The Ring of Malcev-Neumann Series and the Residue Theorem

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A New Theory on Residue (Constant Term) Evaluation

The theory of Malcev-Neumann series

The theory of iterated Laurent series

And the residue theorem

Give us a large **freedom** for evaluating constant terms in a general **frame**.

Three Major Applications

- Evaluation of combinatorial sums (G. P. Egorychev, 1977)
- Lattice path enumeration (Gessel, 1980 and Bousquet-Mélou M. and G. Schaeffer, 2002)
- MacMahon's partition analysis (Andrews et al., 2001)

Outline of This Talk

1. The theory of iterated Laurent series.
2. MacMahon's partition analysis as an application of our theory.
3. The theory of MN-series.
4. The Univariate Residue Theorem (URT) for MN-series.

The Field of Iterated Laurent Series

From

▼ Complex Analysis

to

▲ The Field of Iterated Laurent Series

Motivation: Series Expansions

A **formal series expansion** of a rational function $f(x_1, \ldots, x_n)$:

$$f(x_1, \ldots, x_n) = \sum_{i_1, \ldots, i_n = -\infty}^{\infty} a_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n}$$

is used in many situations. There is no trouble in the one variable case.

In the multivariate case, we do not have trouble in expanding

$$\frac{x+y}{1-x-y-x^2} = (x+y) \sum_{n \geq 0} (x+y+x^2)^n,$$

But we **do** have trouble in expanding a rational function like

$$\frac{x+y}{x-y}.$$

Series Expansion of $\dfrac{1}{A-B}$ For $A, B \in \mathbb{C}$

Let $A \neq B$. We can express $\dfrac{1}{A-B}$ in terms of A and B:

$$\frac{1}{A-B} = \begin{cases} \frac{1}{B}\frac{-1}{1-A/B} = -\sum_{n\geq 0} A^n/B^{n+1}, & \text{if } A < B, \\ \frac{1}{A}\frac{1}{1-B/A} = \sum_{n\geq 0} B^n/A^{n+1}, & \text{if } A > B. \end{cases} \tag{1}$$

Observation: in order to get the expansion of $1/(A-B)$, A and B must be "comparable".

Note: the above expansions do not hold if we let $A \neq B$ be in an arbitrary field K (e.g. $\mathbb{Z}_p$), since infinite sum of nonzero elements in K makes no sense.

Series expansion of $\frac{1}{A-B}$ in $K((x))$

Let $A \neq B \in K((x))$. If $\eta = B/A \in xK[[x]]$, then

$$\frac{1}{A-B} = \frac{1}{A} \cdot \frac{1}{1-\frac{B}{A}} = \sum_{n\geq 0} B^n/A^{n+1},$$

since $\frac{1}{1-\eta} = \sum_{n\geq 0} \eta^n$ holds by the composition law in $K[[x]]$.

This can be understood as $x = o(1)$, and $\eta \in xK[[x]] \Leftrightarrow \eta = o(1)$, or $B = o(A)$.

Series expansion of $\frac{1}{A-B}$ in $K((x))((t))$

Let $A \neq B$ be two series in $K((x))((t))$, the field of **double Laurent series**.

Similar as before, we want it hold for $B = o(A)$ that

$$\frac{1}{A-B} = \sum_{n\geq 0} B^n/A^{n+1}.$$

Obviously we can let $t = o(x)$. But if $A = x^2, B = t$, we are in trouble expanding $\frac{1}{x^2-t}$.

The solution is simply

$$t = o(c), \forall c \neq 0 \in K((x)).$$

Generalization to the Multivariate Case

◇ The field $K\langle\langle x_1, x_2, \ldots, x_n \rangle\rangle$ of ***iterated Laurent series*** is inductively defined as $K\langle\langle x_1, \ldots x_{n-1} \rangle\rangle((x_n))$, with $K\langle\langle x_1 \rangle\rangle = K((x_1))$.

◇ An element in $K\langle\langle x_1, x_2, \ldots, x_n \rangle\rangle$ is first treated as a Laurent series in x_n, then a Laurent series in x_{n-1}, and so on.

◇ This means: for any positive integers $i < j$ and k, $x_j = o(x_i^k)$. E.g. the expansion of $\dfrac{1}{x_2 - x_4^3}$ is clear.

◇ Equivalently,

$$x_i = o(c), \forall c \neq 0 \in K\langle\langle x_1, \ldots, x_{i-1} \rangle\rangle.$$

Some Embeddings into $K\langle\langle x_1, \ldots, x_n \rangle\rangle$

- ⁎ The ring of polynomials $K[x_1, \ldots, x_n]$.
- ⁎ The ring of Laurent polynomials $K[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-n}]$.
- ⁎ The field of rational functions $K(x_1, \ldots, x_n)$.
- ⁎ The ring of formal power series $K[[x_1, \ldots, x_n]]$.
- ⁎ The ring of multivariate Laurent series

$$K((x_1, \ldots, x_n)) := K[[x_1, \ldots, x_n]][x_1^{-1}, \ldots, x_n^{-n}].$$

Not-So-Well-Known Embeddings into $K\langle\langle x_1, \ldots, x_n \rangle\rangle$

⁕ The ring $K[[x, y/x]]$ was studied by Gessel (1980).

