) \in S$ is defined to be $\Gamma((a, b)) = x^a y^b t$, and the weight of a path $\sigma = s_1 \cdots s_n$ is defined to be $\Gamma(\sigma) = \Gamma(s_1) \cdots \Gamma(s_n)$. The weight of a path is determined by its length and its end point.

For any two paths σ_1 and σ_2 , we have $\Gamma(\sigma_1 \sigma_2) = \Gamma(\sigma_1) \Gamma(\sigma_2)$.

If $Q \subset S^*$, then the generating function of P is:

$$\Gamma(Q) = \sum_{\sigma \in Q} \Gamma(\sigma).$$

Let $H \subset S^*$. If H is closed under multiplication of paths and contains the empty path, then H is a *monoid*. We call a nonempty path $\sigma \in H$ a *prime* of H if it cannot be factored into two nonempty paths in H . We say that H is a *free monoid* if any element in H can be uniquely factored into products of primes in H .

If H is a free monoid, then for any $\sigma \in H$ with its factorization into primes as $\sigma = h_1 h_2 \cdots h_m$, we say that $h_1 h_2 \cdots h_i$ is an *H head* of σ for $i = 0, 1, \dots, m$.

If we let P be the set of primes in H , then

$$\Gamma(H) = 1/(1 - \Gamma(P)).$$

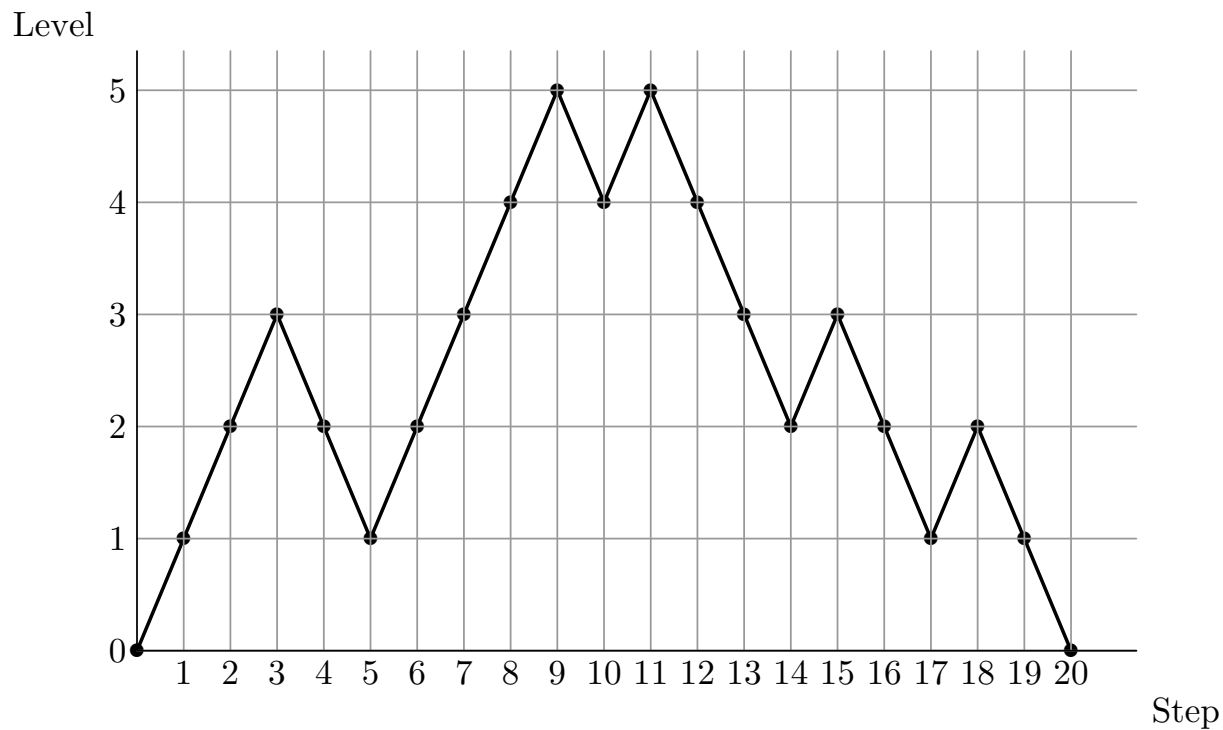
Example 1. S^* is a free monoid, whose primes are all the elements in S .

$$\Gamma(S^*) = \frac{1}{1 - \Gamma(S)}.$$

Example 2. The set S_x of all paths in S^* that end on the x -axis is a free monoid, whose primes are those paths that only return the x -axis at the end point.

$$S_x = [y^0]\Gamma(S^*) := \underset{y}{\text{CT}}\Gamma(S^*).$$

Example 3. The set of all paths in S^* that end on the x -axis and never goes below the x -axis. What are the prime paths?



