

COUNTING LATTICE PATHS BY GESSEL PAIRS

GUOCE XIN

ABSTRACT. We count a large class of lattice paths by using factorizations of free monoids. Besides the classical lattice paths counting problems related to Catalan numbers, we give a new approach to the problem of counting walks on the slit plane (walks avoid a half line) that was first solved by Bousquet-Mélou and Schaeffer. We also solve a problem about walks in the half plane avoiding a half line by subsequently applying the factorizations of two different Gessel pairs, giving a generalization of a result of Bousquet-Mélou.

Keywords: *lattice path, slit plane, generating function, Laurent series*

1. INTRODUCTION

Ira Gessel (1980) [3] connected the factorization of formal Laurent series with that of lattice paths and solved a large class of lattice path enumeration problems, in which the unique factorization lemma (Lemma 2.3 below) plays an important role.

After about two decades, Bousquet-Mélou and Schaeffer [2] remarkably solved the counting problem of walks on the slit plane: lattice paths that start at $(0, 0)$ with steps in a finite subset \mathfrak{S} of \mathbb{Z}^2 and never hit $(-k, 0)$ for any nonnegative integer k after the starting point. Additionally they showed that the complete generating functions of such walks are algebraic in some models, and gave a surprising combinatorial interpretation of Catalan numbers. In solving problems of walks on the slit plane of $[2; 1]$, the unique factorization lemma again plays an important role.

This coincidence strongly suggests the existence of a factorization of lattice paths that applies directly to walks on the slit plane. We generalize Gessel's factorization of lattice paths to that of a Gessel pair: a free monoid together with a homomorphism to \mathbb{Z} . It turns out that such factorizations can be repeatedly applied by choosing different Gessel pairs. This technique yields a new approach to walks on the slit plane, and a solution to the counting problems of lattice paths avoiding a half plane and a half line.

We introduce the unique factorization lemma in Section 2, and the concept of Gessel pairs in Section 3. In Section 4 we study some explicit examples of Gessel pairs including walks on the slit plane and walks on the half plane avoiding a half line. We shall see that Theorem 2.4 is a basic computational tool.

2. THE UNIQUE FACTORIZATION LEMMA

The unique factorization lemma was developed in [3] for the ring $K[[x, y/x]]$ and rediscovered in [2] for the ring $K[x, y, x^{-1}, y^{-1}][[t]]$, where K is a field. It will be rephrased for the field of iterated Laurent series so that we can deal with a larger class of problems.

Let K be a field and let x_1, \dots, x_n be a set of variables. The *field of iterated Laurent series* $K\langle\langle x_1, \dots, x_n \rangle\rangle$ is inductively defined to be $K\langle\langle x_1, \dots, x_{n-1} \rangle\rangle((x_n))$, with $K\langle\langle x_1 \rangle\rangle$ being the field of Laurent series $K((x_1))$. An iterated Laurent series is first regarded as a Laurent series in x_n , then a Laurent series in x_{n-1} , and so on.

A fundamental structure theorem [5] gives an overview of iterated Laurent series: an iterated Laurent series is a formal series that has a well-ordered support, where \mathbb{Z}^n is ordered reverse lexicographically, and the *support* of a formal Laurent series is

$$\text{supp} \sum_{(i_1, \dots, i_n) \in \mathbb{Z}^n} a_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n} := \{(i_1, \dots, i_n) \mid a_{i_1, \dots, i_n} \neq 0\}.$$

For an iterated Laurent series f , we can then define its order $\text{ord}(f)$ to be the minimum of its support, and its initial term to be the term with the minimum order. We say that f has a positive order if $\text{ord}(f) > \text{ord}(1) = (0, \dots, 0)$.

Consequently, the following linear operators are well defined in $\mathbb{C}\langle\langle x_1, \dots, x_n \rangle\rangle$:

$$\text{CT}_x \sum_{n \in \mathbb{Z}} b_n x^n = b_0, \quad \text{PT}_x \sum_{n \in \mathbb{Z}} b_n x^n = \sum_{n \geq 0} b_n x^n, \quad \text{NT}_x \sum_{n \in \mathbb{Z}} b_n x^n = \sum_{n < 0} b_n x^n,$$

where x is one of the variables, and b_n may involve the other variables. We say that $f(x)$ is PT in x if f contains only nonnegative powers in x . Obviously if $f(x)$ is PT in x then $\text{CT}_x f(x) = f(0)$. We will see that these operators have combinatorial meanings in lattice path enumeration.