⁕ The ring $K[x, y, x^{-1}, y^{-1}][[t]]$ was studied by Bousquet-Mélou M. and G. Schaeffer, (2002).

⁕ The ring $K[\lambda_1, \ldots, \lambda_r, \lambda_1^{-1}, \ldots, \lambda_r^{-1}][[x_1, \ldots, x_n]]$ was applied to MacMahon's partition analysis.

Iterated Laurent Series Was Mentioned by ...

⁜ Physicists, e.g., Wilson, 1962, used the notation

$$1 >> x_1 >> \cdots >> x_n$$

to do integration in complex analysis.

⁜ Stanley, 1974, tried this approach in proving his well-known Combinatorial Reciprocity Theorems.

Zeilberger's Identity

Zeilberger (1999) proved a Conjecture of Chan et al. (2000) by showing an identity **equivalent to** the following

$$\underset{x_1}{\mathrm{CT}}\cdots\underset{x_n}{\mathrm{CT}}\,\frac{1}{\prod_{i=1}^{n}(1-x_i)}\,\frac{1}{\prod_{i<j}(x_i-x_j)}=C_1\cdots C_{n-1},$$

where

$$C_n=\frac{1}{n+1}\binom{2n}{n}$$

is the Catalan number.

The Iterated Constant Term Operator

- Baldoni-Silva and Vergne (2001): this identity should be interpreted as taking ***iterated constant terms***; i.e., in applying CT_{x_n} to the displayed rational function, we expand it as a Laurent series in x_n; the result is still a rational function and we can apply $\mathrm{CT}_{x_{n-1}}, \dots, \mathrm{CT}_{x_1}$ to it iteratively.

  Note that in this definition, the operator CT_{x_i} does **not** commute with CT_{x_j}.

- Equivalent to: expand the rational function in $K\langle\langle x_1, \dots, x_n \rangle\rangle$, and take the constant terms in $x_1, \dots, x_n$, where the order does not matter after **s**pecifying **t**he **f**ield.

A Natural Definition of the Constant Term

Definition (Natural Definition).

$$\operatorname*{CT}_{x_j} \sum_{(i_1,\ldots,i_n)\in\mathbb{Z}^n} a_{i_1,\ldots,i_n} x_1^{i_1}\cdots x_n^{i_n} := \sum_{(i_1,\ldots,i_n)\in\mathbb{Z}^n, i_j=0} a_{i_1,\ldots,i_n} x_1^{i_1}\cdots x_n^{i_n},$$

where $a_{i_1,\ldots,i_n}$ *belongs to* K*.*

The set of all **formal** series does not form a ring. We can not do multiplications in it, so that the application of the constant term is limited.

E.g. $\sum_{n=-\infty}^{\infty} x^n \cdot \sum_{n=-\infty}^{\infty} x^n$ does not exist.

The Fundamental Structure of $K\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$

The *support* of a formal series is the set of the powers in its nonzero terms, which is a subset of $\mathbb{Z}^n$. For instance, the support of $2x_1^2 + x_1 x_2 + x_2^{-1}$ is $\{\,(2,0),(1,1),(0,-1)\,\}$.

We use the **reverse** lexicographical order on $\mathbb{Z}^n$.

Proposition (Fundamental structure). *A formal series in* $\mathbf{x}$ *belongs to* $K\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$ *if and only if it has a well-ordered support.*

Fact: ★ *Any subset of a well-ordered set is well-ordered* ★

Consequences of the Fundamental Structure

$P1.$ $\operatorname*{CT}_{x_i} : K\langle\langle x_1, \ldots, x_n \rangle\rangle \to K\langle\langle x_1, \ldots, \hat{x}_i, \ldots, x_n \rangle\rangle.$

—— necessary to make the natural definition applicable.

$P2.$ $\operatorname*{CT}_{x_k} \sum_i F_i = \sum_i \operatorname*{CT}_{x_k} F_i.$

—— the key to converting many enumeration problems into simple algebraic computations.

$P3.$ $\operatorname*{CT}_{x_i} \operatorname*{CT}_{x_j} F = \operatorname*{CT}_{x_j} \operatorname*{CT}_{x_i} F.$

—— the constant term evaluations could be significantly simplified.

Basic Computation Rules for $K\langle\langle x_1, \ldots, x_n \rangle\rangle$

Let $F, G \in K\langle\langle x_1, \ldots, x_n \rangle\rangle$.

1. Linearity. $\mathrm{CT}_{x_i}\, aF + bG = a\,\mathrm{CT}_{x_i}\, F + b\,\mathrm{CT}_{x_i}\, G$,
if a, b are independent of x_i.

