Partial Fraction Algorithm for MacMahon’s Partition Analysis

UCSD seminar

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Outline

1. MacMahon’s Partition Analysis
   History
   Iterated Laurent Series

2. Partial Fraction Algorithm

3. Working by Hand
   To Obtain the Crude Generating Functions
   Evaluating the Constant Term
   Examples of Application

4. Perspectives
Partition Analysis was firstly introduced by Percy MacMahon in his famous book “Combinatorial Analysis” in 1915.
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By using the theory of iterated Laurent series and partial fraction decompositions, I give a speed up of the Omega package in 2004.
Object of Study

Solving combinatorial problems in connection with system of linear Diophantine equations and inequalities. Several examples of applications are

1. Solving system of linear Diophantine equations and inequalities.
2. Counting lattice points in a rational convex polytope.
MacMahon’s Omega Operator

Definition

\[ \Omega \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\ldots,s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := \sum_{s_1=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} A_{s_1,\ldots,s_r}, \]

\[ \Omega = \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\ldots,s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := A_{0,\ldots,0}. \]

Everything is treated analytically. The method relies on the unique Laurent series representations of rational functions.
Basic Idea by an Example 1/2

Problem: Find all nonnegative integral solutions to \( a_1 + a_2 - a_3 = 0 \).

\[
\sum_{a_1, a_2, a_3 \geq 0 \atop a_1 + a_2 - a_3 = 0} x^{a_1} y^{a_2} z^{a_3} = \sum_{a_1, a_2, a_3 \geq 0} \Omega \lambda^{a_1 + a_2 - a_3} x^{a_1} y^{a_2} z^{a_3} = \sum_{a_1 \geq 0} \Omega \lambda^{a_1} x^{a_1} \cdot \sum_{a_2 \geq 0} \lambda^{a_2} y^{a_2} \cdot \sum_{a_3 \geq 0} \lambda^{-a_3} z^{a_3}
\]
Basic Idea by an Example 2/2

Apply the formula for the sum of a geometric series:

\[ \frac{1}{\Omega} = \frac{1}{(1 - \lambda x)(1 - \lambda y)(1 - z/\lambda)}, \]

Eliminate \( \lambda \) using Elliott’s Reduction Procedure to obtain

\[ \frac{1}{(1 - xz)(1 - yz)} \]

If we set \( x = t, y = t, z = t \), we will get the generating function for the number of the above solutions such that \( a_1 + a_2 + a_3 = n \):

\[ \frac{1}{(1 - t^2)^2} \]
Iterated Laurent Series

Definition of Iterated Laurent Series

Let $K$ be a field. Define

$$K \langle \langle x_1 \rangle \rangle = K((x_1)) = \left\{ \sum_{n \geq N_0} a_n x_1^n \mid a_n \in K \right\}$$

to be the field of Laurent series in $x_1$. Inductively define

$$K \langle \langle x_1, x_2, \ldots, x_n \rangle \rangle = K \langle \langle x_1, x_2, \ldots, x_{n-1} \rangle \rangle ((x_n)) = K((x_1)) \cdots ((x_n))$$

to be the field of iterated Laurent series in $x_1, x_2, \ldots, x_n$.

An iterated Laurent series is firstly regarded as a Laurent series in $x_n$, and then (coefficients) regarded as Laurent series in $x_{n-1}$, and so on.
Equivalently, we defined a total order on the variables, or can be understood as $1 >> x_1 >> \cdots >> x_n$. (Wilson, Stanley)

This total order is extended for all monomials:
For a monomial $M$, scan for the smallest variable in it, say $x_i$. Then $M = x_1^{k_1} \cdots x_i^{k_i} x_{i+1}^0 \cdots x_n^0$.

1. Set $M < 1$ if $k_i > 0$. For instance, $x_1 x_3^{-2} > 1$, $x_2^{-3} x_4 < 1$
2. Set $M_1 < M_2 \iff M_1/M_2 < 1$
Series Expansion

If $M_2$ is small then

$$\frac{1}{M_1 - M_2} = \frac{1}{M_1} \frac{1}{1 - \frac{M_2}{M_1}} = \sum_{n \geq 0} \frac{M_2^n}{M_1^{n+1}}.$$ 

If $M_1$ is small then

$$\frac{1}{M_1 - M_2} = -\frac{1}{M_2} \frac{1}{1 - \frac{M_1}{M_2}} = -\sum_{n \geq 0} \frac{M_1^n}{M_2^{n+1}}.$$ 

After fixing a total order on the variables, everything works formally when taking constant terms. This is guaranteed by:

**Theorem (Fundamental Structure)**

A series is an iterated Laurent series if and only if it has a well-ordered support.
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Object of Study

Take constant terms of a given Elliott rational function, i.e., a rational function whose denominator is written as a product of $M_1 - M_2$.

