# *Partial Fraction Algorithm for MacMahon's Partition Analysis*

# *UCSD seminar*

Guoce Xin

Center for Combinatorics
Nankai University

August 16th, 2007

# Outline

# History

1. Partition Analysis was firstly introduced by Percy MacMahon in his famous book "Combinatorial Analysis" in 1915.

# History

1. Partition Analysis was firstly introduced by Percy MacMahon in his famous book "Combinatorial Analysis" in 1915.
2. George Andrews (about 1999) observed that MacMahon's idea can be implemented by computer: the Omega package.

# History

1. Partition Analysis was firstly introduced by Percy MacMahon in his famous book "Combinatorial Analysis" in 1915.
2. George Andrews (about 1999) observed that MacMahon's idea can be implemented by computer: the Omega package.
3. He and his coauthors, Peter Paule, Axel Riese, and Volker Strehl, have written a series of papers with many applications of MacMahon's partition analysis.

# History

1. Partition Analysis was firstly introduced by Percy MacMahon in his famous book "Combinatorial Analysis" in 1915.
2. George Andrews (about 1999) observed that MacMahon's idea can be implemented by computer: the Omega package.
3. He and his coauthors, Peter Paule, Axel Riese, and Volker Strehl, have written a series of papers with many applications of MacMahon's partition analysis.
4. By using the theory of iterated Laurent series and partial fraction decompositions, I give a speed up of the Omega package in 2004.

# Object of Study

Solving combinatorial problems in connection with system of linear Diophantine equations and inequalities. Several examples of applications are

1. Solving system of linear Diophantine equations and inequalities.
2. Counting lattice points in a rational convex polytope.
3. Computing certain Hilbert series.

# MacMahon's Omega Operator

## Definition

$$\Omega_{\geq} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\dots,s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := \sum_{s_1=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} A_{s_1,\dots,s_r},$$

$$\Omega_{=} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\dots,s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := A_{0,\dots,0}.$$

Everything is treated analytically. The method relies on the unique Laurent series representations of rational functions.

# Basic Idea by an Example 1/2

Problem: Find all nonnegative integral solutions to $a_1 + a_2 - a_3 = 0$.

$$\sum_{\substack{a_1,a_2,a_3\geq 0\\ a_1+a_2-a_3=0}} x^{a_1} y^{a_2} z^{a_3}$$

$$= \sum_{a_1,a_2,a_3\geq 0} \underset{=}{\Omega} \lambda^{a_1+a_2-a_3} x^{a_1} y^{a_2} z^{a_3}$$

$$= \underset{=}{\Omega} \sum_{a_1\geq 0} \lambda^{a_1} x^{a_1} \cdot \sum_{a_2\geq 0} \lambda^{a_2} y^{a_2} \cdot \sum_{a_3\geq 0} \lambda^{-a_3} z^{a_3}$$

# Basic Idea by an Example 2/2

Apply the formula for the sum of a geometric series:

$$= \underset{=}{\Omega} \frac{1}{(1-\lambda x)(1-\lambda y)(1-z/\lambda)},$$

Eliminate $\lambda$ using Elliott's Reduction Procedure to obtain

$$\frac{1}{(1-xz)(1-yz)}$$

If we set $x=t, y=t, z=t$, we will get the generating function for the number of the above solutions such that $a_1+a_2+a_3=n$: $\frac{1}{(1-t^2)^2}$

# Definition of Iterated Laurent Series

Let $K$ be a field. Define

$$K\langle\langle x_1\rangle\rangle = K((x_1)) = \left\{ \sum_{n\geq N_0} a_n x_1^n \mid a_n \in K \right\}$$

to be the field of Laurent series in $x_1$. Inductively define

$$K\langle\langle x_1, x_2, \ldots, x_n\rangle\rangle = K\langle\langle x_1, x_2, \ldots, x_{n-1}\rangle\rangle((x_n)) = K((x_1))\cdots((x_n))$$

to be the field of iterated Laurent series in $x_1, x_2, \ldots, x_n$.
An iterated Laurnet series is firstly regarded as a Laurent series in $x_n$, and then (coefficients) regarded as Laurent series in $x_{n-1}$, and so on.

# Total Order on Monomials

Equivalently, we defined a total order on the variables, or can be understood as $1 >> x_1 >> \cdots >> x_n$. (Wilson, Stanley)

This total order is extended for all monomials:
For a monomial $M$, scan for the smallest variable in it, say $x_i$. Then $M = x_1^{k_1} \cdots x_i^{k_i} x_{i+1}^0 \cdots x_n^0$.

