Enumeration of k-Stack Sortable Matchings and Partitions

2007 Permutation Pattern

Guoce Xin (joint with William Y. C. Chen and Terence Y.J. Zhang)

Center for Combinatorics
Nankai University

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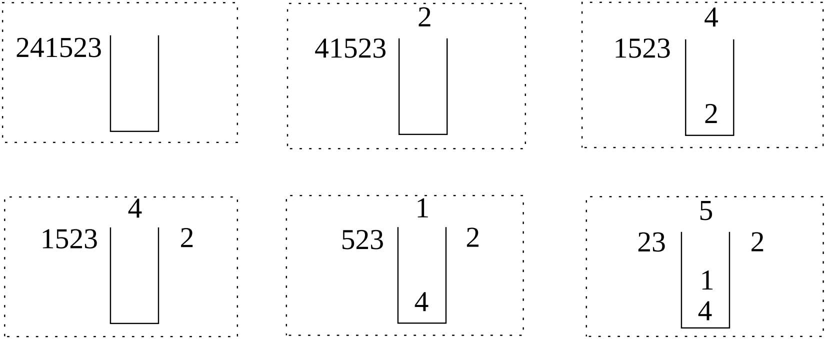
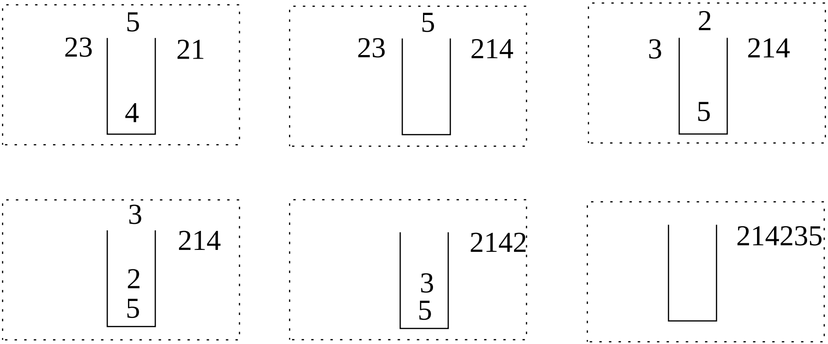
Outline

Knuth's stack sort: An example (part 1/2)



Knuth's stack sort: An example (part 2/2)



A recursive definition

Definition

Let p be separated by its largest element n as $p = w_1 n w_2 n \cdots n w_k$. Then we have

$$s(p) = s(w_1)s(w_2)\cdots s(w_k)N,$$

where N is consist of $(k-1)$ consecutive n's.

Define $s^k(p) = s(s^{k-1}(p))$ for $k \geq 2$. Then an integer sequence p is called k-stack sortable if $s^k(p)$ is weakly increasing.

Results on k-stack sortable permutations

Let $SS_k(n)$ be the number of k-stack sortable permutations of order n.

Knuth $SS_1(n)$ is the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Zeilberger

$$SS_2(n) = \frac{2(3n)!}{(n+1)!(2n+1)!}.$$

- $SS_k(n)$ are unknown for $k \geq 3$ except for some special values.

Set partitions

A partition π of $[n] = \{1, 2, \ldots, n\}$ can be written as $\pi = B_1 - B_2 - \cdots - B_k$, such that

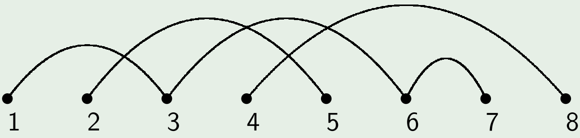
1. $B_i \neq \varnothing$
2. $B_i \cap B_j = \varnothing$, if $i \neq j$
3. $B_1 \cup B_2 \cup B_k = [n]$

By convention, the least element of B_i are in increasing order.
For example $P = \{1, 3, 6\} - \{2, 4\} - \{5, 8\} - \{7\}$ is a partition of [8].

