

On the Kronecker Powers of $s_{d,d}$

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A joint work with

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The Outline of This Talk

The main object of this talk

Zabrocki's Symmetric function approach.

Wallach's Invariant theory approach.

A remarkable linear diophantine system.

Approach by MacMahon's partition analysis.

The main object

Definition: Let $k, d \geq 1$.

$$c_d(k) = \sum_{\rho \vdash 2d} \frac{\left(\chi_\rho^{(d,d)}\right)^k}{z_\rho}$$

$\chi_\rho^{(d,d)}$: Young's irreducible character

$n!/z_\rho$: the size of the corresponding conjugate class.

We want to study the [Wallach series](#) $W_k(q)$:

$$W_k(q) = 1 + \sum_{d \geq 1} c_d(k) q^d.$$

Main object: the Wallach series for small k .

$$W_1(q) = 1$$

$$W_2(q) = \frac{1}{1 - q}$$

$$W_3(q) = \frac{1}{1 - q^2}$$

$$W_4(q) = \frac{1}{(1 - q)(-q^2 + 1)^2(-q^3 + 1)}$$

$$W_5(q) = \frac{(1 - q) \times NN}{(-q^2 + 1)^8(-q^4 + 1)^3(-q^3 + 1)^6(-q^5 + 1)},$$

where $NN = q^{34} + q^{33} - 2q^{32} - 7q^{31} + 14q^{30} + \dots$

Main Object: A better way to write $W_5(q)$

$W_5(q) = \frac{N}{(1-q^2)^4(1-q^3)(1-q^4)^6(1-q^5)(1-q^6)^5}$, where N is:

$$\begin{aligned} & q^{54} + q^{52} + 16q^{50} + 9q^{49} + 98q^{48} + 154q^{47} + 465q^{46} + 915q^{45} + 2042q^{44} \\ & + 3794q^{43} + 7263q^{42} + 12688q^{41} + 21198q^{40} + 34323q^{39} + 52205q^{38} \\ & + 77068q^{37} + 108458q^{36} + 147423q^{35} + 191794q^{34} + 241863q^{33} \\ & + 292689q^{32} + 342207q^{31} + 386980q^{30} + 421057q^{29} + 443990q^{28} \\ & + 451398q^{27} + 443990q^{26} + 421057q^{25} + 386980q^{24} + 342207q^{23} + 292689q^{22} \\ & + 241863q^{21} + 191794q^{20} + 147423q^{19} + 108458q^{18} + 77068q^{17} \\ & + 52205q^{16} + 34323q^{15} + 21198q^{14} + 12688q^{13} + 7263q^{12} + 3794q^{11} \\ & + 2042q^{10} + 915q^9 + 465q^8 + 154q^7 + 98q^6 + 9q^5 + 16q^4 + q^2 + 1 \end{aligned}$$

Symmetric Function Approach: Basic

The Frobenius Map $F_n : \chi^\lambda \mapsto s_\lambda$.

Now fix n .

Kronecker product:

$$F(\chi^\lambda \chi^\mu) = s_\lambda * s_\mu.$$

$$\text{Or } p_\lambda * p_\mu = \delta(\lambda, \mu) z_\lambda p_\lambda$$

Hall's inner product: $\langle p_\lambda, p_\mu \rangle = \delta(\lambda, \mu) z_\lambda$.

$$\langle s_\lambda, s_\mu \rangle = \delta(\lambda, \mu)$$

$$(\chi^\lambda, \chi^\mu) = \delta(\lambda, \mu)$$

The Wallach series comes in

$$\begin{aligned}
 c_d(k) &= \sum_{\rho \vdash 2d} \frac{\left(\chi_\rho^{(d,d)}\right)^k}{z_\rho} \\
 &= \left\langle \left(\chi^{(d,d)}\right)^k, 1 \right\rangle = \left\langle (s_{(d,d)})^{*k}, s_{2d} \right\rangle \\
 &= \left\langle \left(\chi^{(d,d)}\right)^{k-1}, \chi^{(d,d)} \right\rangle = \left\langle (s_{(d,d)})^{*k-1}, s_{d,d} \right\rangle \\
 &= \left\langle \left(\chi^{(d,d)}\right)^{k-2}, \left(\chi^{(d,d)}\right)^2 \right\rangle = \left\langle (s_{(d,d)})^{*k-2}, (s_{d,d})^{*2} \right\rangle \\
 &= \dots
 \end{aligned}$$