1. Set $\begin{array}{l} M < 1 \text{ if } k_i > 0 \\ M > 1 \text{ if } k_i < 0 \end{array}$. For instance, $x_1 x_3^{-2} > 1$, $x_2^{-3} x_4 < 1$
2. Set

$$M_1 < M_2 \Leftrightarrow M_1/M_2 < 1$$

# Series Expansion

If $M_2$ is small then

$$\frac{1}{M_1 - M_2} = \frac{1}{M_1} \frac{1}{1 - M_2/M_1} = \sum_{n\geq 0} M_2^n / M_1^{n+1}.$$

If $M_1$ is small then

$$\frac{1}{M_1 - M_2} = -\frac{1}{M_2} \frac{1}{1 - M_1/M_2} = -\sum_{n\geq 0} M_1^n / M_2^{n+1}.$$

After fixing a total order on the variables, everything works formally when taking constant terms. This is guaranteed by:

**Theorem (Fundamental Structure)**

*A series is an iterated Laurent series if and only if it has a well-ordered support.*

# Outline

# Object of Study

Take constant terms of a given Elliott rational function, i.e., a rational function whose denominator is written as a product of $M_1 - M_2$.

Elliott rational functions arise naturally in the study of Linear Diophantine system.

Such constant terms can be theoretically evaluated.

**Theorem (Elliott)**

*The constant term of an Elliott rational functions is still Elliott rational.*

The constant term in a set of variables can be taken iteratively.

## Revisit an Example

Compute $\Omega_= F$:

$$F = \frac{\lambda}{(1-x\lambda)(1-y\lambda)(\lambda-z)}$$
$$= \frac{1/x}{(1-x\lambda)(1-y/x)(1/x-z)} + \frac{1/y}{(1-y\lambda)(1-x/y)(1/y-z)} + \frac{z}{(\lambda-z)(1-xy)(1-xz)}$$

$$\text{where } \underset{=}{\Omega} \frac{1}{1-x\lambda} = 1, \underset{=}{\Omega} \frac{1}{1-y\lambda} = 1$$

$$\text{and } \underset{=}{\Omega} \frac{1}{\lambda-z} = \underset{=}{\Omega} \frac{1}{\lambda} \frac{1}{1-z/\lambda} = \underset{=}{\Omega} \sum_{n\geq 1} z^{n-1}/\lambda^n = 0.$$

# The PT Operator Replacing MacMahon's Operator

Fix a working field $K\langle\langle \Lambda, \mathbf{x} \rangle\rangle := K\langle\langle \lambda_1, \dots, \lambda_r, x_1, \dots, x_n \rangle\rangle$.
Define a new operator $\mathrm{PT}_\lambda$ to replace both $\Omega_\geq$ and $\Omega_=$:

$$\mathop{\mathrm{PT}}_{\lambda} \sum_{m=-\infty}^{\infty} a_m \lambda^m := \sum_{m=0}^{\infty} a_m \lambda^m.$$

MacMahon's operators can be realized as:

$$\mathop{\Omega}_{\geq} F(\Lambda, \mathbf{x}) = \mathop{\mathrm{PT}}_{\Lambda} F(\Lambda, \mathbf{x})\Big|_{\Lambda=(1,\dots,1)},$$

$$\mathop{\Omega}_{=} F(\Lambda, \mathbf{x}) = \mathop{\mathrm{CT}}_{\Lambda} F(\Lambda, \mathbf{x}) = \mathop{\mathrm{PT}}_{\Lambda} F(\Lambda, \mathbf{x})\Big|_{\Lambda=(0,\dots,0)}.$$

# Garsia's Notation

For $\Lambda = (\lambda_1, \ldots, \lambda_k)$, Garsia use

$$F\Big|_{\lambda_1^0 \cdots \lambda_k^0} = \operatorname*{CT}_{\Lambda} F = \mathop{\Omega}_{=} F$$

and

$$F\Big|_{\lambda_1^{\geq} \cdots \lambda_k^{\geq}} = \mathop{\Omega}_{\geq} F$$

# Heart Problem

How to compute $\Omega_{=} F(\Lambda, \mathbf{x})$ fast?

By iteration, we reduce the problem to the one variable case: evaluating $\mathrm{PT}_\lambda F(\lambda)$ with

$$F(\lambda) = \frac{\text{polynomial}(\lambda)}{\prod_{1\le i\le n}(\lambda^{j_i} - z_i)},$$

where $j_i \in \mathbb{P}$, and $z_i$'s are independent of $\lambda$. Note that $z_i$'s are allowed to be 0.