Graph representations and sequential forms

Example

If $P = \{1,3,6,7\} - \{2,5\} - \{4,8\}$, then the graph of P is



The *sequential form* or *canonical sequence* of P is $p = 12132113$, which is obtained by successively reading the block number of $1, 2, 3, \ldots, 8$.

Matchings

Matchings are set partitions with each block of size two.

1. A matching must be on $[2n]$ for some n.
2. Matchings have graph representations.
3. The sequential form of a matching on $[2n]$ must be a rearrangement of $11223344 \cdots nn$.

Stack sorting on partitions

A partition or a matching is said to be k-stack sortable if its canonical sequence is.

Example

If $P = \{1,3,6,7\} - \{2,5\} - \{4,8\}$, then its sequential form is $p = 12132113$.
One stack sort gives $s(p) = 11211233$.
Two stack sort gives $s(s(p)) = 11112233$.
So P is two stack sortable.

Why consider matchings?

The set of permutations can be naturally embedded into the set of matchings. The embedding preserves some nice properties. For example

$$\pi_1\pi_2\cdots\pi_n \mapsto 123\cdots n\pi_n\pi_{n-1}\cdots\pi_1$$

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Standard Young tableaux $\mapsto$ oscillating tableaux.
Increasing subsequence ($12\cdots k$-pattern) $\mapsto$ nesting ($12\cdots kk\cdots 21$).
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We hope something nice to happen for matchings when studying stack sortable properties.

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Search by Maple

Our first try is on matchings, however the number of matchings on $[2n]$ is $1 \cdot 3 \cdot \cdots \cdot (2n-1)$. This grows too fast.

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One can guess that the generating function for general k is

$$\frac{1}{1-C_0x-C_1x^2-\cdots-C_kx^{k+1}}.$$

Proving the above leads to ...

The characterization

The *inversion height* $ih(p)$ of p is defined to be
$ih(p) = \max\{\, p_i - p_j \mid i < j \,\}$.

Let p be the canonical sequence of a partition P. Then the following statements are equivalent:

i. The partition P is k-stack sortable.

ii. The canonical sequence p is $23\cdots(k+1)(k+2)1$-avoiding.

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The key fact: if 62 is a subsequence of p then 1234562 is also a subsequence of p.

Outline

Noncrossing matchings

Theorem

For any positive integer k, the generating function for k-stack sortable noncrossing matchings is

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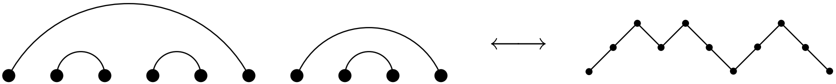
Proof by an example.



Noncrossing partitions

Theorem

The generating function of k-stack sortable noncrossing partitions is

$$\frac{1}{1-\sum_{k\geq s\geq 0, t\geq 0} C_s x^{s+t+1}(-1)^{t-1}\binom{t-1}{k-s}\binom{2s+t}{t}}.$$

Moreover, it is rational.

For example, if $k=1$, the generating function is

$$\frac{1-3\,x+3\,x^2-x^3}{1-4\,x+5\,x^2-3\,x^3} = 1+x+2\,x^2+5\,x^3+13\,x^4+33\,x^5+82\,x^6+206\,x^7+\cdots.$$

Nonnesting Matchings 1/2

A nonnesting matching (i.e., avoiding 1221-pattern) is k-stack sortable if and only if

recall Its inversion height is no more than k.

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By a result of Chen et al. and the new characterization,

k-stack sortable nonnesting matchings $\leftrightarrow$
noncrossing matchings without $(k+2)$-nestings.

Nonnesting Matchings 1/2

Theorem

For any positive integer k, the number of k-stack sortable nonnesting matchings on $[2n]$ is equal to the number of $2n$ step Dyck paths of height at most $k+1$.