The following two propositions follows from a general theory of Malcev-Neumann series. See [5, Theorem 3.1.7 and Proposition 3.2.6].

Proposition 2.1 (Composition Law). *Suppose that F belongs to $K\langle\langle x_1, \dots, x_n \rangle\rangle$ and $\text{ord}(F) > \text{ord}(1)$. Then if $b_i \in K$ for all i ,*

$$\sum_{i=0}^{\infty} b_i f^i$$

is well defined and belong to $K\langle\langle x_1, \dots, x_n \rangle\rangle$, in the sense that all of its coefficients are finite sum of nonzero elements in K .

Proposition 2.2 (Generalized Composition Law). *Let f be the initial term of $F \in K\langle\langle x_1, \dots, x_n \rangle\rangle$. For any $\Phi(x_1, \dots, x_n)$ belongs to $K\langle\langle x_1, \dots, x_n \rangle\rangle$, and any fixed i , $\Phi|_{x_i=F}$ is well defined if and only if $\Phi|_{x_i=f}$ is well defined.*

By the composition law, if $\text{ord}(F) > \text{ord}(1)$, then $\log(1 + F)$ is well defined. The generalized composition law is useful in the application of the kernel method. The other manipulation we will use for iterated Laurent series are given as follows.

For any fixed variable x and an iterated Laurent series $f(x)$, we have that $f(x)$ can be uniquely written as $f_1(x) + f_2(x)$ with f_1 containing only nonnegative powers in x and f_2 containing only negative powers in x . Clearly, $f_1(x) = \text{PT}_x f(x)$ and $f_2(x) = \text{NT}_x f(x)$.

The unique factorization lemma follows from the above fact through taking a logarithm.

Lemma 2.3 (Unique Factorization Lemma). *Let h be an element of $K\langle\langle x_1, \dots, x_n \rangle\rangle$ with initial term 1. Then for each i , h has a unique factorization in $K\langle\langle x_1, \dots, x_n \rangle\rangle$ such that $h = h_- h_0 h_+$, where except for their initial terms, which are 1, h_- contains only negative powers in x_i , h_0 is independent of x_i , and h_+ contains only positive powers in x_i .*

Theorem 2.4 below is a generalization of the Lagrange inversion formula. It plays an important role in the proof of a conjecture about walks on the slit plane [4].

Theorem 2.4. *Let $G(x, t), F(x, t) \in K[[x, t]]$. If $G(x, 0)$ can be written as $ax + \text{higher terms}$, with $a \neq 0$, then*

$$(2.1) \quad \text{CT}_x \frac{x}{G(x, t)} F(x, t) = \left. \frac{F(x, t)}{\frac{\partial}{\partial x} G(x, t)} \right|_{x=X},$$

where $X = X(t)$ is the unique element in $tK[[t]]$ such that $G(X, t) = 0$.

3. FACTORIZATION FOR GESSEL PAIRS

By a *monoid* we mean a set M , equipped with a multiplication which is associative and has a unit element 1. An element in M is said to be a *prime* if it does not have a nontrivial factorization. We say that M is a *free monoid* if every element in M can be uniquely factored as a product of primes in M .

We are going to present factorizations of free monoids with respect to their homomorphisms to \mathbb{Z} . Our objects will be mainly lattice paths: a path σ in \mathbb{Z}^2 is a finite sequence of lattice points $(a_0, b_0), \dots, (a_n, b_n)$ in \mathbb{Z}^2 , in which we call (a_0, b_0) the starting point, (a_n, b_n) the ending point, $(a_i - a_{i-1}, b_i - b_{i-1})$ the steps of σ , and n the length of σ .

In this paper, the starting point of a path is always $(0, 0)$. The theory for other starting points is similar.

Given two paths σ_1 and σ_2 , we define their product $\sigma_1\sigma_2$ to be the path whose steps are those of σ_1 followed by those of σ_2 . Thus the empty path ε is the unit. If $\pi = \sigma_1\sigma_2$, then we call σ_1 a *head* of π , and σ_2 a *tail* of π .

