

# A RESIDUE THEOREM FOR MALCEV-NEUMANN SERIES

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ABSTRACT. In this paper, we establish a residue theorem for Malcev-Neumann series that requires few constraints, and includes previously known combinatorial residue theorems as special cases. Our residue theorem identifies the residues of two formal series that are related by a change of variables. We obtain simple conditions for when a change of variables is possible, and find that the two related formal series in fact belong to two different fields of Malcev-Neumann series. The multivariate Lagrange inversion formula is easily derived and Dyson's conjecture is given a new proof and generalized.

**Keywords:** *Totally ordered group, Malcev-Neumann series, residue theorem, Lagrange inversion*

## 1. INTRODUCTION

Let  $\mathcal{G}$  be a *totally ordered group*, i.e., a group with a total ordering  $\leq$  that is compatible with its group structure. Let  $K$  be a field and let  $K_w[\mathcal{G}]$  be the set of *Malcev-Neumann series* (MN-series for short) on  $\mathcal{G}$  over  $K$  relative to  $\leq$ : an element in  $K_w[\mathcal{G}]$  is a series  $\eta = \sum_{g \in \mathcal{G}} a_g g$  with  $a_g \in K$ , such that the support  $\{g \in \mathcal{G} : a_g \neq 0\}$  of  $\eta$  is a well-ordered subset of  $\mathcal{G}$ .

By a theorem of Malcev [11] and Neumann [12] (see also [13, Th. 13.2.11]),  $K_w[\mathcal{G}]$  is a division algebra that includes the group algebra  $K[\mathcal{G}]$  as a subalgebra. We study the field of MN-series on a totally ordered abelian group, and show that the field of iterated Laurent series  $K\langle\langle x_1, \dots, x_n \rangle\rangle$ , which has been studied in [18, Ch. 2], is a special kind of MN-series. We also obtain a residue theorem for  $K_w[\mathcal{G} \oplus \mathbb{Z}^n]$ , where we identify the generators of  $\mathbb{Z}^n$  with  $x_1, \dots, x_n$  so that we can make a change of variables.

A combinatorial residue formula should at least work for the ring of formal power series. In obtaining such a formula, we usually embed the ring of formal power series into a ring or a field consisting of formal Laurent series, but the embedding is not unique for the multivariate case. Jacobi [10] used the ring  $K((x_1, \dots, x_n))$  of Laurent series, formal series of monomials where the exponents of the variables are bounded from below, to give the following residue formula.

**Theorem 1.1** (Jacobi's Residue Formula). *Let  $f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n)$  be Laurent series. Let  $b_{i,j}$  be integers such that  $f_i(x_1, \dots, x_n)/x_1^{b_{i,1}} \dots x_n^{b_{i,n}}$  is a formal*

power series with nonzero constant term. Then for any Laurent series  $\Phi(y_1, \dots, y_n)$ , we have (provided both sides converge)

$$\operatorname{Res}_{x_1, \dots, x_n} \left| \frac{\partial f_i}{\partial x_j} \right| \Phi(f_1, \dots, f_n) = |b_{i,j}|_{1 \leq i, j \leq n} \operatorname{Res}_{y_1, \dots, y_n} \Phi(y_1, \dots, y_n), \quad (1.1)$$

where  $\operatorname{Res}_{x_1, \dots, x_n}$  means to take the coefficient of  $x_1^{-1} \cdots x_n^{-1}$ .

This formula equates the residues of two formal series related by a change of variables. It has many applications and has been studied by several authors, e.g., Garsia [3], Goldstein [5], Goulden and Jackson [7, p. 19-22], and Henrici [9]. However, Jacobi's formula is rather restricted in application for two reasons: the conditions on the  $f_i$  are too strong, and the condition on  $\Phi$  is not easy to check: given  $f_i$ , when does  $\Phi(f_1, \dots, f_n)$  converge?.

Cheng et al. [1] studied the ring  $K_h((x_1, \dots, x_n))$  of homogeneous Laurent series (formal series of monomials whose total degree is bounded from below), and used homogeneous expansion to give a residue formula. But the above restrictions still exist for the same reason.

Our new residue theorem includes the previous residue theorems of Jacobi and Cheng et al. as special cases. The new formula is easier to apply and more general: the conditions on the  $f_i$  are dropped since we are working in a field; the condition on  $\Phi$  is replaced with a simpler one and we find that the two related formal series in fact belong to two different fields of MN-series. In particular, our theorem applies to any rational function  $\Phi$ .

In section 2 we review some basic properties of MN-series. We give the residue formula in section 3. Then we talk about the (diagonal and non-diagonal) Lagrange inversion formulas in section 4, and give a new proof and a generalization of Dyson's conjecture in section 5.

## 2. BASIC PROPERTIES OF MALCEV-NEUMANN SERIES

A *totally ordered abelian group* or TOA-group is an abelian group  $\mathcal{G}$  (written additively) equipped with a total ordering  $\leq$  that is compatible with the group structure of  $\mathcal{G}$ ; i.e., for all  $x, y, z \in \mathcal{G}$ ,  $x < y$  implies  $x + z < y + z$ . Such ordering  $<$  is also called *translation invariant*. The abelian groups  $\mathbb{Z}, \mathbb{Q}$ , and  $\mathbb{R}$  are all totally ordered abelian groups under the natural ordering.

Let  $K$  be a field. A formal series  $\eta$  on  $\mathcal{G}$  over  $K$  has the form

$$\eta = \sum_{g \in \mathcal{G}} a_g t^g,$$

where  $a_g \in K$  and  $t^g$  is regarded as a symbol. The support  $\operatorname{supp}(\eta)$  of  $\eta$  is defined to be  $\{g \in \mathcal{G} : a_g \neq 0\}$ .