If we let $F_{nn}^{(k)}(x)$ be the corresponding generating function, then

$$F_{nn}^{(k)}(x) = \frac{1}{1 - xF_{nn}^{(k-1)}(x)} = \cdots = \cfrac{1}{1 - \cfrac{x}{\ddots \, \cfrac{}{1 - \cfrac{x}{1-x}}}},$$

where $F_{nn}^{(0)}(x) = \frac{1}{1-x}$.

Nonnesting partitions

Even 1-stack sortable partitions can not be characterized by crossing numbers.

For example, 1212313 is the canonical sequence of the partition $P = \{1,3,6\} - \{2,4\} - \{5,7\}$. This partition has no 3-crossing but contains a 231-pattern.

We use a recursive way to find the generating function of 1-stack sortable nonnesting partitions:

$$\frac{1 - x - x^2 + 2\,x^3}{1 - 3\,x + 4\,x^3 - 3\,x^4} = 1 + 2x + 5x^2 + 13x^3 + 34x^4 + 88x^5 + 227x^6 + \cdots .$$

The results

Let $G_k(x)$ be the generating function of k-stack sortable matchings.

Theorem

For given positive integer k, $G_k(x)$ is rational. In particular,

$$G_1(x) = \frac{1-x}{1-2x-x^2} = 1 + x + 3x^2 + 7x^3 + 17x^4 + 41x^5 + 99x^6 + \cdots,$$

$$G_2(x) = \frac{1-2x-2x^2-x^3}{1-3x-2x^2-5x^3-x^4}$$
$$= 1 + x + 3x^2 + 15x^3 + 57x^4 + 217x^5 + 843x^6 + \cdots,$$

$$G_3(x) = \frac{1-3\,x-5\,x^2-9\,x^3-17\,x^4-5\,x^5+5\,x^6+x^7}{1-4\,x-4\,x^2-8\,x^3-42\,x^4-20\,x^5+4\,x^6+8\,x^7+x^8}$$
$$= 1 + x + 3x^2 + 15x^3 + 105x^4 + 561x^5 + 2931x^6 + \cdots.$$

Recursive method for $k = 1$ and $k = 2$

Let $f(n, k; s)$ be the number of k-stack sortable matchings on $[2n]$ starting with the sequence s.
For $k = 1$, the canonical sequence avoids the 231-pattern. Considering all possible positions of the two 1's, we have

$$\begin{aligned} f(n,1) &= f(n,1;11) + f(n,1;121) + f(n,1;1221) \\ &= f(n-1,1) + f(n-1,1) + f(n-2,1). \end{aligned}$$

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Similar argument works for $k = 2$, but we need to consider reductions like

$$\begin{aligned} f(n,2;1231) &= f(n-1,2;12) = f(n-1,2;1) - f(n-1,2;11) \\ &= f(n-1,2) - f(n-2,2). \end{aligned}$$

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For $k \geq 3$, the cases becomes complicated.

The transfer matrix method

An insertion process for k-stack sortable matchings.

1. Starts with a sequence $(m_1, m_2, \cdots, m_{n+k+1}) = (1, 2, \ldots, n+k+1)$.
2. Sequentially insert $1, 2, \ldots, n$ into the sequence, such that i must be inserted between m_i and m_{i+k+1}.
3. If there are no inserted numbers between m_{n+1} and m_{n+k+1}, then remove $m_{n+1}, \ldots m_{n+k+1}$ and we obtain the canonical sequence of a matching.

Every matching obtained this way can be shown to have maximum inversion hight no more than k and is hence k-stack sortable.