Let \mathfrak{S} be a finite subset of \mathbb{Z}^2 . We are interested in paths all of whose steps lie in \mathfrak{S} . Denote by \mathfrak{S}^* the set of all such paths. Then \mathfrak{S}^* is a free monoid and the primes are elements of \mathfrak{S} . The weight of a step $(a, b) \in \mathfrak{S}$ is defined to be $\Gamma((a, b)) = x^a y^b t$, and the weight of a path $\sigma = s_1 \cdots s_n$ is defined to be $\Gamma(\sigma) = \Gamma(s_1) \cdots \Gamma(s_n)$. It is easy to see that for any two paths σ_1 and σ_2 , we have $\Gamma(\sigma_1\sigma_2) = \Gamma(\sigma_1)\Gamma(\sigma_2)$. If P is a subset of \mathfrak{S}^* , then we define

$$(3.1) \quad \Gamma(P) = \sum_{\sigma \in P} \Gamma(\sigma) = \sum_{n \geq 0} \sum_{i, j \in \mathbb{Z}} a_{i, j}(n) x^i y^j t^n,$$

where $a_{i, j}(n)$ is the number of paths in P of length n that end at (i, j) . We also call $\Gamma(P)$ the generating function of P with respect to the ending points and the lengths.

In the special case that P is the whole set \mathfrak{S}^* , we have

$$\Gamma(\mathfrak{S}^*) = \sum_{n \geq 0} (\Gamma(\mathfrak{S}))^n = (1 - \Gamma(\mathfrak{S}))^{-1},$$

since each term in $(\Gamma(\mathfrak{S}))^n$ corresponds to a path of n steps.

The above equation is interpreted as an identity in $\mathbb{C}[x, x^{-1}, y, y^{-1}][[t]]$, which can be embedded into the field of iterated Laurent series $\mathbb{C}\langle\langle x, y, t \rangle\rangle$. In fact, we can relax the condition on \mathfrak{S} to a well-ordered subset of \mathbb{Z} , and the composition law will guarantee the existence of $\Gamma(\mathfrak{S}^*)$ and hence $\Gamma(P)$.

Since \mathfrak{S} is uniquely determined by $\Gamma(\mathfrak{S})$, sometime we give $\Gamma(\mathfrak{S})$ instead of \mathfrak{S} . Some operators on $\mathbb{C}\langle\langle x, y \rangle\rangle[[t]]$ have simple combinatorial interpretations. Let P be a subset of \mathfrak{S}^* with generating function given by (3.1).

- (1) The generating function for those paths in P that end on the line $y = 0$ is given by $\text{CT}_y \Gamma(P)$.
- (2) The generating function for those paths in P that end above the line $y = -1$ is given by $\text{PT}_y \Gamma(P)$.
- (3) The generating function for those paths in P that end below the line $y = 0$ is given by $\text{NT}_y \Gamma(P)$.

Similar properties hold for x .

Now suppose that H is a set of paths with steps in \mathfrak{S} and that H is a free monoid. Then for any $\sigma \in H$ with its factorization into primes $\sigma = h_1 h_2 \cdots h_m$, we say that $h_1 h_2 \cdots h_i$ is an H -*head* of σ for $i = 0, 1, \dots, m$. If we let \mathcal{P} be the set of primes in H , then $\Gamma(H) = 1/(1 - \Gamma(\mathcal{P}))$.

For example, as we have described, \mathfrak{S}^* is a free monoid; the set of all paths in \mathfrak{S}^* that end on the x -axis is a free monoid, whose primes are those paths that return to the x -axis only at the end point; the set of all paths in \mathfrak{S}^* that end at $(k, 0)$ for some

$k \geq 0$ is a free monoid, whose primes are those paths that only return the nonnegative half of the x -axis at the end point.

Let ρ be a map from H to \mathbb{Z} . We say that ρ is a *homomorphism* from H to \mathbb{Z} if $\rho(\epsilon) = 0$ and for all $\sigma_1, \sigma_2 \in H$, $\rho(\sigma_1\sigma_2) = \rho(\sigma_1) + \rho(\sigma_2)$. The ρ value of a path σ is $\rho(\sigma)$.

If H is a free monoid, then any map from H to \mathbb{Z} defined on the primes of H induces a homomorphism. If in addition, H is a subset of \mathfrak{S}^* , then the natural map to the end point of a path is a homomorphism from H to \mathbb{Z}^2 . Therefore, any homomorphism from \mathbb{Z}^2 to \mathbb{Z} induces a homomorphism from H to \mathbb{Z} through that natural map. The following two homomorphisms are useful. Define $\rho_x(\sigma)$ to be the x coordinate of the ending point of σ . Then ρ_x is clearly a homomorphism. Similarly we can define ρ_y .