2. Easy Fact: If F can be written as $\sum_{k\geq 0} a_k x_i^k$, then

$$\underset{x_i}{\mathrm{CT}}\, F = \left. F\right|_{x_i=0}.$$

3. Well-Known Fact: $\underset{x_i}{\mathrm{Res}}\, \dfrac{\partial}{\partial x_i} F = 0$.

A Quick Example

Show that: $\sum_{k=0}^{q}(-1)^k\binom{p}{k} = (-1)^q\binom{p-1}{q}, \quad p > q.$

By the binomial theorem, we have:

$$\begin{aligned}
\sum_{k=0}^{q}(-1)^k\binom{p}{k} &= \sum_{k=0}^{q}(-1)^k \operatorname*{CT}_x \frac{(1+x)^p}{x^k} \\
&= \operatorname*{CT}_x (1+x)^p \sum_{k=0}^{q}\left(-x^{-1}\right)^k \\
&= \operatorname*{CT}_x x(1+x)^{p-1}(1-(-x)^{-q-1}) \\
&= \operatorname*{CT}_x (-1)^q(1+x)^{p-1}x^{-q} \\
&= (-1)^q\binom{p-1}{q}.
\end{aligned}$$

The Idea of Iterated Laurent Series

▼ The idea of iterated Laurent series is to define a total ordering on its variables.

▲ We can consider $K\langle\langle x_{\sigma(1)}, \ldots, x_{\sigma(n)} \rangle\rangle$, where σ is a permutation on $\{1, \ldots, n\}$.

⧫ Rational functions may have different expansions in $K\langle\langle x_{\sigma(1)}, \ldots, x_{\sigma(n)} \rangle\rangle$ for different σ. For instance: the expansion of $\frac{1}{x+y}$ in $K\langle\langle x, y \rangle\rangle$ is different from that in $K\langle\langle y, x \rangle\rangle$.

★ In $K\langle\langle x_1, \ldots, x_n \rangle\rangle$, the total ordering on its group of monomials leads to the theory of MN-series.

MacMahon’s Partition Analysis

Application of

▼ Iterated Laurent series

▲ Partial fraction decomposition of rational functions

to

♦ MacMahon’s Partition Analysis

To Start With …

· Goal: to count solutions to a system of linear Diophantine equations and inequalities, and lattice points in convex polytopes.

· Idea: to introduce new variables $\lambda_1, \lambda_2, \ldots$ to replace the linear constraints, so that the problems are converted into constant term evaluations.

· It has been given a new life by Andrews et. al. (2001) in a series of papers.

Counting Solutions to a Linear Diophantine Equation

Example: Count the nonnegative integral solutions to the linear Diophantine equation $2a_1 - 3a_2 + 2 = 0$.

The generating function of such solutions is:

$$\sum_{\substack{a_1,a_2\geq 0\\ 2a_1-3a_2+2=0}} x_1^{a_1}x_2^{a_2} = \sum_{a_1,a_2\geq 0} \underset{\lambda}{\mathrm{CT}}\, \lambda^{2a_1-3a_2+2}x_1^{a_1}x_2^{a_2}$$

$$= \underset{\lambda}{\mathrm{CT}} \frac{\lambda^2}{(1-\lambda^2 x_1)(1-\lambda^{-3}x_2)}.$$

⋆ We will come back to this example **later**.

The Definition of Elliott-Rational Function

Obviously, r linear equations can be replaced by introducing r variables $\lambda_1, \ldots, \lambda_r$. Thus the problem is converted into evaluating the C. T. of the special rational functions:

Definition . *An Elliott-rational function is a rational function that can be written in such a way that its denominator can be factored into products of one monomial minus another, with the* 0 *monomial allowed.*

Example. $\dfrac{\lambda^2}{(1-\lambda^2 x_1)(1-\lambda^{-3} x_2)}$ is an Elliott-rational function.

Reducing From Multivariate Case to Univariate Case

The following theorem **reduces** the multivariate case to a univariate case by iteration:

Lemma (Elliott Reduction Identity). *For positive integers j and k,*

$$\frac{1}{(1-x\lambda^{j})(1-y\lambda^{-k})}=\frac{1}{1-xy\lambda^{j-k}}\left(\frac{1}{1-x\lambda^{j}}+\frac{1}{1-y\lambda^{-k}}-1\right).$$

Theorem . *If F is Elliott-rational, then the constant terms of F in a field of iterated Laurent series are still Elliott-rational.*

MacMahon's Ω Operators

Definition .

$$\underset{\geq}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\ldots,s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := \sum_{s_1=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} A_{s_1,\ldots,s_r},$$

$$\underset{=}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\ldots,s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := A_{0,\ldots,0}.$$

A **formal treatment** by Han (2003): everything is in the ring $K[\Lambda, \Lambda^{-1}][[\mathbf{x}]]$, where Λ^{-1} is short for $\lambda_1^{-1}, \ldots, \lambda_r^{-1}$, and $\mathbf{x}$ short for $x_1, \ldots, x_n$.