For a TOA-group  $\mathcal{G}$ , an *Malcev-Neumann series* (MN-series for short) is a formal series on  $\mathcal{G}$  that has a well-ordered support. Recall that a well-ordered set is a totally ordered set such that every nonempty subset has a minimum. We define  $K_w[\mathcal{G}]$  to be the set of all such MN-series.

If  $\eta \in K_w[\mathcal{G}]$ , then we can define the *order* of  $\eta$  to be  $\text{ord}(\eta) = \min(\text{supp}(\eta))$ . The *initial term* of  $\eta$  refers to the term with the smallest order. It is clear that  $\text{ord}(\eta\tau) = \text{ord}(\eta) + \text{ord}(\tau)$ . We denote by  $[t^g]\eta$  the coefficient of  $g$  in  $\eta$ .

By a theorem of Malcev and Neumann [13, Th. 13.2.11],  $K_w[\mathcal{G}]$  is a field for any TOA-group. The idea of the proof will be introduced since we will use some of the facts later.

Let us see some examples of MN-series first.

- (1)  $K_w[\mathbb{Z}] \simeq K((x))$  is the field of Laurent series.
- (2) The generalized Puiseux field [15] with respect to a prime number  $p$  consists all series  $f(x)$  such that  $\text{supp}(f)$  is a well ordered subset of  $\mathbb{Q}$  and there is an  $m$  such that for any  $\alpha \in \text{supp}(f)$  we have  $m\alpha = n_\alpha/p^{i_\alpha}$ .
- (3)  $K_w[\mathbb{Q}]$  strictly contains the field  $K^{\text{fra}}((x))$  of fractional Laurent series [14], and is more complicated.
- (4)  $K_w[\mathbb{R}]$  does not seem to be an interesting field.

The set of MN-series  $K_w[\mathcal{G}]$  is clearly closed under summation. The fact that it is closed under multiplication follows from the following proposition. Hence  $K_w[\mathcal{G}]$  is a ring.

For two subsets  $A$  and  $B$  of  $G$ , we denote by  $A + B$  the set  $\{a + b \mid a \in A, b \in B\}$ , and denote by  $A^{+n}$  the set  $A + A + \dots + A$  of  $n$  copies of  $A$ .

**Proposition 2.1.** *If  $\mathcal{G}$  is a TOA-group and  $A, B$  are two well-ordered subset of  $\mathcal{G}$  then  $A + B$  is also well-ordered.*

For a TOA-group  $\mathcal{G}$ ,  $K_w[\mathcal{G}]$  is a maximal ring in the set of all formal series on  $\mathcal{G}$ : if  $\eta = \sum_{g \in \mathcal{G}} a_g t^g$  is not in  $K_w[\mathcal{G}]$ , then adding  $\eta$  into  $K_w[\mathcal{G}]$  cannot form a ring. For if  $\text{supp}(\eta)$  is not well-ordered, we can assume that  $g_1 > g_2 > \dots$  is an infinite decreasing sequence in  $\text{supp}(\eta)$ . Let  $\tau = \sum_{n \geq 1} a_{g_n}^{-1} g_n^{-1}$ . Note that  $\tau \in K_w[\mathcal{G}]$ , since  $g_1^{-1} < g_2^{-1} < \dots$  is well ordered. But the constant term of  $\eta\tau$  equals an infinite sum of 1's, which diverges.

To see  $K_w[\mathcal{G}]$  is a field takes time. Let  $\eta_1, \eta_2, \dots$  be a series of elements in  $K_w[\mathcal{G}]$ . Then we say that  $\eta_1 + \eta_2 + \dots$  exists or *strictly converges* to  $\eta \in K_w[\mathcal{G}]$ , if for every  $g \in \mathcal{G}$ , there are only finitely many  $i$  such that  $[t^g]\eta_i \neq 0$ , and  $\sum_{i \geq 1} [t^g]\eta_i = [t^g]\eta$ . Note that  $\sum_{n \geq 1} 2^{-n}$  does not strictly converge to 1.

Let  $f(z) = \sum_{n \geq 0} b_n z^n$  be a formal power series in  $K[[z]]$ , and let  $\eta \in K_w[\mathcal{G}]$ . Then we define the composition  $f \circ \eta$  to be

$$f \circ \eta := f(\eta) = \sum_{n \geq 0} b_n \eta^n$$

if it exists.

**Theorem 2.2** (Composition Law). *If  $f \in K[[z]]$  and  $\eta \in K_w[\mathcal{G}]$  with  $\text{ord}(\eta) > 0$ , then  $f \circ \eta$  strictly converges in  $K_w[\mathcal{G}]$ .*

The proof of this composition law consists of two parts: one is to show that the support of  $f \circ \eta$  is well-ordered; the other is to show that for any  $g \in \mathcal{G}$ ,  $[t^g]f \circ \eta$  is a finite sum of elements in  $K$ . The following result is the key to the proof. It does not have a simple proof and will be referred to later.

**Proposition 2.3.** *Let  $\mathcal{G}$  be a TOA-group. If  $A$  is a well-ordered subset of  $\mathcal{G}$ , and  $A > 0$ , i.e.,  $a > 0$  for all  $a \in A$ , then  $\cup_{n \geq 0} A^{+n}$  is also well-ordered.*

**Corollary 2.4.** *For any  $\eta \in K_w[\mathcal{G}]$  with initial term 1,  $\eta^{-1} \in K_w[\mathcal{G}]$ .*

*Proof.* Write  $\eta = 1 - \tau$ . Then  $\tau \in K_w[\mathcal{G}]$  and  $\text{ord}(\tau) > 0$ . By Theorem 2.2,  $\sum_{n \geq 0} \tau^n$  strictly converges in  $K_w[\mathcal{G}]$ . One can check that  $(1 - \tau) \cdot \sum_{n \geq 0} \tau^n = 1$ .  $\square$

So for any  $\eta \in K_w[\mathcal{G}]$  with initial term  $f$ , then  $\eta = f(1 - \tau)$  with  $\text{ord}(\tau) > 0$ , and the expansion of  $\eta$  is given by  $f^{-1} \sum_{n \geq 0} \tau^n$ .