Find the PFD of $F(\lambda)$. Note that the idea of using PFD in this context was first used by Stanley (1974), but was thought to be impractical without using a computer.

# Application of the PFD

Constant term can be read off as soon as we find the PFD.

# Application of the PFD

Constant term can be read off as soon as we find the PFD.

## Theorem

*Suppose that the PFD of $F$ is*

$$F = f(\lambda) + \sum_{1\le s\le n} \frac{p_s(\lambda)}{\lambda^{j_s} - z_s},$$

*where $f(\lambda)$ is a polynomial in $\lambda$, and $p_s(\lambda)$ is a polynomial of degree less than $j_s$ for each $s$. Then*

$$\operatorname*{PT}_{\lambda} F = f(\lambda) + \sum_{s} \frac{p_s(\lambda)}{\lambda^{j_s} - z_s},$$

*where the sum ranges over all $s$ such that $\lambda^{j_s} < z_s$.*

# The Linear Factors Case is Simple

Suppose we are taking constant term in $x_k$ of

$$F = \frac{p(x_k)}{x_k^d \prod_{i=1}^m (1 - x_k/\alpha_i)},$$

where $p(x_k)$ is a polynomial and $\alpha_i$ are independent of $x_k$. If $F$ is proper in $x_k$, then we have the PFD with respect to $x_k$

$$F = \frac{p_1(x_k)}{x_k^d} + \sum_{j=1}^m \frac{1}{1 - x_k/\alpha_j} \left( \frac{p(x_k)}{x_k^d \prod_{i=1, i\neq j}^m (1 - x_k/\alpha_i)} \right) \Bigg|_{x_k=\alpha_j},$$

When taking constant term in $x_k$, we have

$$\mathop{\mathrm{CT}}_{x_k} F = \sum_j \left(F\,(1 - x_k/\alpha_j)\right) \Big|_{x_k=\alpha_j},$$

where the sum ranges over all $j$ such that $x_k/\alpha_j$ is small.

# Outline

# Crude G.F. From the Matrix Form

Problem: Find all nonnegative integer solutions $\alpha \in \mathbb{N}^n$ with $A\alpha = b$:

$$\begin{aligned} a_{1,1}\alpha_1 + a_{1,2}\alpha_2 + \cdots + a_{1,n}\alpha_n &= b_1 \\ a_{2,1}\alpha_1 + a_{2,2}\alpha_2 + \cdots + a_{2,n}\alpha_n &= b_2 \\ \cdots\cdots &= \cdots \\ a_{r,1}\alpha_1 + a_{r,2}\alpha_2 + \cdots + a_{r,n}\alpha_n &= b_r. \end{aligned}$$

The generating function is the constant term of the crude generating function

$$\frac{\lambda_1^{-b_1} \cdots \lambda_r^{-b_r}}{\prod_{i=1}^n (1 - \lambda_1^{a_{1,i}} \lambda_2^{a_{2,i}} \cdots \lambda_r^{a_{r,i}} x_i)}.$$

However, many problems are easy to describe by equations.

# From a System of Equations: An Example

Problem: Find all nonnegative integer $n \times n$ matrices $(a_{ij})$ with every row sum and column sum equal to $r$.
Use $\lambda_i$ for the $i$th row equation, $\mu_j$ for the $j$th column equation. Use $x_{ij}$ for $a_{ij}$, and use $t$ for $r$. For example, the $i$th row equation is

$$a_{i1} + a_{i2} + \cdots + a_{in} - r = 0.$$

The crude generating function is

# From a System of Equations: An Example

Problem: Find all nonnegative integer $n \times n$ matrices $(a_{ij})$ with every row sum and column sum equal to $r$.
Use $\lambda_i$ for the $i$th row equation, $\mu_j$ for the $j$th column equation. Use $x_{ij}$ for $a_{ij}$, and use $t$ for $r$. For example, the $i$th row equation is

$$a_{i1} + a_{i2} + \cdots + a_{in} - r = 0.$$

The crude generating function is

$$\prod_{i,j=1}^{n} \frac{1}{1 - x_{ij}\lambda_i\mu_j} \times \frac{1}{1 - t/(\lambda_1 \cdots \lambda_n \mu_1 \cdots \mu_n)}.$$

Understood as: The $a_{ij}$ only appears in the $\lambda_i$ and $\mu_j$ equations with coefficients 1. The $r$ appears in every equation, with coefficients $-1$.
Actually, we can set $\mu_n = 1$ since it is redundant.