If H is a free monoid, and ρ is a homomorphism from H to \mathbb{Z} , then we call (H, ρ) a *Gessel pair*. For a Gessel pair (H, ρ) , we define:

A *minus-path* is either the empty path or a path whose ρ value is negative and less than the ρ values of all the other H -heads.

A *zero-path* is a path with ρ value 0 and all of whose H -heads have nonnegative ρ values.

A *plus-path* is a path all of whose H -heads (except ϵ) have positive ρ values.

For a Gessel pair (H, ρ) , we denote by H_- , H_0 , and H_+ respectively to be the sets of minus-, zero-, and plus-paths in H . Note that the empty path, but no other path, belongs to all three classes. The path $h_1h_2 \cdots h_n$, where $h_i \in H$, is a minus-path if and only if $h_nh_{n-1} \cdots h_1$ is a plus-path; thus the theories of minus- and plus-paths are identical.

Lemma 3.1. *Let (H, ρ) be a Gessel pair, and let π be a path in H . Then π has a unique factorization $\pi_- \pi_0 \pi_+$, where π_- is a minus-path, π_0 is a zero-path, and π_+ is a plus-path.*

Proof. Let a be the smallest among all the ρ values of the H heads of π . Let π_- be the shortest H -head of π whose ρ value equals a . Then if $\pi = \pi_- \sigma$, let $\pi_- \pi_0$ be the longest H -head of π whose ρ value equals a , and let π_+ be the rest of σ . It is easy to see that this factorization satisfies the required conditions.

To see that it is unique, let $\tau_- \tau_0 \tau_+$ be another factorization of π . By definition, any H -head of $\tau_0 \tau_+$ has a nonnegative ρ value. So the minimum ρ value among all of the H -heads of π is achieved in τ_- . By definition, it equals $\rho(\tau_-)$ and is unique in τ_- . Therefore, $\rho(\tau_-) = a$ and $\tau_- = \pi_-$ by the selection of π_- . The reasons for $\pi_0 = \tau_0$ and $\pi_+ = \tau_+$ are similar.

□

Proposition 3.2. *If (H, ρ) is a Gessel pair, then H_- , H_0 , and H_+ are all free monoids. The map from H to $H_- \times H_0 \times H_+$ defined by $\pi \rightarrow (\pi_-, \pi_0, \pi_+)$ is a bijection.*

Proof. By Lemma 3.1, the map defined by $\pi \rightarrow (\pi_-, \pi_0, \pi_+)$ is clearly a bijection. Now we show that H_- , H_0 , and H_+ are all free monoids.

It is easy to see that they are monoids. We only show that H_- is free. The other parts are similar. Let \mathcal{P} be the subset of H_- such that $\sigma \in \mathcal{P}$ if and only if $\rho(\sigma)$ is negative and every other H -head of σ has nonnegative ρ value. We claim that \mathcal{P} is the set of primes in H_- .

Clearly any $\sigma \in \mathcal{P}$ cannot be factored as the product of two nontrivial elements in H_- . Now let $\pi \in H_-$. In order to factor π into factors in \mathcal{P} , we find the shortest H -head of π that has negative ρ value, and denote it by σ_1 . Then π is factored as $\pi = \sigma_1 \pi'$ for some π' in H . From the definition of minus-path, $\rho(\sigma_1)$ is either less than $\rho(\pi)$, in which case π' is clearly in H_- , or $\rho(\sigma_1) = \rho(\pi)$, in which case π' has to be the unit and $\pi = \sigma_1$ is in \mathcal{P} . So we can inductively obtain a factorization of π into elements in \mathcal{P} .

The uniqueness of this factorization is clear. □

In a Gessel pair (H, ρ) , the weight of an element $\pi \in H$ is defined to be $\Gamma(\pi)z^{\rho(\pi)}$, where z is a new variable. When H is also a subset of \mathfrak{S}^* and we are considering the Gessel pair (H, ρ_x) , the power in z is always the same as the power in x for any π in H . So we can replace z by 1 and let x play the same role as z . Since the factorization in H is with respect to ρ , the factorization of generating function is with respect to z .