An example of the case that $S = \{ (1, 1), (1, -1) \}$.

One way to prove the classical result

When $S = \{ (1, 1), (1, -1) \}$, example 3 is just a set of Dyck paths.

Let $p(x)$ be the generating function for the prime paths. Then

$$c(x) = \frac{1}{1 - p(x)}.$$

And in this case, $p(x) = xc(x)$. Then we get

$$c(x) = 1 + xc^2(x) \Rightarrow c(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}.$$

Since $c(x)$ is a power series,

$$c(x) = \frac{1 - \sqrt{1 - 4x}}{2x} = \sum_{n \geq 0} \frac{1}{n + 1} \binom{2n}{n} x^n$$

Let ρ be a homomorphism from H to \mathbb{Z} . The ρ value of a path σ is defined as $\rho(\sigma)$.

If H is a free monoid, then any map from H to \mathbb{Z} defined on the primes of H induces a homomorphism. If in addition, H is a subset of S^* , then the natural map to the end point of a path is a homomorphism from H to \mathbb{Z}^2 . Therefore, any homomorphism from \mathbb{Z}^2 to \mathbb{Z} induces a homomorphism from H to \mathbb{Z} through that natural map. The following two homomorphisms are useful. Define $\rho_x(\sigma)$ to be the x coordinate of the ending point of σ , then ρ_x is clearly a homomorphism. Similarly we can define ρ_y .

Gessel Pair

If H is a free monoid, and ρ is a homomorphism from H to \mathbb{Z} , then we call (H, ρ) a *Gessel pair*. For a Gessel pair (H, ρ) , we define:

A *minus-path* is either the empty path or a path whose ρ value is negative and less than the ρ values of all the other H heads.

A *zero-path* is a path with ρ value 0 and all of whose H heads have nonnegative ρ values.

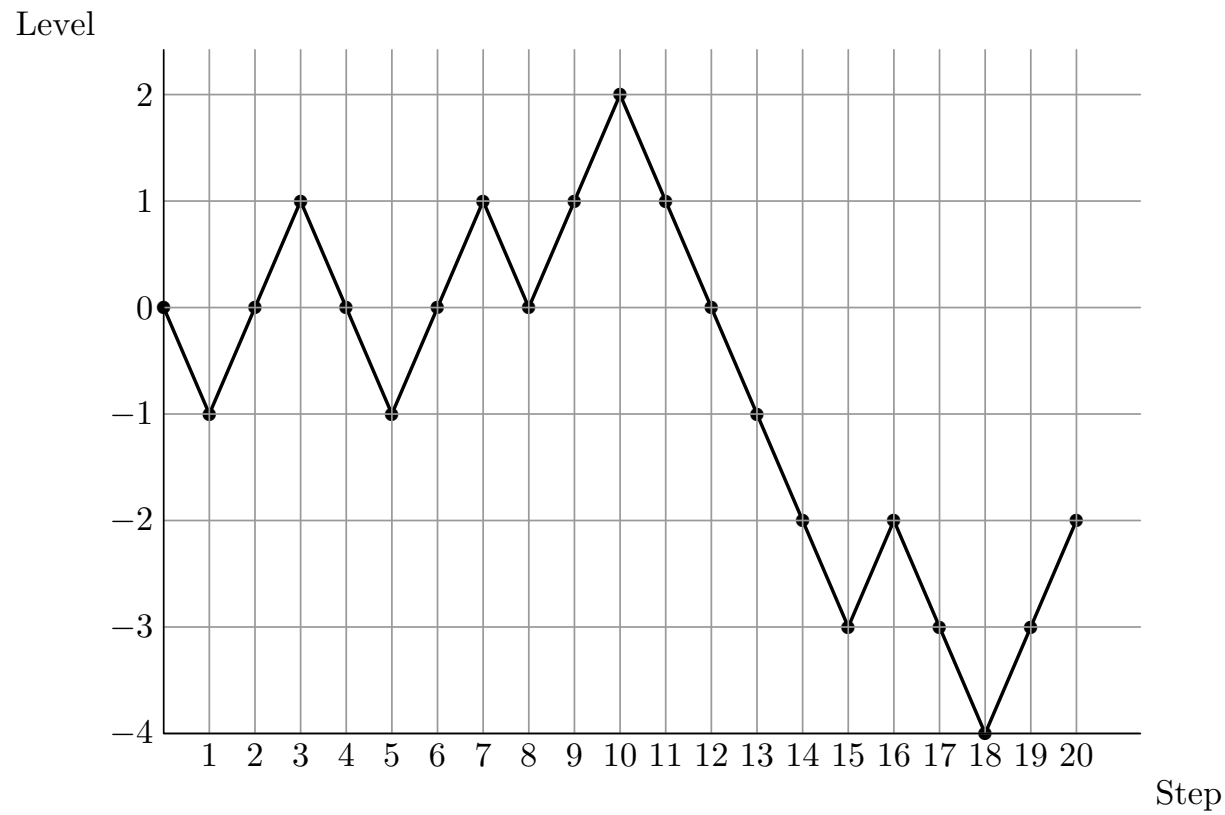
A *plus-path* is a path all of whose H heads (except ϵ) have positive ρ values.