**Remark 2.5.** In the proof of  $K_w[\mathcal{G}]$  being a field, we note that possible cancellations never came into account when considering the support of the sum, product and composition of MN-series.

**Definition 2.6.** If  $\mathcal{G}$  and  $\mathcal{H}$  are two TOA-groups, then the *Cartesian product*  $\mathcal{G} \times \mathcal{H}$  is defined to be the set  $\mathcal{G} \times \mathcal{H}$  equipped with the usual multiplication and the reverse lexicographic order, i.e.,  $(x_1, y_1) \leq (x_2, y_2)$  if and only if  $y_1 <_H y_2$  or  $y_1 = y_2$  and  $x_1 \leq_G x_2$ .

We define  $\mathcal{G}^n$  to be the Cartesian product of  $n$  copies of  $\mathcal{G}$ . It is an easy exercise to show the following.

**Proposition 2.7.** *The Cartesian product of finitely many TO-groups is a TO-group.*

One important example is that  $\mathbb{Z}^n$  is a totally ordered abelian group.

When considering the ring  $K_w(\mathcal{G} \times \mathcal{H})$ , it is natural to treat  $(g, h)$  as  $g + h$ , where  $g$  is identified with  $(g, 0)$  and  $h$  is identified with  $(0, h)$ . With this identification, we have the following.

**Proposition 2.8.** *The ring  $K_w[\mathcal{G} \times \mathcal{H}]$  is the same as the ring  $(K_w[\mathcal{G}])_w[\mathcal{H}]$  of Malcev-Neumann series on  $H$  with coefficients in  $K_w[\mathcal{G}]$ .*

*Proof.* Let  $\eta \in K_w[\mathcal{G} \times \mathcal{H}]$ , and let  $A = \text{supp}(\eta)$ . Let  $\rho$  be the second projection of  $\mathcal{G} \times \mathcal{H}$ , i.e.,  $\rho(g, h) = h$ .

We first show that  $\rho(A)$  is well-ordered. If not, then we have an infinite decreasing sequence in  $H$ , say  $\rho(g_1, h_1) > \rho(g_2, h_2) > \dots$ , which by definition becomes  $h_1 > h_2 > \dots$ . Then in the reverse lexicographic order, this implies that  $(g_1, h_1) > (g_2, h_2) > \dots$  is an infinite decreasing sequence of  $A$ , a contradiction. So  $\rho(A)$  is well-ordered.

Now  $\eta$  can be written as

$$\eta = \sum_{h \in \rho(A)} \left( \sum_{g \in \mathcal{G}, (g, h) \in A} a_{g, h} g \right) h.$$

Since for each  $h \in \rho(A)$ , the set  $\{g \in \mathcal{G} : (g, h) \in A\}$  is a clearly a well-ordered subset of  $\mathcal{G}$ ,  $\sum_{g \in \mathcal{G}, (g, h) \in A} a_{g, h} g$  belongs to  $K_w[\mathcal{G}]$  for every  $h$ , and hence  $\eta \in (K_w[\mathcal{G}])_w[\mathcal{H}]$ .

Now let  $\tau = \sum_{h \in D} b_h h \in (K_w[\mathcal{G}])_w[\mathcal{H}]$ , where  $D = \text{supp}(\tau)$  is a well ordered subset of  $H$ , and  $b_h \in K_w[\mathcal{G}]$ . Let  $B_h$  denote the support of  $b_h$ . We need to show that  $\bigcup_{h \in D} (B_h \times \{h\})$  is well-ordered in  $\mathcal{G} \times \mathcal{H}$ . Let  $A$  be any subset of  $\bigcup_{h \in D} (B_h \times \{h\})$ . We show that  $A$  has a smallest element. Since  $\rho(A)$  is a subset of the well-ordered set  $D$ , we can take  $h_0$  to be the smallest element of  $\rho(A)$ . The set  $A \cap B_{h_0} \times \{h_0\}$  is well-ordered for it is a subset of the well-ordered set  $B_{h_0} \times \{h_0\}$ . Let  $(g_0, h_0)$  be the smallest element of  $A \cap B_{h_0} \times \{h_0\}$ . Then  $(g_0, h_0)$  is also the smallest element of  $A$ .  $\square$

Let  $K$  be a field. We define  $K\langle\langle x_1 \rangle\rangle$  to be the field of Laurent series  $K((x_1))$ , and define the field of iterated Laurent series  $K\langle\langle x_1, \dots, x_n \rangle\rangle$  inductively to be  $K\langle\langle x_1, \dots, x_{n-1} \rangle\rangle((x_n))$ , the field of Laurent series in  $x_n$  with coefficients in  $K\langle\langle x_1, \dots, x_{n-1} \rangle\rangle$ . By Proposition 2.8,  $K_w[\mathbb{Z}^2] \simeq K((x_1))_w[\mathbb{Z}] \simeq K((x_1))((x_2))$ . Using induction, we have

**Corollary 2.9.**

$$K_w[\mathbb{Z}^n] \simeq K\langle\langle x_1, x_2, \dots, x_n \rangle\rangle.$$

The field of iterated Laurent series turns out to be the most useful special kind of MN-series [17; 18].

We conclude this section by the following remark.