The Starting Point of Andrews

In the work by Andrews et al., the problem was reduced to evaluating the constant term (with respect to λ) of

$$\frac{\lambda^k}{\prod_{1\le i\le m}(1-\lambda^{j_i}x_i)\prod_{1\le i\le n}(1-y_i/\lambda^{k_i})}, \tag{2}$$

where $j_i, k_i \in \mathbb{N}, k \in \mathbb{Z}$.

My Approach: The Working Field and the Operators

- Consider the embedding $K[\Lambda, \Lambda^{-1}][[\mathbf{x}]] \hookrightarrow K\langle\langle \Lambda, \mathbf{x} \rangle\rangle$.
- Define a new operator PT_{λ}:

$$\underset{\lambda}{\mathrm{PT}} \sum_{m=-\infty}^{\infty} a_m \lambda^m := \sum_{m=0}^{\infty} a_m \lambda^m.$$

- MacMahon's operators can be realized as:

$$\underset{\geq}{\Omega}\, F(\Lambda, \mathbf{x}) = \underset{\Lambda}{\mathrm{PT}}\, F(\Lambda, \mathbf{x}) \Big|_{\Lambda=(1,\ldots,1)},$$

$$\underset{=}{\Omega}\, F(\Lambda, \mathbf{x}) = \underset{\Lambda}{\mathrm{CT}}\, F(\Lambda, \mathbf{x}) = \underset{\Lambda}{\mathrm{PT}}\, F(\Lambda, \mathbf{x}) \Big|_{\Lambda=(0,\ldots,0)}.$$

⋆ Hence we can compute $\mathrm{PT}_{\Lambda} F$ instead of $\Omega_{\geq} F$ and $\Omega_{=} F$.

My Starting Point

- We reduce the problem into evaluating $\mathrm{PT}_\lambda F(\lambda)$ with

$$F(\lambda) = \frac{P(\lambda)}{\prod_{1\le i\le n}(\lambda^{j_i} - z_i)},$$

where $P(\lambda)$ is a polynomial in λ, $j_i \in \mathbb{P}$, and z_i's are independent of λ. **N**ote that z_i's are allowed to be 0.

- Fact: $F(\lambda)$ is a rational function with coefficients in a **field**. For instance, when $\lambda = \lambda_1$, the coefficients field is $K(\lambda_2, \dots, \lambda_r, \mathbf{x})$.

- Apply the partial fraction decomposition of rational functions to $F(\lambda)$. Note that the idea of using PFD in this context was first used by Stanley (1974), but was thought to be impractical without using a computer.

Application of PFD

Theorem . *Suppose that the factors in the denominator of F are pairwise relatively prime, and that the PFD of F is*

$$F = f(\lambda) + \sum_{1\le s\le n} \frac{p_s(\lambda)}{\lambda^{j_s} - z_s}, \tag{3}$$

where $f(\lambda)$ is a polynomial in λ, and $p_s(\lambda)$ is a polynomial of degree less than j_s for each s. Then

$$\operatorname*{PT}_{\lambda} F = f(\lambda) + \sum_{s} \frac{p_s(\lambda)}{\lambda^{j_s} - z_s},$$

where the sum ranges over all s such that $\lambda^{j_s} = o(z_s)$.

An Example

Example: Recall that the generating function of nonnegative integral solutions to the linear Diophantine equation $2a_1 - 3a_2 + 2 = 0$ is given by $\mathrm{CT}_\lambda\, F(\lambda)$, where

$$\begin{aligned} F(\lambda) &= \frac{\lambda^2}{(1-\lambda^2 x_1)(1-\lambda^{-3} x_2)} = \frac{\lambda^5}{(1-\lambda^2 x_1)(\lambda^3 - x_2)} \\ &= \frac{-1}{x_1} + \frac{\frac{1+{x_1}^2 x_2 \lambda}{x_1(1-{x_1}^3 {x_2}^2)}}{(1-\lambda^2 x_1)} + \frac{\frac{x_2}{1-x_1^3 x_2^2}\left({x_1}^2 {x_2}^2 + x_1 x_2 \lambda + \lambda^2\right)}{\lambda^3\left(1-\frac{x_2}{\lambda^3}\right)} \end{aligned}$$

Thus $$\operatorname*{CT}_{\lambda} F(\lambda) = -\frac{1}{x_1} + \frac{1}{x_1\left(1-{x_1}^3 {x_2}^2\right)} = \frac{x_1^2 x_2^2}{1-x_1^3 x_2^2}$$

Proof of the Previous Theorem

1: The PFD (3) of F is an identity in $K\langle\langle\Lambda, \mathbf{x}\rangle\rangle$.

2: The PT_λ operator is linear.