Theorem 3.3. *For any Gessel pair (H, ρ) , we have $\Gamma(H_-) = [\Gamma(H)]_-$, $\Gamma(H_0) = [\Gamma(H)]_0$, and $\Gamma(H_+) = [\Gamma(H)]_+$.*

Proof. From Proposition 3.2, it follows that $\Gamma(H) = \Gamma(H_-)\Gamma(H_0)\Gamma(H_+)$. Clearly except 1, which is the weight of the empty path, $\Gamma(H_-)$ contains only negative powers in z , $\Gamma(H_0)$ is independent of z , and $\Gamma(H_+)$ contains only positive power in z . The theorem then follows from the unique Factorization Lemma with respect to z . □

Gessel [3] gives many interesting examples involving lattice paths on the plane. We introduce the most classical example as the following:

Example 3.4. Let \mathfrak{S} be $\{(1, r), (1, -1)\}$ with $r \geq 1$, and $H = \mathfrak{S}^*$. Consider the Gessel pair (H, ρ_y) .

Note that in this case the length of a path equals the x coordinate of its end point. Replacing x by 1 will not lose any information.

Clearly we have

$$\Gamma(H) = \Gamma(\mathfrak{S}^*) = \frac{1}{1 - t(y^r + 1/y)}.$$

We see that H_+ is the set of paths in \mathfrak{S}^* that never go below level 1 after the starting point. The set H_0 contains all paths in \mathfrak{S}^* that end on level 0 and never go below level 0. When $r = 1$, these are Dyck paths.

To compute $\Gamma(H_0) := F(t)$, we let $Y(t)$ be the unique positive root of $y - t(1 + y^{r+1})$. It is not hard to show that $F(t) = Y(t)/t$ and hence $F(t) = 1 + t^{r+1}F(t)^{r+1}$. Therefore $F(t)$ equals the generating function of complete $r + 1$ -ary trees.

Example 3.5. Let \mathfrak{S} be $\{(1, 1), (1, -1)\}$, and let $H = \mathfrak{S}^*$. Let ρ be determined by $\rho(1, 1) = r$ and $\rho(1, -1) = -1$.

It is easy to see that this example is isomorphic to the previous one.

Example 3.6. In general if $H = \mathfrak{S}^*$, then (H, ρ_y) is a Gessel pair.

We see that H_+ is the set of paths in \mathfrak{S}^* that never go below the line $y = 1$ after the starting point.

If we let $J = H_+$, then J is also a free monoid. The primes of J are paths that start at $(0, 0)$, end at some positive level d , and never hit level $d - 1$ or lower.

The set H_0 contains all paths in \mathfrak{S}^* that end on the line $y = 0$, and never go below the line $y = 0$. In other words, H_0 contains all paths in \mathfrak{S}^* that stays in the upper half plane and end on the x -axis.

If we let $J = H_0$, then (J, ρ_x) is a Gessel pair. The set J_+ contains all paths in J that avoid the half line $\mathcal{H} = \{(-k, 0); k \in \mathbb{N}\}$ after the starting point. This is the same as walks on the half plane avoiding the half line in [1].

The set J_0 contains all paths in J that end at $(0, 0)$ and never touch the half line \mathcal{H} except $(0, 0)$.

Walks on the slit plane are paths that start at $(0, 0)$ with steps in \mathfrak{S} and never hit the half line \mathcal{H} after the starting point. In solving counting problem of walks on the slit plane [2; 1], it is crucial to obtain the following functional equation (3.2), which will be explained combinatorially in Example 3.8.

$$(3.2) \quad S_0(x, t) \frac{1}{1 - B(x^{-1}, t)} = S_x(x; t) = \text{CT}_y \frac{1}{1 - \Gamma(\mathfrak{S})},$$

where $B(x^{-1}, t)$ is the generating function of paths that start at $(0, 0)$, and only hit $(-k, 0)$ for some $k \geq 0$ at the end point; $S_0(x, t)$ is the generating function of walks on the slit plane that end on the line $y = 0$; $S_x(x; t)$ is the generating function of *bilateral walks* [1]: paths in \mathfrak{S}^* that end on the x -axis.

After obtaining equation (3.2), we can check that $S_x(x;0) = 1$, $S_0(x,0) = 1$, and $B(x^{-1},0) = 1$, and that except for 1, $S_0(x,t)$ contains only positive powers in x , $(1 - B(x^{-1},t))^{-1}$ contains only negative powers in x . Thus the unique factorization lemma applies, and we obtain the following remarkable result in [1], which says that the $S_0(x,t)$, $B(x^{-1},t)$, and the complete generating function for walks on the slit plane $S(x,y;t)$ can be theoretically computed. In practice, computing them is not a easy task. Only special cases have been thoroughly studied.