**Remark 2.10.** MN-series were originally defined on totally ordered group. It was shown in [18] that the results in this section have analogous generalization:  $G$  can be replaced with a totally ordered monoid (a semigroup with a unit), and  $K$  can be replaced with a commutative ring with a unit.

## 3. THE RESIDUE THEOREM

Observe that any subgroup of a TOA-group is still a TOA-group under the induced total ordering. Let  $\mathcal{G}$  be a TOA-group and let  $\mathcal{H}$  be a group. If  $\rho : \mathcal{H} \rightarrow \mathcal{G}$  is an injective homomorphism, then  $\rho(\mathcal{H}) \simeq \mathcal{H}$  is a subgroup of  $\mathcal{G}$ . We can thus regard  $\mathcal{H}$  as a subgroup of  $\mathcal{G}$  through  $\rho$ . The induced ordering  $\leq^\rho$  on  $\mathcal{H}$  is given by  $h_1 \leq^\rho h_2 \Leftrightarrow \rho(h_1) \leq_{\mathcal{G}} \rho(h_2)$ . Thus  $\mathcal{H}$  is a TOA-group under  $\leq^\rho$ . Clearly a subset  $A$  of  $(\mathcal{H}, \leq^\rho)$  is well-ordered if and only if  $\rho(A)$  is well-ordered in  $(\mathcal{G}, \leq_{\mathcal{G}})$ .

Let  $\mathcal{G}$  be a TOA-group. We can give  $\mathcal{G}$  a different ordering so that under this new ordering  $\mathcal{G}$  is still a TOA-group. For instance, the total ordering  $\hat{\leq}$  defined by  $g_1 \leq g_2 \Leftrightarrow g_2 \hat{\leq} g_1$  is clearly such an ordering. One special class of total orderings is interesting for our purpose. If  $\rho : \mathcal{G} \rightarrow \mathcal{G}$  is an injective endomorphism, then the induced ordering  $\leq^\rho$  is also a total ordering on  $\mathcal{G}$ . We denote the corresponding field of MN-series by  $K_w^\rho[\mathcal{G}]$ .

For example, if  $\mathcal{G} = \mathbb{Z}^n$ , then any nonsingular matrix  $M \in GL(\mathbb{Z}^n)$  induces an injective endomorphism. In particular,  $K_w[\mathbb{Z}^2] \simeq K\langle\langle x, t \rangle\rangle$  is the field of double Laurent series, and  $K_w^\rho[\mathbb{Z}^2] \simeq K\langle\langle x^{-1}, t \rangle\rangle$ , where the matrix corresponding to  $\rho$  is the diagonal matrix  $\text{diag}(-1, 1)$ . It is easy to see that  $K\langle\langle x_1^{e_1}, \dots, x_n^{e_n} \rangle\rangle$  with  $e_i = \pm 1$  are special fields of MN-series  $K^\rho\langle\langle x_1, \dots, x_n \rangle\rangle$ , where the corresponding matrix for  $\rho$  is the diagonal matrices with entries  $e_i$ .

Series expansions in a field of MN-series depend on the total ordering  $\leq^\rho$ . When comparing monomials, it is convenient to use  $\preceq^\rho$ . We shall call attention to the expansions in the following example.

Let  $\rho$  be defined by  $\rho(x) = x^2y$  and  $\rho(y) = xy^2$ , and consider  $K^\rho\langle\langle x, y \rangle\rangle$ . The expansion of  $1/(x - y)$  is given by

$$1/x \cdot 1/(1 - y/x) = 1/x \sum_{k \geq 0} y^k/x^k,$$

since  $\rho(y/x) = \rho(y)/\rho(x) = y/x \succ 1$ , which implies  $1 \prec^\rho y/x$ .

Now notice the expansion of  $1/(x^2 - y)$  is given by

$$-1/y \cdot 1/(1 - x^2/y) = -1/y \sum_{k \geq 0} x^{2k}/y^k,$$

since  $\rho(y/x^2) = \rho(y)/\rho(x^2) = 1/x^3 \prec 1$ , which implies  $1 \prec^\rho x^2/y$ .

In order to state the residue theorem, we need more concepts. Consider the following situation. Let  $\mathcal{G}$  and  $\mathcal{H}$  be groups with  $\mathcal{H} \simeq \mathbb{Z}^n$ , and suppose that we have a total ordering  $\leq$  on the direct sum  $\mathcal{G} \oplus \mathcal{H}$  such that  $\mathcal{G} \oplus \mathcal{H}$  is a TOA-group. We identify  $\mathcal{G}$  with  $\mathcal{G} \oplus 0$  and  $\mathcal{H}$  with  $0 \oplus \mathcal{H}$ . Let  $e_1, e_2, \dots, e_n$  be a basis of  $\mathcal{H}$ . Let  $\rho$  be the endomorphism on  $\mathcal{G} \oplus \mathcal{H}$  that is generated by  $\rho(e_i) = g_i + \sum_j m_{ij}e_j$  for all  $i$ , where  $g_i \in \mathcal{G}$ , and  $\rho(g) = g$  for all  $g \in \mathcal{G}$ . Then  $\rho$  is injective if the matrix  $M = (m_{ij})_{1 \leq i, j \leq n}$  belongs to  $GL(\mathbb{Z}^n)$ .

It is natural to use new variables  $x_i$  to denote  $t^{e_i}$  for all  $i$ . Thus monomials in  $K_w[\mathcal{G} \oplus \mathcal{H}]$  can be represented as  $t^g x_1^{k_1} \cdots x_n^{k_n}$ . Correspondingly,  $\rho$  acts on monomials by  $\rho(t^g) = t^g$  for all  $g \in \mathcal{G}$ , and  $\rho(x_i) = t^{g_i} x_1^{m_{i1}} \cdots x_n^{m_{in}}$ .