# From a System of Equations: A Complicated Example

Problem: Find all labelings on the vertices of a $k$ dimensional cube, such that the labels on every facet sum to $r$.
The vertices of the cube is identified by 01 sequences $w = w_1 \cdots w_k$.
A facet is either $\{w_1 \cdots w_k \mid w_i = 0\}$ or $\{w_1 \cdots w_k \mid w_j = 1\}$. Index them by $\lambda_i^{(0)}$ and $\lambda_i^{(1)}$ respectively. For example, the equation for the $\lambda_i^{(0)}$ is

$$\sum_{w_i=0} a_{w_1\cdots w_k} - r = 0.$$

The Crude G.F. is given by

# From a System of Equations: A Complicated Example

Problem: Find all labelings on the vertices of a $k$ dimensional cube, such that the labels on every facet sum to $r$.
The vertices of the cube is identified by 01 sequences $w = w_1 \cdots w_k$.
A facet is either $\{w_1 \cdots w_k \mid w_i = 0\}$ or $\{w_1 \cdots w_k \mid w_j = 1\}$. Index them by $\lambda_i^{(0)}$ and $\lambda_i^{(1)}$ respectively. For example, the equation for the $\lambda_i^{(0)}$ is

$$\sum_{w_i=0} a_{w_1\cdots w_k} - r = 0.$$

The Crude G.F. is given by

$$\prod_{w_1\cdots w_k} \frac{1}{1 - x_{w_1\cdots w_k}\lambda_1^{(w_1)}\cdots\lambda_k^{(w_k)}} \times \frac{1}{1 - t/\lambda_1^{(0)}\cdots\lambda_k^{(0)}\lambda_1^{(1)}\cdots\lambda_k^{(1)}}.$$

Of course, one can set $\lambda_2^{(1)}, \ldots, \lambda_k^{(1)}$ all equal to 1, since they are redundant.

# The Set Up for Evaluating Constant Term

Our Goal: evaluate the constant term in a set of variables of a crude G.F.:

$$F = \frac{P}{(1-m_1)(1-m_2)\cdots(1-m_n)},$$

where $P$ is a Laurent polynomial, and $m_i$ are monomials.

1. First choose an order of the variables compatible with our problem. There may be many choices!
2. Choose a variable and eliminate, but do NOT combine the result.
3. For each term obtained, choose another variable and eliminate.
4. It is NOT a good strategy to decide before hand in with order the variables are to be eliminated.

The choice of the variable for elimination is guided by the following . . .

# For the One Variable Linear Factors Case

We often meet rational function of the form:

$$F(\lambda) = \frac{P(\lambda)}{\prod_{i=1}^{n} (1-(\lambda/m_i)^{e_i})} \quad \text{(with } e_i = \pm 1).$$

**Lemma**

*Suppose the above $F$ is proper in $\lambda$ and $\lim_{\lambda \to 0} F(\lambda) = 0$. Then*

$$\begin{aligned} \operatorname*{CT}_{\lambda} F(\lambda) &= \sum_{\lambda/m_i<1} e_i \left(F(\lambda)(1-(\lambda/m_j)^{e_i})\right)\Big|_{\lambda=m_i} \\ &= -\sum_{\lambda/m_i>1} e_i \left(F(\lambda)(1-(\lambda/m_j)^{e_i})\right)\Big|_{\lambda=m_i} \end{aligned}$$

# For the One Variable Linear Factors Case

We often meet rational function of the form:

$$F(\lambda) = \frac{P(\lambda)}{\prod_{i=1}^{n} (1 - (\lambda/m_i)^{e_i})} \quad (\text{with } e_i = \pm 1).$$

## Lemma

*Suppose the above F is proper in λ and* $\lim_{\lambda \to 0} F(\lambda) = 0$. *Then*

$$\operatorname*{CT}_{\lambda} F(\lambda) = \sum_{\lambda/m_i < 1} e_i \left(F(\lambda)(1 - (\lambda/m_j)^{e_i})\right)\Big|_{\lambda=m_i}$$
$$= - \sum_{\lambda/m_i > 1} e_i \left(F(\lambda)(1 - (\lambda/m_j)^{e_i})\right)\Big|_{\lambda=m_i}$$

A denominator factor $1 - (\lambda/m_i)^{e_i}$ is said to be contributing if $\lambda/m_i < 1$ and to be dually contributing if $\lambda/m_i > 1$.