Unique Factorization Lemma of Paths

Proposition 8. *If (H, ρ) is a Gessel pair, then H_- , H_0 , and H_+ are all free monoids. The map from H to $H_- \times H_0 \times H_+$ defined by $\pi \rightarrow (\pi_-, \pi_0, \pi_+)$ is a bijection.*

In a Gessel pair (H, ρ) , the weight of an element $\pi \in H$ is defined to be $\Gamma(\pi)z^{\rho(\pi)}$, where z is a new variable. When H is also a subset of S^* and we are considering the Gessel pair (H, ρ_x) , the power in z is always the same as the power in x for any π in H . So we can replace z by 1 and let x play the same role as z .

An example: $H = \{(1, \pm 1)\}^*$, $\rho = \rho_y$.



Since the factorization in H is with respect to ρ , the factorization of the generating function is with respect to z .

Theorem 9. *For any Gessel pair (H, ρ) , we have $\Gamma(H_-) = [\Gamma(H)]_-$, $\Gamma(H_0) = [\Gamma(H)]_0$, and $\Gamma(H_+) = [\Gamma(H)]_+$.*

Proof. We have $\Gamma(H) = \Gamma(H_-)\Gamma(H_0)\Gamma(H_+)$. It is easy to check this is the unique factorization of $\Gamma(H)$ with respect to z . \square

Example 10. Let S be $\{(1, r), (1, -1)\}$ with $r \geq 1$, and $H = S^*$. Consider the Gessel pair (H, ρ_y) .

Note that in this case the length of a path equals the x coordinate of its end point. Replacing x by 1 will not lose any information.