**Notation:** If  $f_i$  are monomials, we use  $\mathbf{f}$  to denote the homomorphism  $\rho$  generated by  $\rho(x_i) = f_i$ .

An element  $\eta$  of  $K_w[\mathcal{G} \oplus \mathcal{H}]$  can be written as

$$\eta = \sum_{\mathbf{k} \in \mathbb{Z}^n} \sum_{g \in \mathcal{G}} a_{g,\mathbf{k}} t^g x_1^{k_1} \cdots x_n^{k_n} = \sum_{\mathbf{k} \in \mathbb{Z}^n} b_{\mathbf{k}} \mathbf{x}^{\mathbf{k}},$$

where  $a_{g,\mathbf{k}} \in K$  and  $b_{\mathbf{k}} \in K_w[\mathcal{G}]$ . We call the  $b_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}$  an  $x$ -term of  $\eta$ . Since the set  $\{\text{ord}(b_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}) : \mathbf{k} \in \mathbb{Z}^n\}$  is a subset of  $\text{supp}(\eta)$ , it is well-ordered and hence has a least element. Because of the different powers in the  $x$ 's, no two of  $\text{ord}(b_{\mathbf{k}} \mathbf{x}^{\mathbf{k}})$  are equal. So we can define the  $x$ -initial term of  $\eta$  to be the  $x$ -term that has the least order.

To define the operators  $\frac{\partial}{\partial x_i}$ ,  $\text{CT}_{x_i}$ ,  $\text{Res}_{x_i}$ ,  $\text{PT}_{x_i}$ ,  $\text{NT}_{x_i}$ , it suffices to consider the case  $\mathcal{H} = \mathbb{Z}$ . These operators are defined as usual:

$$\begin{aligned} \frac{\partial}{\partial x} \sum_{n \in \mathbb{Z}} b_n x^n &= \sum_{n \in \mathbb{Z}} n b_n x^{n-1}, \\ \text{CT}_x \sum_{n \in \mathbb{Z}} b_n x^n &= b_0, & \text{Res}_x \sum_{n \in \mathbb{Z}} b_n x^n &= b_{-1}, \\ \text{PT}_x \sum_{n \in \mathbb{Z}} b_n x^n &= \sum_{n \geq 0} b_n x^n, & \text{NT}_x \sum_{n \in \mathbb{Z}} b_n x^n &= \sum_{n < 0} b_n x^n. \end{aligned}$$

Multivariate operators are defined by iteration. All these operators work nicely in the field of MN-series  $K_w[\mathcal{G} \oplus \mathcal{H}]$ , because an MN-series has a well-ordered support, and still has a well-ordered support after taking these operators.

Computational Rules: In a field of MN-series  $K_w[\mathcal{G} \oplus \mathcal{H}]$  with  $\mathcal{H} \simeq \mathbb{Z}$ , we identify  $t^{(0,1)}$  with  $x$ , where  $(0,1) \in \mathcal{G} \oplus \mathcal{H}$ . Let  $F$  and  $G$  be two elements in  $K_w[\mathcal{G} \oplus \mathcal{H}]$ .

Rule 1: (linearity) For any  $a, b$  that are independent of  $x$ ,

$$\text{CT}_x (aF(x) + bG(x)) = a \text{CT}_x F(x) + b \text{CT}_x G(x).$$

Rule 2: If  $F$  can be written as  $\sum_{k \geq 0} a_k x^k$ , then we say that  $F$  is PT in  $x$  and

$$\text{CT}_x F = F|_{x=0}.$$

Rule 3:

$$\text{Res}_x \frac{\partial F(x)}{\partial x} G(x) = - \text{Res}_x F(x) \frac{\partial G(x)}{\partial x}.$$

Rule 4: Suppose that  $F$  is PT in  $x$ . If  $G$  can be factored as  $(x - u)H$  such that  $u$  is independent of  $x$  and  $\text{ord}(u) > \text{ord}(x)$ , and  $1/H$  is PT in  $x$ , then

$$\text{CT}_x F(x) \frac{x}{G(x)} = \frac{F(x)}{\frac{\partial G(x)}{\partial x}} \Bigg|_{x=u}$$

These computational rules are useful in evaluating constant term. But we are not going to concentrate on this topic. See [18] for further information.

Now we come back to the multivariate case, and suppose  $F_i \in K_w[\mathcal{G} \oplus \mathcal{H}]$  for all  $i$ .

**Definition 3.1.** The Jacobian determinant (or simply Jacobian) of  $\mathbf{F}$  with respect to  $\mathbf{x}$  is defined to be

$$J(\mathbf{F}|\mathbf{x}) := J \left( \frac{F_1, F_2, \dots, F_n}{x_1, x_2, \dots, x_n} \right) = \det \left( \frac{\partial F_i}{\partial x_j} \right)_{1 \leq i, j \leq n}.$$

When the  $x$ 's are clear, we write  $J(F_1, F_2, \dots, F_n)$  for short.

**Definition 3.2.** If the  $x$ -initial term of  $F_i$  is  $a_i x_1^{b_{i1}} \cdots x_n^{b_{in}}$ , then the Jacobian number of  $\mathbf{F}$  with respect to  $\mathbf{x}$  is defined to be

$$j(\mathbf{F}|\mathbf{x}) := j \left( \frac{F_1, F_2, \dots, F_n}{x_1, x_2, \dots, x_n} \right) = \det (b_{ij})_{1 \leq i, j \leq n}.$$

**Definition 3.3.** The log Jacobian of  $F_1, \dots, F_n$  is defined to be

$$LJ(F_1, \dots, F_n) := \frac{x_1 \cdots x_n}{F_1 \cdots F_n} J(F_1, \dots, F_n).$$

We call it the log Jacobian because formally it can be written as [16]

$$LJ(F_1, \dots, F_n) = J \left( \frac{\log F_1, \dots, \log F_n}{\log x_1, \dots, \log x_n} \right),$$

since

$$\frac{\partial \log F}{\partial \log x} = \frac{\partial \log F}{\partial F} \frac{\partial F}{\partial \log x} = \frac{1}{F} \frac{\partial F}{\partial x} \frac{\partial x}{\partial \log x} = \frac{x}{F} \frac{\partial F}{\partial x}.$$

**Remark 3.4.** The Jacobian is convenient in residue evaluation, while the log Jacobian is convenient in constant term evaluation.