# Choosing the Variable for Elimination

Since for $m_j$ with $m_j > 1$ we can write

$$\frac{1}{1-m_j} = \frac{-\frac{1}{m_j}}{1-\frac{1}{m_j}},$$

Our crude G.F. $F$ can be written in *proper form*

$$F = P \times \prod_{m_i<1} \frac{1}{1-m_i}.$$

With respect to $\lambda$, $1-m_i$ is contributing if $\deg_\lambda m_i > 0$, and is dually contributing if $\deg_\lambda m_i < 0$.

The variable we shall eliminate is the one that has the smallest number of contributing factors or dually contributing factors.

# A Simple Example: Weak Magic Squares of Order 2

The Crude G.F. is

$$\frac{1}{(1-\lambda_1\mu_1)(1-\lambda_1)(1-\lambda_2\mu_1)(1-\lambda_2)(1-t/\lambda_1\lambda_2\mu_1)}.$$

Taking constant term in $\mu_1$ gives

# A Simple Example: Weak Magic Squares of Order 2

The Crude G.F. is

$$\frac{1}{(1-\lambda_1\mu_1)(1-\lambda_1)(1-\lambda_2\mu_1)(1-\lambda_2)(1-t/\lambda_1\lambda_2\mu_1)}.$$

Taking constant term in $\mu_1$ gives

$$\frac{1}{(1-t/\lambda_2)(1-\lambda_1)(1-t/\lambda_1)(1-\lambda_2)}.$$

Taking constant term in $\lambda_1$ gives

# A Simple Example: Weak Magic Squares of Order 2

The Crude G.F. is

$$\frac{1}{(1-\lambda_1\mu_1)(1-\lambda_1)(1-\lambda_2\mu_1)(1-\lambda_2)(1-t/\lambda_1\lambda_2\mu_1)}.$$

Taking constant term in $\mu_1$ gives

$$\frac{1}{(1-t/\lambda_2)(1-\lambda_1)(1-t/\lambda_1)(1-\lambda_2)}.$$

Taking constant term in $\lambda_1$ gives

$$\frac{1}{(1-t/\lambda_2)(1-t)(1-\lambda_2)}.$$

Taking constant term in $\lambda_2$ gives

# A Simple Example: Weak Magic Squares of Order 2

The Crude G.F. is

$$\frac{1}{(1-\lambda_1\mu_1)(1-\lambda_1)(1-\lambda_2\mu_1)(1-\lambda_2)(1-t/\lambda_1\lambda_2\mu_1)}.$$

Taking constant term in $\mu_1$ gives

$$\frac{1}{(1-t/\lambda_2)(1-\lambda_1)(1-t/\lambda_1)(1-\lambda_2)}.$$

Taking constant term in $\lambda_1$ gives

$$\frac{1}{(1-t/\lambda_2)(1-t)(1-\lambda_2)}.$$

Taking constant term in $\lambda_2$ gives

$$\frac{1}{(1-t)^2} = \sum_{r\geq 0}(r+1)t^r.$$

# Magic Labeling of the Cube 1/4

The crude G.F. for the 3-dimensional case:

$$\frac{1}{(1-\lambda_1\lambda_2\lambda_3)(1-\lambda_1\lambda_2)(1-\lambda_1\lambda_3)(1-\lambda_1)} \times \frac{1}{(1-\mu_1\lambda_2\lambda_3)(1-\mu_1\lambda_2)(1-\mu_1\lambda_3)(1-\mu_1)} \times \frac{1}{1-\frac{t}{\lambda_1\lambda_2\lambda_3\mu_1}}$$

Choose $t < \lambda_1 < \lambda_2 < \lambda_3 < \mu_1 < 1$. Eliminating $\mu_1$ gives

$$\frac{1}{(1-\lambda_1\lambda_2\lambda_3)(1-\lambda_1\lambda_2)(1-\lambda_1\lambda_3)(1-\lambda_1)} \times \frac{1}{(1-t/\lambda_1)(1-t/\lambda_1\lambda_3)(1-t/\lambda_1\lambda_2)(1-t/\lambda_1\lambda_2\lambda_3)}$$