A yet unsolved problem

Open: find a combinatorial interpretation of the integer

$$c_{\lambda\mu\nu} = \sum_{\rho \vdash n} \chi_{\rho}^{\lambda} \chi_{\rho}^{\mu} \chi_{\rho}^{\nu}$$

$$s_{\mu} * s_{\nu} = \sum_{\lambda} c_{\lambda\mu\nu} s_{\lambda}$$

The problem is known for some special cases.

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The problem is known for some special cases.

Theorem.

$$s_{d,d} * s_{d,d} = \sum_{\lambda \vdash 2d} \chi(\lambda \in EO_4) s_{\lambda},$$

where EO_4 denotes the set of partitions of length 4 whose parts are ≥ 0 and all even or all odd.

$W_k(q)$ for k up to 4

$$c_d(1) = \langle s_{d,d}, s_{2d} \rangle = 0 \text{ for } d \geq 1.$$

$$c_d(2) = \langle s_{d,d} * s_{d,d}, s_{2d} \rangle = \langle s_{d,d}, s_{d,d} \rangle = 1.$$

$$c_d(3) = \langle s_{d,d} * s_{d,d}, s_{d,d} \rangle = 1 \text{ if } d \text{ is even, and } 0 \text{ if } d \text{ is odd.}$$

$$c_d(4) = \langle s_{d,d} * s_{d,d}, s_{d,d} * s_{d,d} \rangle = \#\{\lambda \vdash 2d : \lambda \in EO_4\}$$

$$\sum_{\lambda} q^{|\lambda|/2} \chi(\lambda \in E_{\leq 4}) = \frac{1}{(1-q)(1-q^2)(1-q^3)(1-q^4)}$$

$$\sum_{\lambda} q^{|\lambda|/2} \chi(\lambda \in O_{=4}) = \frac{q^2}{(1-q)(1-q^2)(1-q^3)(1-q^4)}$$

Invariant Theory Approach

Let $A = (a_{ij})$ be an $n \times n$ matrix.

A act on polynomials $P(x_1, \dots, x_n)$ by

$$T_A P(x) = P(xA).$$

Let G be a group of $n \times n$ matrices. Then P is G -invariant iff

$$T_A P(x) = P(x), \quad \forall A \in G.$$

Notation: the subspace of G -invariant polynomials is denoted by $\mathbb{C}[x]^G$.

The Hilbert series of $\mathbb{C}[x]^G$

$$\mathbb{C}[x]^G = \mathcal{H}_0(\mathbb{C}[x]^G) \oplus \mathcal{H}_1(\mathbb{C}[x]^G) \oplus \mathcal{H}_2(\mathbb{C}[x]^G) \oplus \cdots ,$$

where $\mathcal{H}_d(\mathbb{C}[x]^G)$ is of homogeneous degree d .

The Hilbert series is defined by

$$H_G(q) = \sum_{d \geq 0} q^d \dim (\mathcal{H}_d(\mathbb{C}[x]^G)) .$$

Example: If G is the set of the identity matrix, then

$$H_G(q) = \sum_{d \geq 0} \binom{d + n - 1}{n - 1} q^d = \frac{1}{(1 - q)^n} .$$