Theorem 3.7 (Bousquet-Mélou). *Let \mathfrak{S} be a well-ordered subset in \mathbb{Z}^2 . Using notation as above, we have:*

$$(3.3) \quad S_0(x,t) = (S_x(x,t))_+,$$

$$(3.4) \quad \frac{1}{1 - B(x^{-1},t)} = (S_x(x,t))_0 (S_x(x,t))_-,$$

$$(3.5) \quad S(x,y;t) = \frac{1}{(1 - \Gamma(\mathfrak{S})) (S_x(x,t))_0 (S_x(x,t))_-}.$$

Walks on the slit plane can be counted by a factorization of Gessel pair.

Example 3.8. For any \mathfrak{S} , let H be the set of paths that end on the x -axis. Then (H, ρ_x) is a Gessel pair.

The set H_+ contains all paths that end on the x axis and never hit the half line $\mathcal{H} = \{(-k,0) \mid k \geq 0\}$ after the starting point. This is exactly the walks on the slit plane that end on the x -axis.

The set H_0 , which was called the set of loops in [1], consists of all paths that end at $(0,0)$, and never touch $(-k,0)$ for $k = 1, 2, \dots$

The combinatorial explanation of equation (3.2) is as follows. The set H_-H_0 is a free monoid. It contains all paths that end at $(-k,0)$ for some $k \geq 0$. Its primes are all paths that hit $(-k,0)$ only once at the end point. Clearly, the primes are counted by $B(x^{-1};t)$. So we have

$$\Gamma(H_-H_0) = \frac{1}{1 - B(x^{-1};t)}, \quad \text{and } \Gamma(H_+) = S_0(x,t).$$

Equation (3.2) then follows.

Remark 3.9. Note that equation (3.2) can also be explained combinatorially by using the cycle lemma as in [1].

Example 3.10. For any \mathfrak{S} , let H be the set of paths that end on the x -axis and never go below the line $y = -d$ for some given $d > 0$. Then it is easy to check that (H, ρ_x) is a Gessel pair.

The set H_+ contains all paths that end on the x -axis, and never hit the half line \mathcal{H} after the starting point, and never go below the line $y = -d$.

The set H_0 can be similarly described.

Example 3.11. For any \mathfrak{S} , let H be the set of paths that end on the x -axis and never go below the line $y = -d$ and never go above the line $y = f + 1$ for some given positive integers d and e . Then it is easy to see that (H, ρ_x) is a Gessel pair.

This example is similar to the previous one.

4. EXPLICIT EXAMPLES

We will discuss two explicit examples that were proposed in [1]. Taking walks on the slit plane as an example, we see that $\log S_0(x, t) = \text{PT}_x \log S_x(x; t)$. Now if $\log S_0(x, t)$ has the form $b(t)x^p + \text{higher degree terms}$, then so does $S_0(x, t) - 1$.

Proposition 4.1 (Proposition 4, [1]). *Let p be the smallest positive integer such that there is a walk on the slit plane with respect to a finite set \mathfrak{S} that ends at $(p, 0)$. Then the generating function for such walks ending at $(p, 0)$ is D -finite and is given by*

$$S_{p,0}(t) = [x^p] \log S_x(x; t).$$

Bousquet-Mélou shows in addition that $S_{k,0}$ is D -finite for every k .

Our task is to find a formula for $\log \Gamma(H_+)$ for a given algebraic $\Gamma(H)$ as described in last section. The idea is as follows. Let $P(x, y, t)$ be a polynomial and let $Y(x; t)$ be the unique root of positive order of $y - tP(x, y, t)$ as a polynomial in y . The problem will be reduced to finding the unique factorization of a rational function $Q(x, Y(t), t)$ with respect to x . We are especially interested in $[x^p]Q_+(x, Y(t), t)$ for a certain integer p , which is D -finite by Proposition 4.1. This generating function can be obtained if we can get a nice form for $\frac{\partial}{\partial t} \log Q(x, Y(t), t)$. Our approach to finding such a nice form is to do all the computation implicitly. It is best illustrated by examples.

Example 4.2. Let \mathfrak{S} be the set $\{(1, 0), (-1, 0), (0, 2), (0, -1)\}$, or equivalently, $\Gamma(\mathfrak{S}) = t(x + x^{-1} + y^2 + y^{-1})$. Bousquet-Mélou proposed in [1] the problem of solving walks on the slit plane in this model, or even replacing the 2 by a general positive integer q .