3: $$z_s = o(\lambda^{j_s}) \Rightarrow \frac{p_s(\lambda)}{\lambda^{j_s} - z_s} = \frac{p_s(\lambda)}{\lambda^{j_s}} \frac{1}{1 - \frac{z_s}{\lambda^{j_s}}} = \frac{p_s(\lambda)}{\lambda^{j_s}} \sum_{m\geq 0} \frac{z_i s^m}{\lambda^{m j_s}}$$

$$\Rightarrow \underset{\lambda}{\mathrm{PT}} \frac{p_s(\lambda)}{\lambda^{j_s} - z_s} = 0$$

4: $$\lambda^{j_s} = o(z_s) \Rightarrow \frac{p_s(\lambda)}{\lambda^{j_s} - z_s} = \frac{p_s(\lambda)}{-z_s} \frac{1}{1 - \frac{\lambda^{j_s}}{z_s}} = \frac{p_s(\lambda)}{-z_s} \sum_{m\geq 0} \frac{\lambda^{m j_s}}{z_s^m}$$

$$\Rightarrow \underset{\lambda}{\mathrm{PT}} \frac{p_s(\lambda)}{\lambda^{j_s} - z_s} = \frac{p_s(\lambda)}{\lambda^{j_s} - z_s}.$$

The Evaluation of $p_s(\lambda)$, as in the Theorem

A general algorithm is needed to find $p_s(\lambda)$.

- Suppose that the coefficient field of $F(\lambda)$ is $K(y_1, \ldots, y_m)$. The traditional algorithm for finding the PFD is exponential to m, since the general division algorithm is so.

- We are applying a new algorithm for PFD, which we will not discuss here, to find $p_s(\lambda)$. This algorithm is fast in obtaining $p_s(\lambda)$, since it only involves:

The Division Algorithm

The **remainder** of $p(\lambda)$ when divided by $(\lambda^j - a)$ can be obtained by replacing λ^d with $\lambda^{\mathrm{rmd}(d,j)} a^{\lfloor d/j \rfloor}$ in $p(\lambda)$ for all d, where $\mathrm{rmd}(d,j)$ is the remainder of d divided by j.

Example. The remainder of λ^{10} divided by $\lambda^4 - a$ is $a^2\lambda^2$. In this case, $d = 10$ and $j = 4$.

In particular, when $j = 1$, we just replace λ by a.

The Solution of $p_s(\lambda)$

Let

$$\mathcal{F}(\lambda^j - a, \lambda^k - b) = \frac{\sum_{i=0}^{j'-1} \lambda^{ik'} b^{j'-1-i}}{a^{k'} - b^{j'}},$$

where $j' = j/\gcd(j,k)$ and $k' = k/\gcd(j,k)$.

Theorem . *The polynomial $p_s(\lambda)$, as in the PFD of F, equals the remainder of*

$$P(\lambda) \prod_{i=1, i \neq s}^{n} \mathcal{F}(\lambda^{j_s} - z_s, \lambda^{j_i} - z_i),$$

when divided by $\lambda^{j_s} - z_s$ as a polynomial in λ.

An Example

$$F(\lambda) = \frac{\lambda^2}{(1-\lambda^2 x_1)(1-\lambda^{-3}x_2)} = \frac{\lambda^5}{-x_1(\lambda^2 - 1/x_1)(\lambda^3 - x_2)}$$

Let $a = 1/x_1$, $b = x_2$.

$$\begin{aligned}\frac{a^3-b^2}{(\lambda^2-a)(\lambda^3-b)} &= \frac{\lambda^6-b^2}{(\lambda^2-a)(\lambda^3-b)} - \frac{\lambda^6-a^3}{(\lambda^2-a)(\lambda^3-b)} \\ &= \frac{\lambda^3+b}{\lambda^2-a} + \frac{*}{\lambda^3-b}\end{aligned}$$

$$\lambda^5 \frac{\lambda^3+b}{\lambda^2-a} = \frac{\lambda^8+b\lambda^5}{\lambda^2-a} = \text{poly} + \frac{a^4+a^2b\lambda}{\lambda^2-a}$$

The Steps of Finding $\operatorname*{CT}_{\Lambda} F(\Lambda, \mathbf{x})$

1. Choose a filed, e.g., $K\langle\langle\lambda_1, \ldots, \lambda_r, x_1, \ldots, x_n\rangle\rangle$, to work with.

2. Choose an i and take the constant term in λ_i of F.

3. Apply the previous Theorem to find the PFD of F w.r.t λ_i.
Recall that we will not need all the summands.

4. For each summand that we have got, repeat the above procedure.

Another Example

Verify Zeilberger's identity for $n = 3$ in $K\langle\langle x_1, x_2, x_3\rangle\rangle$.

$$\begin{aligned}
&\underset{x_1,x_2,x_3}{\mathrm{CT}} \frac{1}{(1-x_1)(1-x_2)(1-x_3)(x_1-x_2)(x_2-x_3)(x_1-x_3)} \\
&\qquad = \underset{x_2,x_3}{\mathrm{CT}} \frac{1}{(1-x_2)^2(1-x_3)^2(x_2-x_3)} \\
&\qquad = \underset{x_2}{\mathrm{CT}} \frac{1}{(1-x_2)^2 x_2} = 2,
\end{aligned}$$

since

$$\frac{1}{(1-x_1)(x_1-x_2)(x_1-x_3)} = \frac{\frac{1}{(1-x_2)(1-x_3)}}{1-x_1} + \frac{*}{x_1-x_2} + \frac{*}{x_1-x_3}.$$

Summary of MacMahon's Partition Analysis

⋇ MacMahon's partition analysis convert counting system of linear Diophantine equation into the evaluation of Elliott-rational function.