Elliott rational functions arise naturally in the study of Linear Diophantine system.

Such constant terms can be theoretically evaluated.

**Theorem (Elliott)**

*The constant term of an Elliott rational functions is still Elliott rational.*

The constant term in a set of variables can be taken iteratively.
Revisit an Example

Compute $\Omega = F$:

$$F = \frac{\lambda}{(1 - x\lambda)(1 - y\lambda)(\lambda - z)}$$

$$= \frac{1/x}{(1 - x\lambda)(1 - y/x)(1/x - z)} + \frac{1/y}{(1 - y\lambda)(1 - x/y)(1/y - z)}$$

$$+ \frac{z}{(\lambda - z)(1 - xy)(1 - xz)}$$

where $\Omega = \frac{1}{1 - x\lambda} = 1, \Omega = \frac{1}{1 - y\lambda} = 1$

and $\Omega = \frac{1}{\lambda - z} = \Omega = \frac{1}{\lambda} \frac{1}{1 - z/\lambda} = \Omega = \sum_{n \geq 1} z^{n-1}/\lambda^n = 0.$
The PT Operator Replacing MacMahon’s Operator

Fix a working field $K \langle \Lambda, x \rangle := K \langle \lambda_1, \ldots, \lambda_r, x_1, \ldots, x_n \rangle$.

Define a new operator $\text{PT}_\lambda$ to replace both $\Omega_\geq$ and $\Omega_\leq$:

$$\text{PT}_\lambda \sum_{m=-\infty}^{\infty} a_m \lambda^m := \sum_{m=0}^{\infty} a_m \lambda^m.$$ 

MacMahon’s operators can be realized as:

$$\Omega F(\Lambda, x) = \text{PT}_\Lambda F(\Lambda, x) \bigg|_{\Lambda=(1,\ldots,1)} ,$$

$$\Omega F(\Lambda, x) = \text{CT}_\Lambda F(\Lambda, x) = \text{PT}_\Lambda F(\Lambda, x) \bigg|_{\Lambda=(0,\ldots,0)} .$$
Garsia’s Notation

For $\Lambda = (\lambda_1, \ldots, \lambda_k)$, Garsia use

$$F \bigg|_{\lambda_1^0 \cdots \lambda_k^0}^{\Lambda} = CT_F = \Omega F$$

and

$$F \bigg|_{\lambda_1^\geq \cdots \lambda_k^\geq}^{\geq} = \Omega F$$
Heart Problem

How to compute $\Omega = F(\Lambda, x)$ fast?

By iteration, we reduce the problem to the one variable case: evaluating $\text{PT}_\lambda F(\lambda)$ with

$$F(\lambda) = \frac{\text{polynomial}(\lambda)}{\prod_{1 \leq i \leq n}(\lambda^{j_i} - z_i)},$$

where $j_i \in \mathbb{P}$, and $z_i$’s are independent of $\lambda$. Note that $z_i$’s are allowed to be 0.

Find the PFD of $F(\lambda)$. Note that the idea of using PFD in this context was first used by Stanley (1974), but was thought to be impractical without using a computer.
Application of the PFD

Constant term can be read off as soon as we find the PFD.
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Constant term can be read off as soon as we find the PFD.

**Theorem**

Suppose that the PFD of $F$ is

$$F = f(\lambda) + \sum_{1 \leq s \leq n} \frac{p_s(\lambda)}{\lambda^{j_s} - z_s},$$

where $f(\lambda)$ is a polynomial in $\lambda$, and $p_s(\lambda)$ is a polynomial of degree less than $j_s$ for each $s$. Then

$$\lambda^{PT} F = f(\lambda) + \sum_s \frac{p_s(\lambda)}{\lambda^{j_s} - z_s},$$

where the sum ranges over all $s$ such that $\lambda^{j_s} < z_s$. 
The Linear Factors Case is Simple