Molien's Formula

If G is a finite group, then

$$H_G(q) = \frac{1}{|G|} \sum_{A \in G} \frac{1}{\det(I - qA)}.$$

If G possesses a unit invariant measure ω , then

$$H_G(q) = \int_{A \in G} \frac{1}{\det(I - qA)} \omega(A).$$

The Wallach series

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \otimes \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} := \begin{bmatrix} a_{1,1}B & a_{1,2}B \\ a_{2,1}B & a_{2,2}B \end{bmatrix}$$

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$$SL[2]^{\otimes k} = \{A_1 \otimes \cdots \otimes A_k : A_i \in SL[2]\}.$$

Theorem (N. Wallach)

For all $k \geq 1$, we have

$$W_k(q^2) = H_{SL[2]^{\otimes k}}(q) = H_{SU[2]^{\otimes k}}(q)$$

A related result

$$\text{Let } T_2 = \left\{ \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} : t\bar{t} = 1 \right\}$$

Theorem

$$H_{T_2^{\otimes k}}(q) = \sum_{d \geq 0} q^{2d} \langle (h_d h_d)^{*k}, s_{2d} \rangle.$$

A remarkable linear Diophantine system: $k = 2$ case

$\langle h_d h_d * h_d h_d, s_{2d} \rangle = \langle h_d, h_d \rangle$ equals

- the number of 2×2 nonnegative integer matrix with row sum and column sum equals d .
- the number of labeling of the vertices of the square, such that the sum of every face equals d .

The $k = 2$ case

$$\begin{bmatrix} p_{1,1} & p_{1,2} \\ p_{2,1} & p_{2,2} \end{bmatrix}, \text{ with}$$

$$p_{1,1} + p_{1,2} = d$$

$$p_{1,1} + p_{2,1} = d,$$

$$p_{2,1} + p_{2,2} = d.$$

We must have

$$\begin{bmatrix} r & 2d - r \\ 2d - r & r \end{bmatrix}$$

The labeling interpretation

$\langle \underbrace{h_d h_d * h_d h_d * \cdots * h_d h_d}_{k \text{ times}}, s_{2d} \rangle$ equals

the number of labeling on the vertices of the k cube such that the sum of the labels on each face are equal to d .

the number of nonnegative integer solutions of $A_k p = 0$, with $p_1 + p_2 + \cdots + p_{2^k} = 2d$, where A_k is ...

The defining equations

$$A_2 = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}$$

Descriptions for A_k

$$A_1 = [1, -1]$$

$$A_k = \begin{bmatrix} \mathbf{1} & -\mathbf{1} \\ A_{k-1} & A_{k-1} \end{bmatrix}$$

The column i of A_k is related to the binary representation of $i - 1$.

Count solutions via MacMahon's partition analysis

Example: the $k = 2$ case

$$A_2 = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

We need

$$p_1 + p_2 - p_3 - p_4 = 0$$

$$p_1 - p_2 + p_3 - p_4 = 0$$

Example of $k = 2$

$$\begin{aligned}
 F_2(x) &= \sum_{p_i \geq 0, A_2 p = 0} x_1^{p_1} x_2^{p_2} x_3^{p_3} x_4^{p_4} \\
 &= \sum_{p_i \geq 0,} \underset{a_1}{\text{CT}} a_1^{p_1 + p_2 - p_3 - p_4} \underset{a_2}{\text{CT}} a_2^{p_1 - p_2 + p_3 - p_4} x_1^{p_1} x_2^{p_2} x_3^{p_3} x_4^{p_4} \\
 &= \underset{a_1}{\text{CT}} \underset{a_2}{\text{CT}} \sum_{p_i \geq 0,} a_1^{p_1 + p_2 - p_3 - p_4} a_2^{p_1 - p_2 + p_3 - p_4} x_1^{p_1} x_2^{p_2} x_3^{p_3} x_4^{p_4} \\
 &= \underset{a_1, a_2}{\text{CT}} \frac{1}{(1 - x_1 a_1 a_2)(1 - x_2 a_1 a_2^{-1})(1 - x_3 a_1^{-1} a_2)(1 - x_4 a_1^{-1} a_2^{-1})}
 \end{aligned}$$

The crude generating function for $Ap = 0$

In general, the generating function for nonnegative integral solutions of $Ap = 0$ is the constant term of the *crude generating function*

$$\frac{1}{\prod_i (1 - x_i a^{\text{column}_i(A)})}$$

This is a power series in the x 's, but not a power series in the a 's.