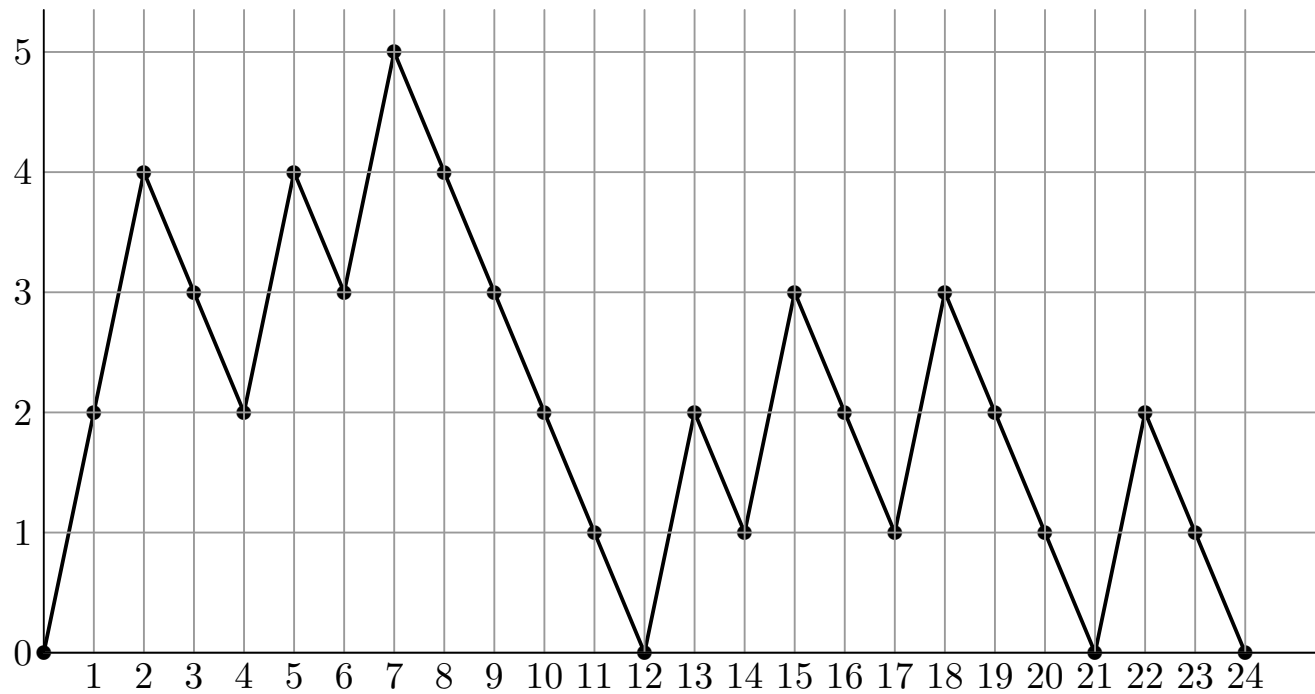
Clearly we have

$$\Gamma(H) = \Gamma(S^*) = \frac{1}{1 - t(y^r + 1/y)}.$$

We see that H_+ is the set of paths in S^* that never go below level 1 after the starting point. The set H_0 contains all paths in S^* that end on level 0 and never go below level 0. When $r = 1$, this becomes Dyck paths.

An example of H_0 : of the case $r = 2$.

Level



Step

The case $r = 2$: To compute $\Gamma(H_0) := F(t)$, we let $Y_1(t)$ be the unique positive root of $y - t(1 + y^3)$ and let Y_2 and Y_3 be the other roots. Then

$$\frac{y - t(1 + y^3)}{y} = (1 - Y_1/y) \cdot A(t) \cdot (1 - y/Y_2)(1 - y/Y_3) \quad (2)$$

is the desired unique factorization. Thus $F(t) = 1/A(t)$.

Equate coefficients of y^{-1} on both sides of (2). Then $-t = -A(t)Y_1(t)$.

So $F(t) = Y_1(t)/t$, and $F(t) = 1 + t^{r+1}F(t)^{r+1}$.

More Examples

Example 11. Let S be $\{(1, 1), (1, -1)\}$, and let $H = S^*$. Let ρ be determined by $\rho(1, 1) = r$ and $\rho(1, -1) = -1$.

It is easy to see that this example is isomorphic to the previous one.

Or let $H = (a, b)^*$, the free monoid generated by a and b . And let ρ be generated by $\rho(a) = 1, \rho(b) = -1$.

Example 12. *In general if $H = S^*$, then (H, ρ_y) is a Gessel pair.*

We see that H_+ is the set of paths in S^* that never go below the line $y = 1$ after the starting point.

If we let $J = H_+$, then J is also a free monoid. The primes of J are paths that start at $(0, 0)$, end at some positive level d , and never hit level $d - 1$ or lower.

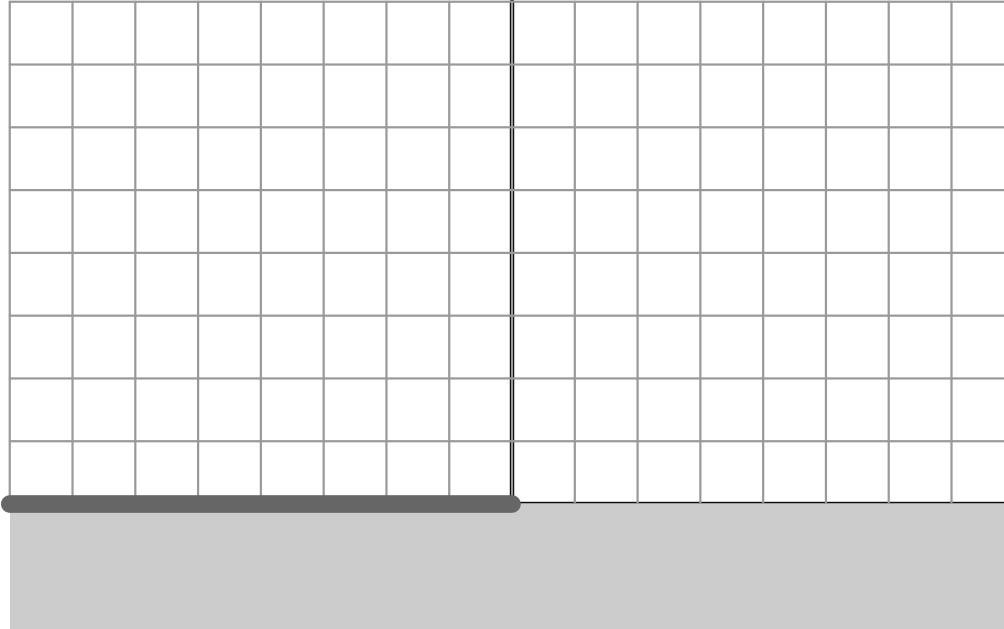
The set H_0 contains all paths in S^* that end on the line $y = 0$, and never go below the line $y = 0$. In other words, H_0 contains all paths in S^* that stays in the upper half plane and end on the x -axis.