# Magic Labeling of the Cube 2/4

Eliminating $\lambda_3$ gives a sum of two terms

$$T1 = \frac{1}{(1-\lambda_1\lambda_2)(1-1/\lambda_2)(1-\lambda_1)(1-t/\lambda_1)(1-t\lambda_2)(1-t/\lambda_1\lambda_2)(1-t)}$$

$$T2 = \frac{1}{(1-\lambda_2)(1-\lambda_1\lambda_2)(1-\lambda_1)(1-t/\lambda_1)(1-t)(1-t/\lambda_1\lambda_2)(1-t/\lambda_2)}$$

For $T1$, eliminating $\lambda_2$ gives

# Magic Labeling of the Cube 2/4

Eliminating $\lambda_3$ gives a sum of two terms

$$T1 = \frac{1}{(1-\lambda_1\lambda_2)(1-1/\lambda_2)(1-\lambda_1)(1-t/\lambda_1)(1-t\lambda_2)(1-t/\lambda_1\lambda_2)(1-t)}$$

$$T2 = \frac{1}{(1-\lambda_2)(1-\lambda_1\lambda_2)(1-\lambda_1)(1-t/\lambda_1)(1-t)(1-t/\lambda_1\lambda_2)(1-t/\lambda_2)}$$

For $T1$, eliminating $\lambda_2$ gives

$$\frac{1}{(1-t)(1-\lambda_1/t)(1-\lambda_1)(1-t/\lambda_1)(1-t^2/\lambda_1)(1-t)}$$

Eliminating $\lambda_1$ gives our

# Magic Labeling of the Cube 2/4

Eliminating $\lambda_3$ gives a sum of two terms

$$T1 = \frac{1}{(1-\lambda_1\lambda_2)(1-1/\lambda_2)(1-\lambda_1)(1-t/\lambda_1)(1-t\lambda_2)(1-t/\lambda_1\lambda_2)(1-t)}$$

$$T2 = \frac{1}{(1-\lambda_2)(1-\lambda_1\lambda_2)(1-\lambda_1)(1-t/\lambda_1)(1-t)(1-t/\lambda_1\lambda_2)(1-t/\lambda_2)}$$

For $T1$, eliminating $\lambda_2$ gives

$$\frac{1}{(1-t)(1-\lambda_1/t)(1-\lambda_1)(1-t/\lambda_1)(1-t^2/\lambda_1)(1-t)}$$

Eliminating $\lambda_1$ gives our

$$out_1 = \frac{1}{(1-t)(1-1/t)(1-t)(1-t^2)(1-t)} = \frac{-t}{(1-t)^4(1-t^2)}.$$

# Magic Labeling of the Cube 3/4

For T2, eliminating $\lambda_2$ gives two terms

$$U1 = \frac{1}{(1-\lambda_1)(1-\lambda_1)(1-t/\lambda_1)(1-t)(1-t/\lambda_1)(1-t)}$$

$$U2 = \frac{1}{(1-1/\lambda_1)(1-\lambda_1)(1-t/\lambda_1)(1-t)(1-t)(1-t\lambda_1)}$$

For $U2$ eliminating $\lambda_1$ gives our

$$out_2 = \frac{1}{(1-1/t)(1-t)(1-t)(1-t)(1-t^2)} = \frac{-t}{(1-t)^4(1-t^2)}$$

For $U1$ we can not apply our formula directly.

# Magic Labeling of the Cube 4/4

Introduce a new variable $z$, that will be taken to 1.

$$U1 = \frac{1}{(1-\lambda_1)(1-\lambda_1/z)(1-t/\lambda_1)(1-t)(1-t/\lambda_1)(1-t)}$$

Eliminating $\lambda_1$ gives

$$\frac{1}{(1-1/z)(1-t)^4} + \frac{1}{(1-z)(1-t/z)^2(1-t)^2}$$

Combining and setting $z=1$, we get our

$$out_3 = \frac{1-t^2}{(1-t)^6}.$$

Putting all the above together, we get the Final generating function

$$out_1 + out_2 + out_3 = \frac{1+t^2}{(1-t)^4(1-t^2)}.$$

# An Example From Invariant Theory 1/4

A recent joint work with Garsia, Wallach, and Zabrocki includes the following identity:

$$\operatorname*{CT}_{x} \prod_{i=1}^{n} \frac{1}{(1-qx_i)(1-q/x_i)} \prod_{1\le i<j\le n} \frac{1-x_j/x_i}{(1-qx_j/x_i)}\Big|_{x_1\cdots x_{n-1}x_n=1} = \frac{1+q^{\binom{n+1}{2}}}{(q^2)_n(1-q^{\binom{n+1}{2}})}$$

This is related to the Hilbert series of certain invariant graded ring.