This is the work by MacMahon (about 1905).

Andrews et al. applied computer algebra to MacMahon's partition analysis: the Omega package. (about 1999)

The special crude generating functions

The crude generating function for $A_k p = 0$ is:

$$\mathcal{F}_k(q) = \prod_{\beta \in (a_1 + a_1^{-1}) \cdots (a_k + a_k^{-1})} \frac{1}{1 - \beta q}$$

The crude generating function for the Wallach series is:

$$\mathcal{W}_k(q^2) = \mathcal{F}_k(q) \cdot \prod_{i=1}^k (1 + a_i^2).$$

This is because $s_{d,d} = h_d h_d - h_{d+1} h_{d-1}$.

Results: the complete generating function, the x -series

$$F_2(x) = \frac{1}{(-1+x_2x_3)(-1+x_4x_1)}$$

$$F_3(x) =$$

$$-\frac{x_4x_1x_6x_7x_2x_8x_5x_3-1}{(-1+x_2x_3x_8x_5)(-1+x_4x_5)(-1+x_6x_3)(-1+x_2x_7)(-1+x_4x_1x_6x_7)(-1+x_1x_8)}$$

$F_4(x)$ is a sum of 60 simple rational functions (not in the form of power series).

Results: the generating function, the q -series

$$F_2(\sqrt{q}) = \frac{1}{(-1+q)^2}$$

$$F_3(\sqrt{q}) = -\frac{q^2+1}{(-1+q)^4(-1+q^2)}$$

$$F_4(\sqrt{q}) = \frac{q^{10}+q^9+21q^8+36q^7+74q^6+86q^5+74q^4+36q^3+21q^2+q+1}{(-1+q)^7(-1+q^2)^4(-1+q^3)}$$

$$F_5(\sqrt{q}) = -\frac{Num}{(q-1)^9(q^2-1)^8(q^5-1)(q^4-1)^3(q^3-1)^6}, \text{ where}$$

The numerator of the q -series for $k = 5$

$$\begin{aligned} & q^{44} + 7q^{43} + 220q^{42} + 2606q^{41} + 24229q^{40} + 169840q^{39} + 951944q^{38} \\ & + 4391259q^{37} + 17128360q^{36} + 57582491q^{35} + 169556652q^{34} \\ & + 442817680q^{33} + 1036416952q^{32} + 2192191607q^{31} + 4219669696q^{30} \\ & + 7433573145q^{29} + 12041305271q^{28} + 18003453305q^{27} + 24921751416q^{26} \\ & + 32017113319q^{25} + 38243274851q^{24} + 42524815013q^{23} + 44052440432q^{22} \\ & + 42524815013q^{21} + 38243274851q^{20} + 32017113319q^{19} + 24921751416q^{18} \\ & + 18003453305q^{17} + 12041305271q^{16} + 7433573145q^{15} + 4219669696q^{14} \\ & + 2192191607q^{13} + 1036416952q^{12} + 442817680q^{11} + 169556652q^{10} \\ & + 57582491q^9 + 17128360q^8 + 4391259q^7 + 951944q^6 + 169840q^5 \\ & + 24229q^4 + 2606q^3 + 220q^2 + 7q + 1. \end{aligned}$$

Why so hard? The number of extreme rays

The number of Extreme rays, orbits

$k = 2$: 2 Extreme rays, 1 orbits

$k = 3$: 6 Extreme rays, 2 orbits

$k = 4$: 64 Extreme rays, 4 orbits, take about 1.5 seconds to get the q -series.

$k = 5$: 4664 Extreme rays, 14 orbits, take about 15 minutes to get the q -series.