Our method works for general q , but so far we have found a reasonable formula only for $q = 2$. We have:

Proposition 4.3. *The number of walks on the slit plane, of length N , ending at $(1, 0)$, and with steps in $\{(1, 0), (-1, 0), (0, 2), (0, -1)\}$ equals*

$$(4.1) \quad a_{1,0}(N) = \binom{N}{\frac{N-1}{2}} + \sum_{n=1}^{\lfloor N/3 \rfloor} \frac{3^{3n-1}}{n2^{2n}} \binom{N-1}{3n-1} \binom{N-3n}{\frac{N-3n}{2}} +$$

$$\sum_{n,m,k} \frac{3^{3m+2}}{nN2^{2m+2}} \binom{n}{k, 2k+1, n-3k-1} \binom{N-n}{3m+2} \binom{N-3m-3k-3}{\frac{N-3m-3k-4}{2}},$$

where $\binom{A}{B+1/2}$ is interpreted as 0 for all integers A, B , and the second sum ranges over all n, m, k such that $1 \leq n \leq N$, $0 \leq m \leq \frac{N-n-2}{3}$, and $0 \leq k \leq \frac{n-1}{3}$.

Proof. We proceed by computing $S_x(x; t)$. Let $b = x+x^{-1}$. Then $\Gamma(\mathfrak{S}) = t(b+y^2+y^{-1})$. Applying Theorem 2.4, we get

$$S_x(x; t) = \text{CT}_y \frac{y}{y - t(y^3 + by + 1)} = \frac{1}{1 - tb - 3tY^2},$$

where $Y = Y(t) = Y(b, t) = Y(x, t)$ is the unique root of positive order of the denominator for y . More precisely, Y is the unique power series in t with constant term 0 that satisfies

$$(4.2) \quad Y(t) - t(Y(t)^3 + bY(t) + 1) = 0.$$

Using the Lagrange inversion formula we get

$$(4.3) \quad Y(t) = \sum_{n \geq 1} \sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} \binom{n}{k, 2k+1, n-3k-1} b^{n-3k-1} t^n.$$

We can compute $\log S_x(x; t)$ explicitly in order to obtain $\log S_0(x, t)$. We have

$$(4.4) \quad \begin{aligned} \frac{\partial}{\partial t} \log S_x(x; t) &= \frac{b + 3(Y(t))^2 + 6tY(t) \frac{\partial}{\partial t} Y(t)}{1 - tb - 3tY(t)^2} \\ &= \frac{b - tb^2 + 3(Y(t))^2 - 3(Y(t))^4 t + 6tY(t)}{(1 - tb - 3tY(t)^2)^2} \end{aligned}$$

where

$$\frac{\partial}{\partial t} Y(t) = \frac{1 + bY(t) + Y(t)^3}{1 - tb - 3tY(t)^2}$$

is determined implicitly by equation (4.2).

Since $Y(t)$ satisfies (4.2), we can rewrite (4.4) as $C_0 + C_1 Y(t) + C_2 Y(t)^2$, where C_i are rational functions of b and t . These can be found by Maple, and we get

$$(4.5) \quad \frac{\partial}{\partial t} \log S_x(x; t) = \frac{(4b^3 + 27)t^2 - 8tb^2 + 4b}{4(1-bt)^3 - 27t^3} + \frac{9tY(t)}{4(1-bt)^3 - 27t^3}.$$

Now we need to integrate to get $\log S_x(x; t)$. The first term has a simple form:

$$\int \frac{(4b^3 + 27)t^2 - 8tb^2 + 4b}{4(1-bt)^3 - 27t^3} dt = \log(4(1-bt)^3 - 27t^3)^{-1/3} + C,$$

where C is independent of t . After some manipulation, we get

$$\int \frac{(4b^3 + 27)t^2 - 8tb^2 + 4b}{4(1-bt)^3 - 27t^3} dt = \log \frac{1}{1-bt} + \sum_{N \geq 1} \sum_{n=1}^{\lfloor N/3 \rfloor} \frac{3^{3n-1}}{n2^{2n}} \binom{N-1}{3n-1} b^{N-3n} t^N + C.$$

For the second term, we have

$$\frac{9t}{4(1-bt)^3 - 27t^3} = \frac{9t}{4(1-bt)^3} \frac{1}{1 - 27t^3/(4(1-bt)^3)}.$$