Suppose we are taking constant term in $x_k$ of

$$F = \frac{p(x_k)}{x_k^d \prod_{i=1}^{m} (1 - x_k/\alpha_i)},$$

where $p(x_k)$ is a polynomial and $\alpha_i$ are independent of $x_k$. If $F$ is proper in $x_k$, then we have the PFD with respect to $x_k$

$$F = \frac{p_1(x_k)}{x_k^d} + \sum_{j=1}^{m} \frac{1}{1 - x_k/\alpha_j} \left( \frac{p(x_k)}{x_k^d \prod_{i=1, i \neq j}^{m} (1 - x_k/\alpha_i)} \right) \bigg|_{x_k = \alpha_j},$$

When taking constant term in $x_k$, we have

$$\text{CT} F = \sum_{j} (F (1 - x_k/\alpha_j)) \bigg|_{x_k = \alpha_j},$$

where the sum ranges over all $j$ such that $x_k/\alpha_j$ is small.
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Problem: Find all nonnegative integer solutions \( \alpha \in \mathbb{N}^n \) with \( A\alpha = b \):

\[
\begin{align*}
    a_{1,1}\alpha_1 + a_{1,2}\alpha_2 + \cdots + a_{1,n}\alpha_n &= b_1 \\
    a_{2,1}\alpha_1 + a_{2,2}\alpha_2 + \cdots + a_{2,n}\alpha_n &= b_2 \\
    \quad &\quad \quad \quad \cdots \quad = \cdots \\
    a_{r,1}\alpha_1 + a_{r,2}\alpha_2 + \cdots + a_{r,n}\alpha_n &= b_r.
\end{align*}
\]

The generating function is the constant term of the crude generating function

\[
\ell_1^{-b_1} \cdots \ell_r^{-b_r} \prod_{i=1}^n (1 - \ell_1^{a_{1,i}} \ell_2^{a_{2,i}} \cdots \ell_r^{a_{r,i}} x_i).
\]

However, many problems are easy to describe by equations.
To Obtain the Crude Generating Functions

From a System of Equations: An Example

Problem: Find all nonnegative integer $n \times n$ matrices $(a_{ij})$ with every row sum and column sum equal to $r$.

Use $\lambda_i$ for the $i$th row equation, $\mu_j$ for the $j$th column equation. Use $x_{ij}$ for $a_{ij}$, and use $t$ for $r$. For example, the $i$th row equation is

$$a_{i1} + a_{i2} + \cdots + a_{in} - r = 0.$$

The crude generating function is
To Obtain the Crude Generating Functions

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The crude generating function is

$$\prod_{i,j=1}^{n} \frac{1}{1 - x_{ij} \lambda_i \mu_j} \times \frac{1}{1 - t/(\lambda_1 \cdots \lambda_n \mu_1 \cdots \mu_n)}.$$ 

Understood as: The $a_{ij}$ only appears in the $\lambda_i$ and $\mu_j$ equations with coefficients 1. The $r$ appears in every equation, with coefficients $-1$. Actually, we can set $\mu_n = 1$ since it is redundant.
Problem: Find all labelings on the vertices of a $k$ dimensional cube, such that the labels on every facet sum to $r$.

The vertices of the cube is identified by 01 sequences $w = w_1 \cdots w_k$. A facet is either $\{w_1 \cdots w_k \mid w_i = 0\}$ or $\{w_1 \cdots w_k \mid w_j = 1\}$. Index them by $\lambda_i^{(0)}$ and $\lambda_i^{(1)}$ respectively. For example, the equation for the $\lambda_i^{(0)}$ is

$$\sum_{w_i=0} a_{w_1 \cdots w_k} - r = 0.$$ 

The Crude G.F. is given by
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$$\prod_{w_1 \cdots w_k} \frac{1}{1 - x_{w_1 \cdots w_k} \lambda_1^{(w_1)} \cdots \lambda_k^{(w_k)}} \times \frac{1}{1 - t/\lambda_1^{(0)} \cdots \lambda_k^{(0)} \lambda_1^{(1)} \cdots \lambda_k^{(1)}}.$$

Of course, one can set $\lambda_2^{(1)}, \ldots, \lambda_k^{(1)}$ all equal to 1, since they are redundant.
Evaluating the Constant Term

The Set Up for Evaluating Constant Term

Our Goal: evaluate the constant term in a set of variables of a crude G.F.:

\[ F = \frac{P}{(1-m_1)(1-m_2) \cdots (1-m_n)}, \]

where \( P \) is a Laurent polynomial, and \( m_i \) are monomials.

1. First choose an order of the variables compatible with our problem. There may be many choices!
2. Choose a variable and eliminate, but do NOT combine the result.
3. For each term obtained, choose another variable and eliminate.
4. It is NOT a good strategy to decide before hand in with order the variables are to be eliminated.