Extreme rays for $k = 5$

The representatives are

$$\begin{aligned}
 & x_{20}x_{22}x_1^2x_{23}x_{16}^3x_{25}^2, x_2x_3x_8x_{12}x_{20}x_{29}^3, \\
 & x_{10}x_{17}^2x_8x_{11}x_5x_{32}^2, x_{24}x_2x_3x_{12}x_{29}^2, x_{10}x_{17}x_8x_5x_{32}x_{27}, \\
 & x_{15}x_{20}x_{22}x_1x_{16}x_{25}, \sqrt{x_{17}x_{31}}\sqrt{x_{22}x_4}^{3/2}\sqrt{x_{26}x_{13}}, \\
 & x_9\sqrt{x_{14}}\sqrt{x_{31}}x_{20}\sqrt{x_{22}}\sqrt{x_7}, x_9x_{15}x_{20}x_{22}, x_{10}x_{17}x_{31}x_8, \\
 & x_{14}^{2/3}\sqrt[3]{x_{11}}\sqrt[3]{x_1}x_{23}\sqrt[3]{x_4}x_{26}^{2/3}, x_{14}\sqrt{x_{20}}\sqrt{x_1}\sqrt{x_{23}}\sqrt{x_{27}}, \\
 & \sqrt[3]{x_6}\sqrt[3]{x_{24}}x_{11}\sqrt[3]{x_{18}}\sqrt[3]{x_{21}}\sqrt[3]{x_{30}}, x_{14}x_{19}
 \end{aligned}$$

The idea of iterated Laurent series

Let K be a field. The field of iterated Laurent series
 $K\langle\langle x_1, \dots, x_n \rangle\rangle := K((x_1))((x_2)) \cdots ((x_n))$.

Idea: define a total ordering on the variables.

Why? To handle the series expansion of something like $1/(x - y)$

Effect: We can do partial fraction decomposition freely.

For MacMahon's partition analysis, the working field is
 $\mathbb{C}\langle\langle a_1, a_2, \dots, a_r, x_1, \dots, x_n \rangle\rangle$.

This leads to ...

The basic tool: Xin's fast algorithm for MacMahon's partition analysis

The basic idea by an example:

$$\begin{aligned}
 F &= \frac{\lambda}{(1-x\lambda)(1-y\lambda)(\lambda-z)} \\
 &= \frac{1/x}{(1-x\lambda)(1-y/x)(1-zx)} + \frac{1/y}{(1-y\lambda)(1-x/y)(1-zy)} \\
 &\quad + \frac{z}{(\lambda-z)(1-xy)(1-xz)}
 \end{aligned}$$

$$\text{where } \text{CT}_{\lambda} \frac{1}{1-x\lambda} = 1, \text{CT}_{\lambda} \frac{1}{1-y\lambda} = 1$$

$$\text{and } \text{CT}_{\lambda} \frac{1}{\lambda-z} = \text{CT}_{\lambda} \frac{1}{\lambda} \frac{1}{1-z/\lambda} = \text{CT}_{\lambda} \sum_{n \geq 1} z^{n-1}/\lambda^n = 0.$$

Our first method to compute $F_5(q)$

Explain by the $k = 3$ case

$$A_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}$$

has the same null space as

$$B_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}$$

Complex analysis for the $k = 5$ case

Direct computation runs out of memory.

The crude generating function for B_k is

$$\frac{1}{\prod_{S \in [2 \dots k]} \left(1 - \frac{x}{\lambda_1 \Lambda_S}\right) (1 - x \lambda_1 \Lambda_S)}.$$

Next

$$\frac{1}{\prod_{S \in [3 \dots k]} \left(1 - \frac{x}{\lambda_1 \Lambda_S}\right) (1 - x \lambda_1 \Lambda_S)} \frac{1}{\prod_{S \in [3 \dots k]} \left(1 - \frac{x}{\mu \Lambda_S}\right) (1 - x \mu \Lambda_S)},$$

where μ stands for $\lambda_1 \lambda_2$.