⋇ Elliott's reduction identity allows us to reduce the multivariate case to the univariate case.

⊛ For the univariate case, we deal with the general situation

$$F(\lambda) = \frac{P(\lambda)}{\prod_{1 \le i \le n} (\lambda^{j_i} - z_i)}$$

in the working field $K\langle\langle \Lambda, \mathbf{x} \rangle\rangle$.

⊛ The last thing is to obtain $\mathrm{PT}_\lambda F$ in the 4 steps as we just saw.

The Field of Malcev-Neumann Series

A Generalized Setting of

Iterated Laurent Series

Leads to

A Generalized Residue Theorem

The History

Let K be a field, and let G be a group. Then

$$K[G] = \{ \sum_{i=1}^{m} r_i g_i \mid r_i \in K, g_i \in G \}$$

is a *group algebra*.

A Historical Problem: Does $K[G]$ has a zero divisor?

Malcev (1948) and Neumann (1949) showed:

★ $K[G]$ has no zero divisors if G is a totally ordered group.

The Definition

A ***totally ordered group*** is a group G equipped with a total ordering $\leq$ that is compatible with the multiplication of G; i.e., for all $x, y, z \in G$, $x < y$ implies that $zx < zy$ and that $xz < yz$.

$$K_w[G] := \{ \sum_{g \in S} a_g g \mid a_g \in K,\ S \text{ is a well-ordered subset of } G \}$$

Elements in $K_w[G]$ are called ***MN-series***, short for ***Malcev-Neumann series***.

The Field of MN-Series

Theorem (Malcev-Neumann). *If K is a field, and G is a totally ordered group, then $K_w[G]$ is a K-division algebra, which includes $K[G]$ as a subalgebra.*

★ The theorem ⇒ if G is abelian, then $K_w[G]$ is a field, called **the field of MN-series**.

· This theorem will be proved, as we shall see, by the lemma of **Composition Law**.

Iterated Laurent Series Are Special MN-series

Let G be an abelian group written additively. We can identify g with t^g, so that $t^g t^h = t^{g+h}$.

$\circledast$ $\mathbb{Z}$ is a totally ordered additive group with the natural ordering.

$K_w[\mathbb{Z}] \simeq K((x))$ is the field of Laurent series.

$\circledast$ $\mathbb{Z}^n$ is a totally ordered additive group with the reverse lexicographical ordering.

Proposition . $K_w[\mathbb{Z}^n] \simeq K\langle\langle x_1, x_2, \ldots, x_n \rangle\rangle$, *is a field of iterated Laurent series.*

· Iterated Laurent series are the most useful special class of MN-series.

The Order of An MN-Series

· Let $\eta \in K_w[G]$. The order of η is defined to be

$$\operatorname{ord}(\eta) := \min_{g \in G}\{\, g \in G \mid \eta \text{ has nonzero coefficient in } g \,\},$$

· The ***initial term*** of η is the term having order $\operatorname{ord}(\eta)$.

· B is the initial term of η means $\eta = B +$ higher ordered terms, or $\eta = B + o(B)$.

Similar to what we have in complex analysis:

· For all $\eta, \tau \in K_w[G]$,

$$\operatorname{ord}(\eta) = \operatorname{ord}(\tau) \Leftrightarrow \eta = O(\tau),$$

$$\operatorname{ord}(\eta) > \operatorname{ord}(\tau) \Leftrightarrow \eta = o(\tau).$$

The Composition Law

Lemma (Composition Law for MN-Series). *If* $\eta \in K_w[G]$ *with* $\mathrm{ord}(\eta) > 1$, *then* $\sum_{i\geq 0} b_i \eta^i$ *is in* $K_w[G]$, *where* $b_i \in K$.

In other words, if $\eta = o(1)$ then the composition law holds. However, this natural result is far from being trivial.

Note that any $\eta \neq 0$ can be written as $A - B$ with $B = o(A)$,

$$\frac{1}{\eta} = \frac{1}{A-B} = \frac{1}{A} \sum_{m \geq 0} (B/A)^m.$$

$\triangleright$ Thus the lemma $\Rightarrow \dfrac{1}{\eta} \in K_w[G]$ for all $\eta \neq 0$, which is to say that $K_w[G]$ is a K-division algebra.

The Intrinsic Ideas of the Theory of MN-Series

- The theory of MN-series gives us a guide in series expansion by giving a total ordering.
- The total ordering on the group of monomials, which is G if the field is $K_w[G]$, plays a central role in our theory.
- The study of MN-series comes from the study of the residue theorem. As we will see, the residue theorem involves two different fields of MN-series.