The choice of the variable for elimination is guided by the following . . .
**For the One Variable Linear Factors Case**

We often meet rational function of the form:

\[
F(\lambda) = \frac{P(\lambda)}{\prod_{i=1}^{n} (1 - (\lambda/m_i)^{e_i})} \quad \text{(with } e_i = \pm 1\text{)}.
\]

**Lemma**

*Suppose the above F is proper in \( \lambda \) and \( \lim_{\lambda \to 0} F(\lambda) = 0 \). Then*

\[
\text{CT}_\lambda F(\lambda) = \sum_{\lambda/m_i < 1} e_i (F(\lambda)(1 - (\lambda/m_j)^{e_i})) \bigg|_{\lambda = m_i} \\
= - \sum_{\lambda/m_i > 1} e_i (F(\lambda)(1 - (\lambda/m_j)^{e_i})) \bigg|_{\lambda = m_i}
\]
Evaluating the Constant Term

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= - \sum_{\lambda/m_i > 1} e_i (F(\lambda)(1 - (\lambda/m_j)^{e_i})) \bigg|_{\lambda=m_i}
\]

A denominator factor \( 1 - (\lambda/m_i)^{e_i} \) is said to be contributing if \( \lambda/m_i < 1 \) and to be dually contributing if \( \lambda/m_i > 1 \).
Choosing the Variable for Elimination

Since for $m_j$ with $m_j > 1$ we can write

$$\frac{1}{1-m_j} = \frac{-\frac{1}{m_j}}{1 - \frac{1}{m_j}}.$$ 

Our crude G.F. $F$ can be written in *proper form*

$$F = P \times \prod_{m_i < 1} \frac{1}{1 - m_i}.$$ 

With respect to $\lambda$, $1 - m_i$ is contributing if $\deg_\lambda m_i > 0$, and is dually contributing if $\deg_\lambda m_i < 0$.

The variable we shall eliminate is the one that has the smallest number of contributing factors or dually contributing factors.
A Simple Example: Weak Magic Squares of Order 2

The Crude G.F. is

\[
\frac{1}{(1 - \lambda_1 \mu_1)(1 - \lambda_1)(1 - \lambda_2 \mu_1)(1 - \lambda_2)(1 - t/\lambda_1 \lambda_2 \mu_1)}.
\]

Taking constant term in $\mu_1$ gives
A Simple Example: Weak Magic Squares of Order 2

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Taking constant term in \( \mu_1 \) gives

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\frac{1}{(1 - t/\lambda_2)(1 - \lambda_1)(1 - t/\lambda_1)(1 - \lambda_2)}.
\]

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Taking constant term in \( \lambda_2 \) gives
A Simple Example: Weak Magic Squares of Order 2

The Crude G.F. is

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\[ \frac{1}{(1 - t/\lambda_2)(1 - \lambda_1)(1 - t)(1 - \lambda_2)}. \]

Taking constant term in \( \lambda_2 \) gives

\[ \frac{1}{(1 - \lambda_2)^2} = \sum_{r \geq 0} (r + 1)t^r. \]
Examples of Application

**Magic Labeling of the Cube 1/4**

The crude G.F. for the 3-dimensional case:

\[
\frac{1}{(1 - \lambda_1 \lambda_2 \lambda_3)(1 - \lambda_1 \lambda_2)(1 - \lambda_1 \lambda_3)(1 - \lambda_1)} \times \frac{1}{(1 - \mu_1 \lambda_2 \lambda_3)(1 - \mu_1 \lambda_2)(1 - \mu_1 \lambda_3)(1 - \mu_1)} \times \frac{1}{1 - \frac{t}{\lambda_1 \lambda_2 \lambda_3 \mu_1}}
\]

Choose \( t < \lambda_1 < \lambda_2 < \lambda_3 < \mu_1 < 1 \). Eliminating \( \mu_1 \) gives

\[
\frac{1}{(1 - \lambda_1 \lambda_2 \lambda_3)(1 - \lambda_1 \lambda_2)(1 - \lambda_1 \lambda_3)(1 - \lambda_1)} \times \frac{1}{(1 - t/\lambda_1)(1 - t/\lambda_1 \lambda_3)(1 - t/\lambda_1 \lambda_2)(1 - t/\lambda_1 \lambda_2 \lambda_3)}
\]
Examples of Application