The following lemma is devised for the proof of our residue theorem. It is also a kind of generalized composition law.

**Lemma 3.5.** Let  $\Phi$  be a formal series in  $x_1, \dots, x_n$  with coefficients in  $K_w[\mathcal{G}]$ . Then  $\Phi(F_1, \dots, F_n) \in K_w[\mathcal{G} \oplus \mathcal{H}]$  if and only if  $\Phi(f_1, \dots, f_n) \in K_w[\mathcal{G} \oplus \mathcal{H}]$ , where  $f_i$  is the initial term of  $F_i$  for all  $i$ . Moreover if  $j(F_1, \dots, F_n) \neq 0$ , then  $\Phi(F_1, \dots, F_n) \in K_w[\mathcal{G} \oplus \mathcal{H}]$  if and only if  $\Phi(x_1, \dots, x_n) \in K_w^f[\mathcal{G} \oplus \mathcal{H}]$ .

This lemma reduces the convergence of  $\Phi(F_1, \dots, F_n)$  to that of  $\Phi(f_1, \dots, f_n)$ . For example, the ring of formal power series  $K[[x_1, \dots, x_n]]$  is isomorphic to  $K_w[\mathbb{N}^n]$ , where  $\mathbb{N}^n$  itself is well-ordered under the reverse lexicographic ordering. If  $\Phi$  is a formal power series in  $\mathbf{x}$ , then when  $f_i$  are monomials in  $K[[x_1, \dots, x_n]]$ ,  $\Phi(f_1, \dots, f_n)$  is obviously a formal power series. Thus the composition law of  $K[[x_1, \dots, x_n]]$  follows from Lemma 3.5.

*Proof of Lemma 3.5.* Write every  $F_i$  as  $f_i(1 + \tau_i)$ , where  $f_i$  is the initial term and  $\text{ord}(\tau_i) > 0$  or  $\tau_i = 0$ .

For the first part, we show that if  $\Phi(F_1, \dots, F_n) \in K_w[\mathcal{G} \oplus \mathcal{H}]$ , then replacing  $F_i$  by  $F_i(1 + \tau)$  with  $\text{ord}(\tau) > 0$  results in an element of  $K_w[\mathcal{G} \oplus \mathcal{H}]$ . Then the first part follows by replacing  $F_i$  with  $f_i = F_i(1 + \tau_i)^{-1}$ , (or conversely,  $f_i$  with  $F_i = f_i(1 + \tau_i)$ ) one by one for  $i$  from 1 to  $n$ .

We deal with the case  $i = 1$  as follows. The case of arbitrary  $i$  is similar. Let  $A = \text{supp}(\Phi(F_1, \dots, F_n))$  and  $T = \text{supp}(\tau)$ . Then by assumption,  $A$  is well-ordered, and  $T$  is positive and well-ordered. We can write

$$\Phi(F_1, \dots, F_n) = \sum_{k \in \mathbb{Z}} d_k F_1^k,$$

where  $d_k$  is a formal series in  $F_2, \dots, F_n$  with coefficients in  $K_w[\mathcal{G}]$ . Then

$$\Phi(F_1(1 + \tau), \dots, F_n) = \sum_{k \in \mathbb{Z}} d_k F_1^k (1 + \tau)^k = \sum_{k \in \mathbb{Z}} d_k F_1^k \sum_{l \geq 0} \binom{k}{l} \tau^l \quad (3.1)$$

Now we see that the support of  $\Phi(F_1(1 + \tau), \dots, F_n)$  is a subset of

$$\bigcup_{l \geq 0} (A + T^{+l}) = A + \bigcup_{l \geq 0} T^{+l},$$

which is well-ordered by Proposition 2.1 and Proposition 2.3.

To see that the coefficient of  $t^{g+h}$  is a finite sum for every  $g$  and  $h$ , we observe that replacing each  $\binom{k}{l}$  by 1 will not decrease the number of summands. The right side of equation (3.1) then becomes

$$\left( \sum_{k \in \mathbb{Z}} d_k F_1^k \right) \left( \sum_{l \geq 0} \tau^l \right),$$

in which the coefficient of  $t^{g+h}$  is a finite sum, because it is a product of two elements in  $K_w[\mathcal{G} \oplus \mathcal{H}]$ .