A Simple Application to Combinatorial Sums

Lemma . *In the field* $K((\alpha^{-1}))$, *we have*

$$\underset{\alpha}{\mathrm{CT}}\, \frac{(1+\alpha)^n}{\alpha^k} = \begin{cases} \binom{n}{k} & \textit{if } n \geq 0 \\ 0 & \textit{if } n < 0 \end{cases} . \tag{4}$$

Proof: Working in $K((\alpha^{-1}))$ means that we shall treat $\alpha^{-1} = o(1)$, or equivalently $1 = o(\alpha)$. For $n < 0$, we let $m = -n > 0$.

$$\begin{aligned} \frac{(1+\alpha)^n}{\alpha^k} &= \alpha^{-k} \frac{1}{(1+\alpha)^m} = \alpha^{-k-m} \frac{1}{(1+1/\alpha)^m} \\ &= \alpha^{-k-m} \sum_{i \geq 0} \binom{m+i-1}{i} \alpha^{-i}. \end{aligned}$$

A Combinatorial Sum Evaluation

Example . *Evaluate the expectation of the smallest number when choosing r distinct elements from the set $\{1, 2, \ldots, n\}$.*

The steps:

- ◦ We need to evaluate the sum $\sum_{i=1}^{n} i\binom{n-i}{r-1}$, divided by $\binom{n}{r}$.
- ◦ We will show that the sum is $\binom{n+1}{r+1}$, so that the expectation is $\binom{n+1}{r+1}/\binom{n}{r} = (n+1)/(r+1)$.
- ◦ The working field is $K((\alpha^{-1}))$.

The Solution

$$
\begin{aligned}
\sum_{i=1}^{n} i\binom{n-i}{r-1} &= \sum_{i\geq 1} i \operatorname*{CT}_{\alpha^{-1}} \frac{(1+\alpha)^{n-i}}{\alpha^{r-1}} \\
&= \operatorname*{CT}_{\alpha^{-1}} \frac{(1+\alpha)^{n}}{\alpha^{r-1}} \sum_{i\geq 1} i \frac{1}{(1+\alpha)^{i}} \\
&= \operatorname*{CT}_{\alpha^{-1}} \frac{(1+\alpha)^{n}}{\alpha^{r-1}} \frac{1/(1+\alpha)}{\left(1-1/(1+\alpha)\right)^{2}} \\
&= \operatorname*{CT}_{\alpha^{-1}} \frac{(1+\alpha)^{n+1}}{\alpha^{r+1}} \\
&= \binom{n+1}{r+1}
\end{aligned}
$$

The Residue Theorems

From the Residue Theorem in Complex Analysis

To the Residue Theorem for MN-series

From the Univariate Residue Theorem (URT)

To the Multivariate Residue Theorem (MRT)

A Review of Complex Analysis

Recall the **univariate residue theorem** in complex analysis:

Theorem (Classical Result). *Let* γ *be a simple curve in* $\mathbb{C}$*. If* f *is a meromorphic function that has no singularity on* γ*, then*

$$\frac{1}{2\pi i}\oint_{\gamma} f dz = \sum_{a\in E} \operatorname*{Res}_{z=a} f,$$