**Magic Labeling of the Cube 2/4**

Eliminating $\lambda_3$ gives a sum of two terms

$$T_1 = \frac{1}{(1 - \lambda_1 \lambda_2)(1 - 1/\lambda_2)(1 - \lambda_1)(1 - t/\lambda_1)(1 - t \lambda_2)(1 - t/\lambda_1 \lambda_2)(1 - t)}$$

$$T_2 = \frac{1}{(1 - \lambda_2)(1 - \lambda_1 \lambda_2)(1 - \lambda_1)(1 - t/\lambda_1)(1 - t)(1 - t/\lambda_1 \lambda_2)(1 - t/\lambda_2)}$$

For $T_1$, eliminating $\lambda_2$ gives
Examples of Application

Magic Labeling of the Cube 2/4

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For $T_1$, eliminating $\lambda_2$ gives

$$\frac{1}{(1 - t)(1 - \lambda_1/t)(1 - \lambda_1)(1 - t/\lambda_1)(1 - t^2/\lambda_1)(1 - t)}$$

Eliminating $\lambda_1$ gives our
**Examples of Application**

**Magic Labeling of the Cube 2/4**

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For $T_1$, eliminating $\lambda_2$ gives

$$\frac{1}{(1 - t)(1 - \lambda_1/t)(1 - \lambda_1)(1 - t/\lambda_1)(1 - t^2/\lambda_1)(1 - t)}$$

Eliminating $\lambda_1$ gives our

$$out_1 = \frac{1}{(1 - t)(1 - 1/t)(1 - t)(1 - t^2)(1 - t)} = \frac{-t}{(1 - t)^4(1 - t^2)}.$$
For $T2$, eliminating $\lambda_2$ gives two terms

$$U1 = \frac{1}{(1 - \lambda_1)(1 - \lambda_1)(1 - t/\lambda_1)(1 - t)(1 - t/\lambda_1)(1 - t)}$$

$$U2 = \frac{1}{(1 - 1/\lambda_1)(1 - \lambda_1)(1 - t/\lambda_1)(1 - t)(1 - t)(1 - t/\lambda_1)}$$

For $U2$ eliminating $\lambda_1$ gives our

$$out_2 = \frac{1}{(1 - 1/t)(1 - t)(1 - t)(1 - t)(1 - t^2)} = \frac{-t}{(1 - t)^4(1 - t^2)}$$

For $U1$ we can not apply our formula directly.
Examples of Application

**Magic Labeling of the Cube 4/4**

Introduce a new variable $z$, that will be taken to 1.

$$U_1 = \frac{1}{(1 - \lambda_1)(1 - \lambda_1/z)(1 - t/\lambda_1)(1 - t)(1 - t/\lambda_1)(1 - t)}$$

Eliminating $\lambda_1$ gives

$$\frac{1}{(1 - 1/z)(1 - t)^4} + \frac{1}{(1 - z)(1 - t/z)^2(1 - t)^2}$$

Combining and setting $z = 1$, we get our

$$out_3 = \frac{1 - t^2}{(1 - t)^6}.$$

Putting all the above together, we get the Final generating function

$$out_1 + out_2 + out_3 = \frac{1 + t^2}{(1 - t)^4(1 - t^2)}.$$
Examples of Application

An Example From Invariant Theory 1/4

A recent joint work with Garsia, Wallach, and Zabrocki includes the following identity:

\[
\text{CT} \times \prod_{i=1}^{n} \frac{1}{(1 - qx_i)(1 - q/x_i)} \prod_{1 \leq i < j \leq n} \frac{1 - x_j/x_i}{(1 - qx_j/x_i)} \bigg|_{x_1 \cdots x_{n-1}x_n = 1} = \\
\frac{1 + q^{\binom{n+1}{2}}}{(q^2)^n(1 - q^{\binom{n+1}{2}})}
\]

This is related to the Hilbert series of certain invariant graded ring.
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\]

This is related to the Hilbert series of certain invariant graded ring.

Directly applying my algorithm gives no more than \( n \) simple terms for particular \( n \). But the sum of these terms is not easy to be simplified.
An Example From Invariant Theory 2/4

We have a simple proof by working on an intermediate crude G.F.

\[
\frac{1}{1-x_1 \cdots x_n} \prod_{i=1}^{n} \frac{1}{1-qx_i}(1-q/x_i) \prod_{1\leq i<j\leq n} \frac{1-x_j/x_i}{1-qx_j/x_i}
\]