For the second part, if  $j(F_1, \dots, F_n) \neq 0$ , then  $\rho : x_i \rightarrow f_i$  induces an injective endomorphism on  $\mathcal{G} \oplus \mathcal{H}$ . We see that  $\text{supp}(\Phi(f_1, \dots, f_n))$  is well-ordered in  $\mathcal{G} \oplus \mathcal{H}$  if and only if  $\rho(\text{supp}(\Phi(x_1, \dots, x_n)))$  is well-ordered. This, by definition, is to say that  $\Phi(x_1, \dots, x_n) \in K_w^f[\mathcal{G} \oplus \mathcal{H}]$ . The lemma now follows from the first part.  $\square$

**Notation.** Starting with a TOA-group  $\mathcal{G} \oplus \mathcal{H}$  as described above, let  $\Phi$  be a formal series on  $\mathcal{G} \oplus \mathcal{H}$ . When we write  $\text{CT}_{\mathbf{x}}^{\rho} \Phi(x_1, \dots, x_n)$ , we mean both that  $\Phi(x_1, \dots, x_n)$  belongs to  $K_w^{\rho}[\mathcal{G} \oplus \mathcal{H}]$ , and that the constant term is taken in this field. When  $\rho$  is the identity map, it is omitted. When we write  $\text{CT}_{\mathbf{F}} \Phi(F_1, \dots, F_n)$ , it is assumed that  $\Phi(x_1, \dots, x_n) \in K_w^{\mathbf{f}}[\mathcal{G} \oplus \mathcal{H}]$ , and we are taking the constant term of  $\Phi(x_1, \dots, x_n)$  in the ring  $K_w^{\mathbf{f}}[\mathcal{G} \oplus \mathcal{H}]$ . Or equivalently, we always have

$$\text{CT}_{\mathbf{F}} \Phi(F_1, \dots, F_n) = \text{CT}_{\mathbf{x}}^{\mathbf{f}} \Phi(x_1, \dots, x_n).$$

This treatment is particularly useful when dealing with rational functions, as we shall see soon.

Now comes our residue theorem for  $K_w[\mathcal{G} \oplus \mathcal{H}]$ , in which we will see how an element in one field is related to an element in another field through taking the constant terms.

**Theorem 3.6** (Residue Theorem). *Suppose for each  $i$ ,  $F_i \in K_w[\mathcal{G} \oplus \mathcal{H}]$  has  $x$ -initial term  $f_i = a_i x_1^{b_{i1}} \cdots x_n^{b_{in}}$  with  $a_i \in K_w[\mathcal{G}]$ . If  $j(F_1, \dots, F_n) \neq 0$ , then for any  $\Phi(\mathbf{x}) \in K_w^{\mathbf{f}}[\mathcal{G} \oplus \mathcal{H}]$ , we have*

$$\text{CT}_{\mathbf{x}} \Phi(F_1, \dots, F_n) L J(F_1, \dots, F_n) = j(F_1, \dots, F_n) \text{CT}_{\mathbf{F}} \Phi(F_1, \dots, F_n). \quad (3.2)$$

*Proof of Theorem 3.6.* With the hypothesis, both sides of equation (3.2) strictly converge. In fact, Lemma 3.5 is designed for this convergence.

The rest part is similar to the proof of Jacobi's formula: by multilinearity, it suffices to show the theorem is true for monomials  $\Phi$ , and the proof will be completed by showing Lemmas 3.13 and 3.14 below. The proof of the two lemmas will take time and be given later.  $\square$

**Remark 3.7.** When  $j(F_1, \dots, F_n) = 0$ ,  $\Phi(F_1, \dots, F_n)$  is only well defined in some special cases.

**Remark 3.8.** If  $\Phi(x_1, \dots, x_n)$  is a Laurent polynomial, then  $\Phi(F_1, \dots, F_n)$  always exists. In this case, it is not necessary to consider the map  $\mathbf{f}$  and hence  $j(F_1, \dots, F_n)$  is allowed to be 0.

**Remark 3.9.** The theorem holds for any rational function  $\Phi$ , i.e.,  $\Phi(x_1, \dots, x_n)$  belongs to the quotient field of  $K_w[G][H]$ . This follows from the fact that  $K_w^{\mathbf{f}}[\mathcal{G} \oplus \mathcal{H}]$  is a field, which contains  $K_w[G][H]$  as a subring.

The proof of our residue theorem and lemmas basically comes from [1], except for the proof of Lemma 3.14, which use the original idea of Jacobi.

The following properties of Jacobians can be easily checked.

**Lemma 3.10.** *Let the Jacobian be defined as in the previous subsection. Then*

- (1)  $J(F_1, F_2, \dots, F_n)$  is  $K_w[\mathcal{G}]$ -multilinear.
- (2)  $J(F_1, F_2, \dots, F_n)$  is alternating; i.e.,  $J(F_1, F_2, \dots, F_n) = 0$  if  $F_i = F_j$  for some  $i \neq j$ .
- (3)  $J(F_1, F_2, \dots, F_n)$  is anticommutative; i.e.,

$$J(F_1, \dots, F_i, \dots, F_j, \dots, F_n) = -J(F_1, \dots, F_j, \dots, F_i, \dots, F_n).$$

- (4) (Composition Rule) If  $g(z) \in K((z))$  is a series in one variable, then

$$J(g(F_1), F_2, \dots, F_n) = g'(F_1)J(F_1, F_2, \dots, F_n).$$

- (5) (Product Rule)

$$J(F_1 G_1, F_2, \dots, F_n) = F_1 J(G_1, F_2, \dots, F_n) + G_1 J(F_1, F_2, \dots, F_n).$$

- (6)  $J(F_2^{-1}, F_2, \dots, F_n) = 0$ .