# An Example From Invariant Theory 1/4

A recent joint work with Garsia, Wallach, and Zabrocki includes the following identity:

$$\underset{x}{\mathrm{CT}} \prod_{i=1}^{n} \frac{1}{(1-qx_i)(1-q/x_i)} \prod_{1\le i<j\le n} \frac{1-x_j/x_i}{(1-qx_j/x_i)}\Big|_{x_1\cdots x_{n-1}x_n=1} = \frac{1+q^{\binom{n+1}{2}}}{(q^2)_n(1-q^{\binom{n+1}{2}})}$$

This is related to the Hilbert series of certain invariant graded ring.

Directly applying my algorithm gives no more than $n$ simple terms for particular $n$. But the sum of these terms is not easy to be simplified.

# An Example From Invariant Theory 2/4

We have a simple proof by working on an intermediate crude G.F.

$$\frac{1}{1-x_1\cdots x_n}\prod_{i=1}^{n}\frac{1}{(1-qx_i)(1-q/x_i)}\prod_{1\le i<j\le n}\frac{1-x_j/x_i}{(1-qx_j/x_i)}$$

First, if we eliminate $x_n, x_{n-1}, \dots$ successively, then we obtain only one term in each step! For instance, eliminating $x_n$ gives

$$\frac{1}{(1-x_1\cdots x_{n-1}q)(1-q^2)}\prod_{i=1}^{n-1}\frac{1}{(1-qx_i)(1-q/x_i)}\prod_{i<j}^{n-1}\frac{1-x_j/x_i}{(1-qx_j/x_i)}\times\prod_{i=1}^{n-1}\frac{1-q/x_i}{1-q^2/x_i}$$

which simplifies to

# An Example From Invariant Theory 2/4

We have a simple proof by working on an intermediate crude G.F.

$$\frac{1}{1-x_1\cdots x_n}\prod_{i=1}^{n}\frac{1}{(1-qx_i)(1-q/x_i)}\prod_{1\le i<j\le n}\frac{1-x_j/x_i}{(1-qx_j/x_i)}$$

First, if we eliminate $x_n, x_{n-1}, \ldots$ successively, then we obtain only one term in each step! For instance, eliminating $x_n$ gives

$$\frac{1}{(1-x_1\cdots x_{n-1}q)(1-q^2)}\prod_{i=1}^{n-1}\frac{1}{(1-qx_i)(1-q/x_i)}\prod_{i<j}^{n-1}\frac{1-x_j/x_i}{(1-qx_j/x_i)}\times\prod_{i=1}^{n-1}\frac{1-q/x_i}{1-q^2/x_i}$$

which simplifies to

$$\frac{1}{(1-q^2)(1-x_1\cdots x_{n-1}q)}\prod_{i=1}^{n-1}\frac{1}{(1-qx_i)(1-q^2/x_i)}\prod_{1\le i<j\le n-1}\frac{1-x_j/x_i}{(1-qx_j/x_i)}.$$

# An Example From Invariant Theory 3/4

We have a simple proof by working on an intermediate crude G.F.

$$\frac{1}{1-x_1\cdots x_n}\prod_{i=1}^{n}\frac{1}{(1-qx_i)(1-q/x_i)}\prod_{1\le i<j\le n}\frac{1-x_j/x_i}{(1-qx_j/x_i)}$$

Second, it has a simple connection with our problem: Taking constant term in $x_1$ gives a sum of two terms. The first one is

$$\prod_{i=1}^{n}\frac{1}{(1-qx_i)(1-q/x_i)}\prod_{1\le i<j\le n}\frac{1-x_j/x_i}{(1-qx_j/x_i)}\Big|_{x_1=1/x_2\cdots x_n},$$

# An Example From Invariant Theory 3/4

We have a simple proof by working on an intermediate crude G.F.

$$\frac{1}{1-x_1\cdots x_n}\prod_{i=1}^{n}\frac{1}{(1-qx_i)(1-q/x_i)}\prod_{1\le i<j\le n}\frac{1-x_j/x_i}{(1-qx_j/x_i)}$$

Second, it has a simple connection with our problem: Taking constant term in $x_1$ gives a sum of two terms. The first one is

$$\prod_{i=1}^{n}\frac{1}{(1-qx_i)(1-q/x_i)}\prod_{1\le i<j\le n}\frac{1-x_j/x_i}{(1-qx_j/x_i)}\Big|_{x_1=1/x_2\cdots x_n},$$

the constant term of which is what we want.
Now the second one is also simple ...