The running time

For $k = 5$, this reduces to the constant term of $(G_1 + G_2 + \cdots + G_8)^2$. This divides to 36 pieces, each manageable.

The total running time for our first computation took 2 hours and 45 minutes.

After updated my package, this idea takes about 23 minutes.

Second method: by symmetry

The idea: The k cube has a group of automorphism \mathcal{B}_k .

Fact: Let V_i and V_{2^k+1-i} be diagonal vertices. Then

- a) label them by 1, and all others by 0 is a solution. Call it the diagonal solution.
- b) by subtracting a multiple of diagonal solution, we need only to consider solutions with either the label $p_i = 0$ or $p_{2^k+1-i} = 0$, or both.

The algebraic idea

- a) · permute the rows of A_k will not change the solution.
· multiply a row of A_k by -1 will not change the solution.

The results of the above two operation has the same effect by switching certain columns of A_k . This is the same as the \mathcal{B}_k acting on the vertices of the k -cube.

b)

$$\begin{aligned} \frac{1 - x_i x_{2^k+1-i}}{(1 - x_i a) \left(1 - \frac{x_{2^k+1-i}}{a}\right)} &= \frac{x_i a}{(1 - x_i a)} + \frac{1}{1 - x_{2^k+1-i}/a} \\ &= \frac{x_i a}{(1 - x_i a)} + \frac{x_{2^k+1-i}/a}{1 - x_{2^k+1-i}/a} + 1 \end{aligned}$$

Two encodings for $k = 2$

$$\text{Let } X_i = \frac{x_i a^{\text{column}_i(A_5)}}{1 - x_i a^{\text{column}_i(A_5)}}, Y_i = \frac{1}{x_{33-i} a^{\text{column}_{33-i}(A_5)}}.$$

The the crude generating function is $X_1 X_2 \dots X_{32}$.

Group $X_i X_{33-i}$ and factor out a suitable factor. It becomes

M21:

$$\prod_{i=1}^{16} (X_i + Y_i)$$

or M22:

$$\prod_{i=1}^{16} (1 + X_i + X_{33-i})$$

The complexity of M21

Expand M21 will give 2^{16} terms, each has 16 factors. The computation of the constant term of 16 factors is similar to the $k = 4$ case.

The computation of $F_4(q)$ takes about 1.5 seconds. So the computation of $F_5(q)$ will estimates 27.3 hours.

The complexity of M21

Expand M21 will give 2^{16} terms, each has 16 factors. This is similar to the $k = 4$ case.

The computation of $F_4(q)$ takes about 1.5 seconds. So the computation of $F_5(q)$ will estimates 27.3 hours.

Fortunately, the constant term is invariant under the group action of \mathcal{B}_4 , and there are *only* 402 orbits. 371 of them have contribution. This estimates to 10 minutes. The actual running time is about 15 minutes.

The complexity of M22

Expand M22 will give 3^{16} terms, each has no more than 16 factors. This is similar to the $k = 4$ case.

The constant term is invariant under the group action of \mathcal{B}_5 .

Finding the orbits is not easy: the group \mathcal{B}_5 has $5! \cdot 2^5 = 3840$ elements. There are

1, 1, 4, 8, 35, 87, 264, 582, 1243, 2000, 2798, 2964, 2606, 1638, 832, 272, 83 orbits, ordered by the number of factors. The total number of orbits is 15418.

Many of them have no contribution. The good orbits are 1, 0, 0, 0, 2, 1, 11, 34, 155, 435, 936, 1343, 1462, 1078, 604, 212, 67, with total number 6341.

The actual time for computing $F_5(q)$ is also about 15 minutes.

The complete generating function $F_4(x)$

The method of M22 is much better in the $k = 4$ case:

There are only 62 orbits.

The number of good orbits is only 10.

We are able to obtain the complete generating function $F_4(x)$, with 10 representatives.

End of the talk

Thanks for coming!