After some manipulation, we get

$$\frac{9t}{4(1-bt)^3 - 27t^3} = \sum_{m \geq 0} \frac{3^{3m+2}}{2^{2m+2}} \sum_{r \geq 0} \binom{3m+r+2}{3m+2} b^r t^{3m+r+1}.$$

Thus together with the expansion of $Y(t)$ given by (4.3), we obtain

$$\int \frac{9tY(t)}{4(1-bt)^3 - 27t^3} dt = C + \sum_{N \geq 1} \sum_{n=1}^N \sum_{m=0}^{\lfloor \frac{N-n-2}{3} \rfloor} \sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} \frac{3^{3m+2}}{nN2^{2m+2}} \binom{n}{k, 2k+1, n-3k-1} \binom{N-n}{3m+2} b^{N-3m-3k-3} t^N.$$

Since $S_x(x, 0) = 1$, it is easy to check that the sum of the two constants C must be 0.

Note that the powers in b is always nonnegative. It is easy to separate the negative powers and positive powers in x of $b^M = (x + x^{-1})^M$ for every nonnegative integer M . Thus we can obtain a formula for $\log S_0(x, t)$. In particular, from the formulas $[x](x + x^{-1})^M = \binom{M}{\frac{M-1}{2}}$ and $S_{1,0}(t) = [x] \log S_x(x; t)$, we get (4.1). \square

Example 4.4. We consider walks on the half plane avoiding a half line; more precisely, walks that never touch the half line \mathcal{H} and never hit a point (i, j) with $j < 0$. This is a continuation of Example 3.6. We denote by $HS(x, y; t)$ the generating function for such paths.

It turns out that this case is simpler than the previous one. We obtain the following result, which includes [1, Proposition 25] as a special case.

Theorem 4.5. *For any well-ordered set \mathfrak{S} , let p be the smallest positive number such that there is an \mathfrak{S} -path end at $(p, 0)$. Then the number of walks on the half plane avoiding the half line that end at $(p, 0)$ and are of length n is equal to $1/n$ times the number of \mathfrak{S} -paths that end at $(p, 0)$ and are of length n .*

Proof. We use the notation of Example 3.6. From the Gessel pair (\mathfrak{S}^*, ρ_y) , we have $\Gamma(H_0) = (\Gamma(\mathfrak{S}^*))_0$ and

$$\log \Gamma(H_0) = \text{CT}_y \log \Gamma(\mathfrak{S}^*) = \text{CT}_y \log \frac{1}{1 - \Gamma(\mathfrak{S})}.$$

Now let $J = H_0$ and consider the Gessel pair (J, ρ_x) . Then

$$\log \Gamma(J_0 J_+) = \text{PT}_x \log \Gamma(J).$$

In particular, we have

$$[x^p]\Gamma(J_+) = [x^p]\log \Gamma(J) = [x^p]\log \Gamma(H_0) = [x^p] \underset{y}{\text{CT}} \log \Gamma(\mathfrak{S}^*).$$

Therefore,

$$[x^p t^n]\Gamma(J_+) = [x^p y^0 t^n] \frac{1}{n} \Gamma(\mathfrak{S})^n.$$

This prove the theorem. □

Acknowledgment. The author is very grateful to Ira Gessel and Mireille Bousquet-Mélou.

REFERENCES

1. M. Bousquet-Mélou, *Walks on the slit plane: other approaches*, Adv. in Appl. Math. **27** (2001), 243–288, Special issue in honor of Dominique Foata’s 65th birthday (Philadelphia, PA, 2000).
2. M. Bousquet-Mélou and G. Schaeffer, *Walks on the slit plane*, Probab. Theory Related Fields **124** (2002), 305–344.
3. I. M. Gessel, *A factorization for formal Laurent series and lattice path enumeration*, J. Combin. Theory Ser. A **28** (1980), 321–337.
4. G. Xin, *Proof of a conjecture about walks on the slit plane*, Disc. Math. **282** (2004), 281–287.
5. G. Xin, *The Ring of Malcev-Neumann Series and the Residue Theorem*, Ph.D. thesis, Brandeis University, 2004, arXiv:math.CO/0405133.

DEPARTMENT OF MATHEMATICS, BRANDEIS UNIVERSITY, WALTHAM MA 02454-9110

E-mail address: guoce.xin@gmail.com