An example

Take 1-stack sortable matching 121332454665. The process is:

$$
\begin{aligned}
m_1 m_2 m_3 m_4 m_5 m_6 \mid m_7 m_8 &\to m_1 m_2 1 m_3 m_4 m_5 m_6 \mid m_7 m_8 \\
&\to m_1 m_2 1 m_3 2 m_4 m_5 m_6 \mid m_7 m_8 \\
&\to m_1 m_2 1 m_3 32 m_4 m_5 m_6 \mid m_7 m_8 \\
&\to m_1 m_2 1 m_3 32 m_4 m_5 4 m_6 \mid m_7 m_8 \\
&\to m_1 m_2 1 m_3 32 m_4 m_5 4 m_6 5 \mid m_7 m_8 \\
&\to m_1 m_2 1 m_3 32 m_4 m_5 4 m_6 65 \mid m_7 m_8 \\
&\to m_1 m_2 1 m_3 32 m_4 m_5 4 m_6 65 \\
&= 121332454665
\end{aligned}
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m_1 m_2 m_3 m_4 m_5 m_6 m_7 m_8 &\to \underbrace{m_1 m_2 1 m_3}\, m_4 m_5 m_6 m_7 m_8 \\
&\to m_1\, \underbrace{m_2 1 m_3 2 m_4}\, m_5 m_6 m_7 m_8 \\
&\to m_1 m_2 1\, \underbrace{m_3 3 2 m_4 m_5}\, m_6 m_7 m_8 \\
&\to m_1 m_2 1 m_3 3 2\, \underbrace{m_4 m_5 4 m_6}\, m_7 m_8 \\
&\to m_1 m_2 1 m_3 3 2 m_4\, \underbrace{m_5 4 m_6 5 m_7}\, m_8 \\
&\to m_1 m_2 1 m_3 3 2 m_4 m_5 4\, \underbrace{m_6 6 5 m_7 m_8} \\
&\to m_1 m_2 1 m_3 3 2 m_4 m_5 4 m_6 6 5 \\
&= 121332454665
\end{aligned}
$$

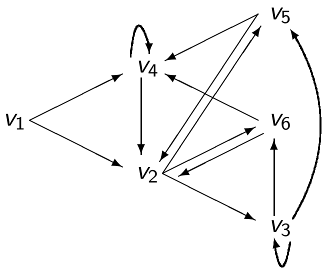
Digraph generating k-stack sortable matchings

In general k-stack sortable matchings are in one-to-one correspondence with certain walks on a digraph D.

1. The vertices are sequences containing $m_0, m_1, \ldots, m_{k+1}$ in increasing order.
2. The vertices possibly contain $-k, -(k-1), \ldots, -1$.
3. The starting point is $v_1 = m_0 m_1 \cdots m_{k+1}$.
4. The vertices and edges can be described in a complicated way.
5. The ending points for k-stack sortable matchings are those vertices of the form $m_0 * * * m_1 m_2 \cdots m_{k+1}$.

The digraph generating 1-stack sortable matchings

The starting point is $v_1 = m_0m_1m_2$. Let v_2, v_3, v_4, v_5, v_6 index $m_0m_10m_2$, $m_0(-1)m_10m_2$, $m_00m_1m_2$, $m_00(-1)m_1m_2$ and $m_0(-1)0m_1m_2$ respectively.



The corresponding matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

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The matrix for $k = 2$ has order 31 and rank 7;

The matrix for $k = 3$ has order 230 and rank 37.

Conclusion:

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Conclusion: The transfer matrix method shows that the generating function is rational, but in general is not suitable for computing the generating function.

What can we expect?

We hope the transfer matrix method works in a similar way, but we can see there are some troubles.

The recursive method works for $k = 1$. Hopefully works for $k = 2$.

Can we expect more?

The result

Theorem

For any positive integer k, the number of k-stack sortable partitions of $[n]$ is

$$S(n,1)+S(n,2)+\cdots+S(n,k+2),$$

where $S(n,r)$ is the Stirling number of second kind.

The corresponding generating function is

$$1+\sum_{t=1}^{k+2}\frac{x^t}{(1-x)(1-2x)\cdots(1-tx)},$$

which is again a rational power series.

The End

Thank you!