# An Example From Invariant Theory 4/4

We have a simple proof by working on an intermediate crude G.F.

$$\frac{1}{1-x_1\cdots x_n}\prod_{i=1}^{n}\frac{1}{(1-qx_i)(1-q/x_i)}\prod_{1\le i<j\le n}\frac{1-x_j/x_i}{(1-qx_j/x_i)}$$

Second, it has a simple connection with our problem: Taking constant term in $x_1$ gives a sum of two terms. The first one . . . , the second one:

$$\frac{1}{(1-q^{-1}x_2\cdots x_n)(1-q^2)}\prod_{i=2}^{n}\frac{1}{(1-qx_i)(1-q/x_i)}\prod_{2\le i<j\le n}\frac{1-x_j/x_i}{(1-qx_j/x_i)}\times\prod_{j=2}^{n}\frac{1-x_jq}{1-x_jq^2}$$

which simplifies to

# An Example From Invariant Theory 4/4

We have a simple proof by working on an intermediate crude G.F.

$$\frac{1}{1-x_1\cdots x_n}\prod_{i=1}^{n}\frac{1}{(1-qx_i)(1-q/x_i)}\prod_{1\le i<j\le n}\frac{1-x_j/x_i}{(1-qx_j/x_i)}$$

Second, it has a simple connection with our problem: Taking constant term in $x_1$ gives a sum of two terms. The first one . . . , the second one:

$$\frac{1}{(1-q^{-1}x_2\cdots x_n)(1-q^2)}\prod_{i=2}^{n}\frac{1}{(1-qx_i)(1-q/x_i)}\prod_{2\le i<j\le n}\frac{1-x_j/x_i}{(1-qx_j/x_i)}\times\prod_{j=2}^{n}\frac{1-x_jq}{1-x_jq^2}$$

which simplifies to

$$\frac{1}{(1-q^2)(1-q^{-1}x_2\cdots x_n)}\prod_{i=2}^{n}\frac{1}{(1-q^2x_i)(1-q/x_i)}\prod_{2\le i<j\le n}\frac{1-x_j/x_i}{(1-qx_j/x_i)}$$

# Outline

# Those Have Been Done

1. The partial fraction algorithm has been implemented by the Maple package Ell2.mpl (an update of Ell.mpl) at http://www.combinatorics.net.cn/homepage/xin/MPA.mht

# Those Have Been Done

1. The partial fraction algorithm has been implemented by the Maple package Ell2.mpl (an update of Ell.mpl) at http://www.combinatorics.net.cn/homepage/xin/MPA.mht
2. The package is used to work on problems relating to invariant theory. (Joint with A. Garsia, G. Musiker, N. Wallach, and M. Zabrocki)

# Those Have Been Done

1. The partial fraction algorithm has been implemented by the Maple package Ell2.mpl (an update of Ell.mpl) at http://www.combinatorics.net.cn/homepage/xin/MPA.mht
2. The package is used to work on problems relating to invariant theory. (Joint with A. Garsia, G. Musiker, N. Wallach, and M. Zabrocki)
3. The partial fraction algorithm has been used to prove Andrews' $q$-Dyson conjecture (also Zeilberger-Bressoud thoerem) and related constant term evaluation. (Joint with I. Gessel, Joint with L. Lv and Y. Zhou)

# Those Have Been Done

1. The partial fraction algorithm has been implemented by the Maple package Ell2.mpl (an update of Ell.mpl) at http://www.combinatorics.net.cn/homepage/xin/MPA.mht
2. The package is used to work on problems relating to invariant theory. (Joint with A. Garsia, G. Musiker, N. Wallach, and M. Zabrocki)
3. The partial fraction algorithm has been used to prove Andrews' $q$-Dyson conjecture (also Zeilberger-Bressoud thoerem) and related constant term evaluation. (Joint with I. Gessel, Joint with L. Lv and Y. Zhou)
4. Iterated Laurent series is a special case of Malcev-Neumann series arose from algebra. Using MN-series, I gave a simplification of Stanley's monster reciprocity theorem.

# Other Packages

1. Omega package (Mathematica) by G. Andrews, P. Paule, A. Riese.
2. Maple packages by S. Corteel, G. Han, C. Savage and others
3. J. Stembridge's posets package based on Stanley's work.
4. LattE by J.A. De Loera, R. Hemmecke, R. Tanzer, R. Yoshida, based on Barvinok's work for lattice points in a convex rational polytope.

## The End

Thank you!