(continue of last example)

If we let $J = H_0$, then (J, ρ_x) is a Gessel pair. The set J_+ contains all paths in J that avoiding the half line \mathcal{H} after the starting point. This is the same as walks on the half plane avoiding half line. (Bousquet-Mélou).

The set J_0 contains all paths in J that ending at $(0, 0)$ and never touch the half line \mathcal{H} except $(0, 0)$.

Walks avoiding half plane and half line



Example 13. *For any S , let H be the set of paths that end on the x -axis. Then (H, ρ_x) is a Gessel pair.*

The set H_+ contains all paths that end on the x axis and never hit the half line $\mathcal{H} = \{(-k, 0) | k \geq 0\}$ after the starting point. This is exactly the walks on the slit plane that end on the x -axis.

The set H_0 contains all paths that end at $(0, 0)$, and never touch $(-k, 0)$ for $k = 1, 2, \dots$. This was called the set of loops by Bousquet-Mélou.

Example 14. *For any S , let H be the set of paths that end on the x -axis and never go below the line $y = -d$ for some given $d > 0$. Then it is easy to check that (H, ρ_x) is a Gessel pair.*

The set H_+ contains all paths that end on the x -axis, and never hit the half line \mathcal{H} after the starting point, and never go below the line $y = -d$.

The set H_0 can be similarly described.

Example 15. *About walks on the half plane avoiding half line. More precisely, walks that never touch the half line \mathcal{H} and never hit a point (i, j) with $j < 0$. This is a continuation of Example 12. We denote by $HS(x, y; t)$ the generating function for such paths.*

We obtain the following result, which includes

Theorem 16. *For any well-ordered set \mathfrak{G} . Let p be the smallest positive number such that there is an \mathfrak{G} -path end at $(p, 0)$. Then the number of walks on the half plane avoiding half line that end at $(p, 0)$ and is of length n is equal to one n th of the number of paths that end at $(p, 0)$ and is of length n .*

Proof. We use the notations of Example 12. From the Gessel pair (\mathfrak{S}^*, ρ_y) , we have $\Gamma(H_0) = (\Gamma(\mathfrak{S}^*))_0$ and

$$\log \Gamma(H_0) = C_y^T \log \Gamma(\mathfrak{S}^*).$$

Now let $J = H_0$ and consider the Gessel pair (J, ρ_x) . Then

$$\log \Gamma(J_0 J_+) = P_x^T \log \Gamma(J).$$

In particular, we have

$$\begin{aligned} [x^p] \Gamma(J_+) &= [x^p] \log \Gamma(J) = [x^p] \log \Gamma(H_0) \\ &= [x^p] C_y^T \log \Gamma(\mathfrak{S}^*). \end{aligned}$$

Therefore,

$$[x^p t^n] \Gamma(J_+) = [x^p y^0 t^n] \frac{1}{n} \Gamma(\mathfrak{S})^n.$$

This prove the theorem. □

Example 17. If $S = \{ (1, 0), (-1, 0), (0, 1), (0, -1) \}$, then $\Gamma(S) = t(x + y + x^{-1} + y^{-1})$.

$$\begin{aligned} \mathbf{CT}_y \Gamma(S^*) &= \mathbf{CT}_y \frac{y}{y - t(x + x^{-1})y - ty^2 - t} \\ &= \frac{1}{1 - t(x + x^{-1}) - 2tY}, \end{aligned}$$

where $Y = Y(x, t)$ is the unique positive root of $y - t(x + x^{-1})y - ty^2 - t$.

$$Y = \frac{1 - t(x + x^{-1}) - \sqrt{(1 - t(x + x^{-1}))^2 - 4t^2}}{2t}.$$

After simplifying, the desired generating function can be written as

$$\mathbf{CT}_y \Gamma(S^*) = \left[(1 - t(x + x^{-1}))^2 - 4t^2 \right]^{-1/2}.$$