First, if we eliminate \(x_n, x_{n-1}, \ldots\) successively, then we obtain only one term in each step! For instance, eliminating \(x_n\) gives

\[
\frac{1}{(1-x_1 \cdots x_{n-1}q)(1-q^2)} \prod_{i=1}^{n-1} \frac{1}{1-qx_i}(1-q/x_i) \prod_{i<j} \frac{1-x_j/x_i}{1-qx_j/x_i} \times \prod_{i=1}^{n-1} \frac{1-q/x_i}{1-q^2/x_i}
\]

which simplifies to
An Example From Invariant Theory 2/4

We have a simple proof by working on an intermediate crude G.F.

\[
\frac{1}{1-x_1 \cdots x_n} \prod_{i=1}^{n} \frac{1}{(1-qx_i)(1-q/x_i)} \prod_{1 \leq i < j \leq n} \frac{1-x_j/x_i}{(1-qx_j/x_i)}
\]

First, if we eliminate \(x_n, x_{n-1}, \ldots\) successively, then we obtain only one term in each step! For instance, eliminating \(x_n\) gives

\[
\frac{1}{(1-x_1 \cdots x_{n-1}q)(1-q^2)} \prod_{i=1}^{n-1} \frac{1}{(1-qx_i)(1-q/x_i)} \prod_{i<j}^{n-1} \frac{1-x_j/x_i}{(1-qx_j/x_i)} \times \prod_{i=1}^{n-1} \frac{1-q/x_i}{1-q^2/x_i}
\]

which simplifies to

\[
\frac{1}{(1-q^2)(1-x_1 \cdots x_{n-1}q)} \prod_{i=1}^{n-1} \frac{1}{(1-qx_i)(1-q^2/x_i)} \prod_{1 \leq i < j \leq n-1} \frac{1-x_j/x_i}{(1-qx_j/x_i)}.
\]
Examples of Application

An Example From Invariant Theory 3/4

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\frac{1}{1-x_1 \cdots x_n} \prod_{i=1}^{n} \frac{1}{(1-qx_i)(1-q/x_i)} \prod_{1 \leq i < j \leq n} \frac{1-x_j/x_i}{(1-qx_j/x_i)}
\]

Second, it has a simple connection with our problem: Taking constant term in \(x_1\) gives a sum of two terms. The first one is

\[
\prod_{i=1}^{n} \frac{1}{(1-qx_i)(1-q/x_i)} \prod_{1 \leq i < j \leq n} \frac{1-x_j/x_i}{(1-qx_j/x_i)} \bigg|_{x_1=1/x_2 \cdots x_n}
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An Example From Invariant Theory 3/4

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\]

the constant term of which is what we want. Now the second one is also simple . . .
We have a simple proof by working on an intermediate crude G.F.

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\frac{1}{1-x_1 \cdots x_n} \prod_{i=1}^{n} \frac{1}{(1-qx_i)(1-q/x_i)} \prod_{1 \leq i < j \leq n} \frac{1-x_j/x_i}{(1-qx_j/x_i)}
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Second, it has a simple connection with our problem: Taking constant term in \( x_1 \) gives a sum of two terms. The first one . . . , the second one:

\[
\frac{1}{(1-q^{-1}x_2 \cdots x_n)(1-q^2)} \prod_{i=2}^{n} \frac{1}{(1-qx_i)(1-q/x_i)} \prod_{2 \leq i < j \leq n} \frac{1-x_j/x_i}{(1-qx_j/x_i)} \times \prod_{j=2}^{n} \frac{1-x_jq}{1-x_jq^2}
\]

which simplifies to
We have a simple proof by working on an intermediate crude G.F.

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\frac{1}{1 - x_1 \cdots x_n} \prod_{i=1}^{n} \frac{1}{(1 - q x_i)(1 - q / x_i)} \prod_{1 \leq i < j \leq n} \frac{1 - x_j / x_i}{(1 - q x_j / x_i)}
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\]
Outline

1. MacMahon’s Partition Analysis
   History
   Iterated Laurent Series

2. Partial Fraction Algorithm

3. Working by Hand
   To Obtain the Crude Generating Functions
   Evaluating the Constant Term
   Examples of Application

4. Perspectives
Those Have Been Done

The partial fraction algorithm has been implemented by the Maple package Ell2.mpl (an update of Ell.mpl) at http://www.combinatorics.net.cn/homepage/xin/MPA.mht
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4. Iterated Laurent series is a special case of Malcev-Neumann series arose from algebra. Using MN-series, I gave a simplification of Stanley’s monster reciprocity theorem.
Other Packages

2. Maple packages by S. Corteel, G. Han, C. Savage and others
3. J. Stembridge’s posets package based on Stanley’s work.
The End

Thank you!