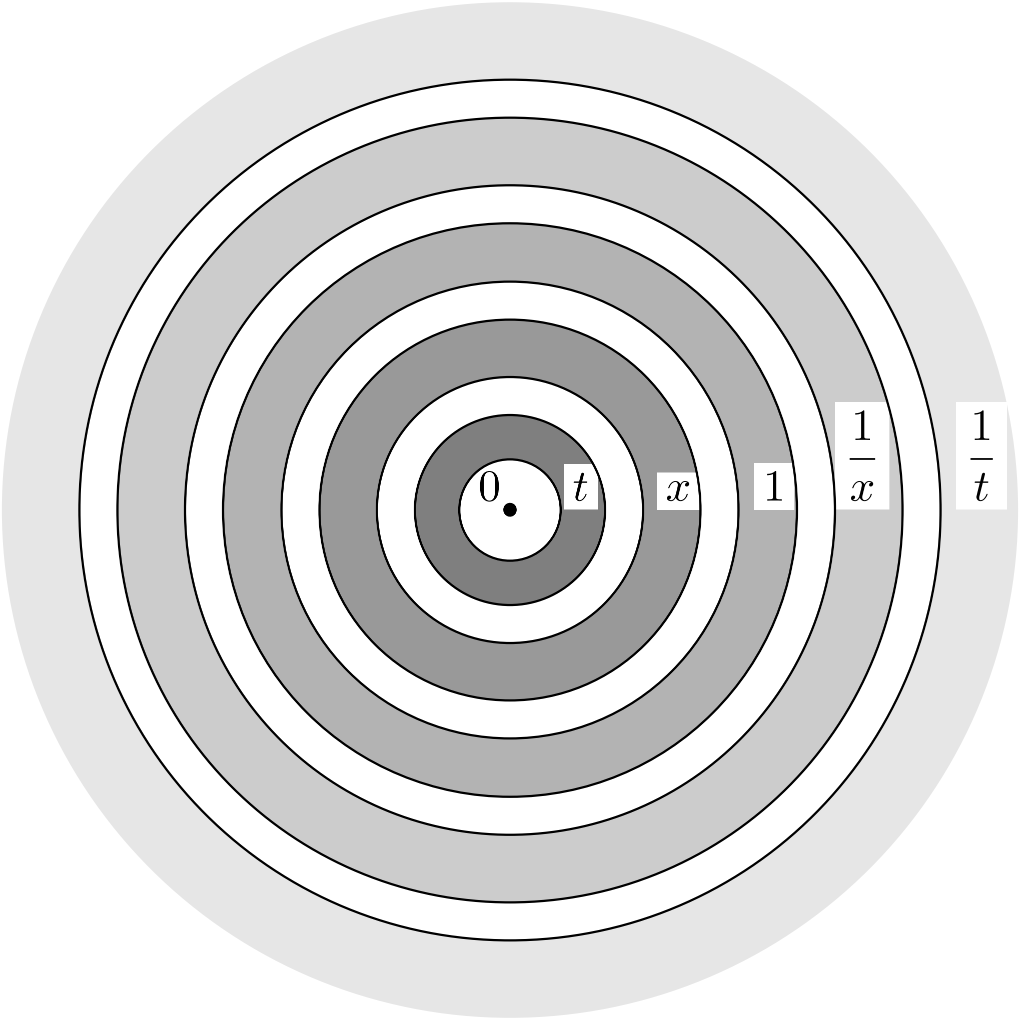
where E *is the set of singularities of* f *that lie inside* γ*.*

Residue of An MN-series

In our settings, e.g., for $f = f(x,t)$ in $K((x))((t))$, $\operatorname{Res}_x f$ is defined to be $[x^{-1}]f$. It can be thought of as

$$\operatorname*{Res}_x F = \frac{1}{2\pi i} \int_\gamma F(z,t)dz,$$

where γ is the curve $|z| = x$, and the plane of complex numbers should be replaced with the plane:



$$\begin{array}{c} \{\,0\,\} \\ \upuparrows \\ tK((x))[[t]] \\ \upuparrows \\ tK((x))[[t]] \cup xK[[x,t]] \\ \upuparrows \\ tK((x))[[t]] \cup K[[x,t]] \\ \upuparrows \\ K((x))[[t]] \\ \upuparrows \\ K((x))((t)) \end{array}$$

The plane of $K((x))((t))$

URT in $K((x))((t))$

Theorem . *Let* $G(x,t), F(x,t) \in K[[x,t]]$. *If* $G(x,0)$ *can be written as* $ax +$ *higher ordered terms with* $a \neq 0 \in K$, *then*

$$\operatorname*{CT}_{x} \frac{x}{G(x,t)} F(x,t) = \left. \frac{F(x,t)}{\frac{\partial}{\partial x} G(x,t)} \right|_{x=X}, \qquad (5)$$

where $X = X(t)$ *is the unique element in* $tK[[t]]$ *such that* $G(X,t) = 0$.

URT In the Field $K_w[G \oplus \mathbb{Z}]$

Let $G \oplus \mathbb{Z}$ is a totally ordered additive group. We let $t^{(0,1)} := x$.

Proposition . *Suppose that $G(x)$ is an MN-series, and that $G(x)$ is a polynomial in x. If $G(x)$ has a unique root X for x such that* $\operatorname{ord}(X) > \operatorname{ord}(x)$, *and if $F(x)$ contains only nonnegative powers in x, then*

$$\operatorname*{CT}_{x} \frac{x}{G(x)} F(x) = \left. \frac{F(x)}{\frac{\partial}{\partial x} G(x)} \right|_{x=X}, \tag{6}$$

· This is the one simple singularity case.

An Example from Lattice Path Enumeration

Compute $\operatorname*{CT}_{y} \dfrac{1}{1-t(x+x^{-1}+y+y^{-1})}$.

$$\begin{aligned}\operatorname*{CT}_{y} \frac{1}{1-t(x+x^{-1}+y+y^{-1})} &= \operatorname*{CT}_{y} \frac{y}{y-t((x+x^{-1})y+y^2+1)}\\ &= \frac{1}{1-(x+x^{-1})t-2tY}\\ &= \frac{1}{\sqrt{(1-(x+x^{-1})t)^2-4t^2}},\end{aligned}$$

where the only root $Y = Y(t) = o(y)$ for y in $y - t(x + x^{-1} + y^2 + 1)$ is:

$$\frac{1-(x+x^{-1})t-\sqrt{(1-(x+x^{-1})t)^2-4t^2}}{2t}.$$

Thank You For Coming and Listening :-)