**Lemma 3.11.** *If all  $F_i$  are  $x$ -monomials, then*

$$LJ(F_1 \dots, F_n) = j(F_1, \dots, F_n). \quad (3.3)$$

*Proof.* Suppose that for every  $i$ ,  $F_i = a_i x_1^{b_{i1}} \dots x_n^{b_{in}}$ , where  $a_i$  is in  $K_w[\mathcal{G}]$ . Factoring  $F_i = a_i x_1^{b_{i1}} \dots x_n^{b_{in}}$  from the  $i$ th row of the Jacobian matrix for all  $i$  and then factoring  $x_j^{-1}$  from the  $j$ th column for all  $j$ , we get

$$J(F_1, F_2, \dots, F_n) = \frac{F_1 \dots F_n}{x_1 \dots x_n} \det(b_{ij}).$$

Equation (3.3) is just a rewriting of the above equation. □

**Lemma 3.12.**

$$\operatorname{Res}_{\mathbf{x}} J(F_1, \dots, F_n) = 0.$$

*Proof.* By multilinearity, it suffices to check monomials  $F_i$ . Suppose that they are given as in Lemma 3.11. Then equation (3.3) gives us

$$J(F_1, \dots, F_n) = j(F_1, \dots, F_n) \frac{F_1 \dots F_n}{x_1 \dots x_n}.$$

More explicitly,

$$J(F_1, \dots, F_n) = \det(b_{ij}) a_1 \dots a_n x_1^{-1+\sum b_{i1}} \dots x_n^{-1+\sum b_{in}}.$$

If  $\sum b_{i1} = \sum b_{i2} = \dots = \sum b_{in} = 0$ , then the Jacobian number is 0, and therefore the residue is 0. Otherwise, at least one of the  $x_i$ 's has exponent  $\neq -1$ , so the residue is 0 by definition. □

**Lemma 3.13.** *For all integers  $e_i$  with at least one of  $e_i \neq -1$ , we have*

$$\operatorname{Res}_{\mathbf{x}} F_1^{e_1} \dots F_n^{e_n} J(F_1, \dots, F_n) = 0. \quad (3.4)$$

*Proof.* The clever proof in [1, Theorem 1.4] also works here.

Permuting the  $F_i$  and using (3) of Lemma 3.10, we may assume that  $e_1 \neq -1, \dots, e_j \neq -1$ , but  $e_{j+1} = \dots = e_n = -1$ , for some  $j$  with  $1 \leq j \leq n-1$ . Setting  $G_i = \frac{1}{e_i+1} F_i^{e_i+1}$  for  $i = 1, \dots, j$ , we have

$$F_1^{e_1} F_2^{e_2} \cdots F_n^{e_n} J(F_1, F_2, \dots, F_n) = F_{j+1}^{-1} \cdots F_n^{-1} J(G_1, \dots, G_j, F_{j+1}, \dots, F_n).$$

Then applying the formula

$$F_{j+1}^{-1} J(G_1, \dots, G_j, F_{j+1}, \dots, F_n) = J(F_{j+1}^{-1} G_1, G_2, \dots, G_j, F_{j+1}, \dots, F_n)$$

repeatedly for  $j+1, j+2, \dots, n$ , we get

$$J(F_{j+1}^{-1} \cdots F_n^{-1} G_1, G_2, \dots, G_j, F_{j+1}, \dots, F_n).$$

The result now follows from Lemma 3.12.  $\square$

For the case  $e_1 = e_2 = \dots = e_n = -1$ , we have

**Lemma 3.14.**

$$\operatorname{Res}_x F_1^{-1} \cdots F_n^{-1} J(F_1, \dots, F_n) = j(F_1, \dots, F_n). \quad (3.5)$$

The simple proof for this case in [1] does not apply in our situation. The reason will be explained in Proposition 3.15.

Note that Lemma 3.14 is equivalent to saying that

$$\operatorname{CT}_x L J(F_1, \dots, F_n) = j(F_1, \dots, F_n). \quad (3.6)$$

*Proof.* Let  $f_i := a_i x_1^{b_i^{i1}} \cdots x_n^{b_i^{in}}$  be the  $x$ -initial term of  $F_i$ . Then  $F_i = f_i B_i$ , where  $B_i \in K_w[\mathcal{G} \oplus \mathcal{H}]$  has  $x$ -initial term 1. By the composition law,  $\log(B_i) \in K_w[\mathcal{G} \oplus \mathcal{H}]$ . Now applying the product rule, we have

$$\begin{aligned} F_1^{-1} \cdots F_n^{-1} J(F_1, F_2, \dots, F_n) &= f_1^{-1} F_2^{-1} \cdots F_n^{-1} J(f_1, F_2, \dots, F_n) + B_1^{-1} F_2^{-1} \cdots F_n^{-1} J(B_1, F_2, \dots, F_n) \\ &= f_1^{-1} F_2^{-1} \cdots F_n^{-1} J(f_1, F_2, \dots, F_n) + F_2^{-1} \cdots F_n^{-1} J(\log(B_1), F_2, \dots, F_n). \end{aligned}$$

From Lemma 3.13, the last term in the above equations has no contribution to the residue in  $x$ , and hence can be discarded.

The same procedure can be applied to  $F_2, F_3, \dots, F_n$ . Finally we will get

$$\operatorname{Res}_x F_1^{-1} \cdots F_n^{-1} J(F_1, F_2, \dots, F_n) = \operatorname{Res}_x f_1^{-1} \cdots f_n^{-1} J(f_1, f_2, \dots, f_n),$$

which is equal to the Jacobian number by Lemma 3.11.  $\square$

Up to now, the proof of our residue theorem is completed.

The next result gives a good reason for using the log Jacobian.

**Proposition 3.15.** *The  $x$ -initial term of the log Jacobian  $LJ(F_1, \dots, F_n)$  equals the Jacobian number  $j(F_1, \dots, F_n)$  when it is nonzero.*

*Proof.* From the definition,

$$LJ(F_1, \dots, F_n) = \frac{x_1 \cdots x_n}{F_1 \cdots F_n} J(F_1, \dots, F_n) = \frac{x_1 \cdots x_n}{F_1 \cdots F_n} \sum_{\mathbf{g}} J(g_1, \dots, g_n),$$

where the sum ranges over all  $x$ -terms  $g_i$  of  $F_i$ . Applying Lemma 3.14 gives us

$$LJ(F_1, \dots, F_n) = \sum_{\mathbf{g}} \frac{g_1 \cdots g_n}{F_1 \cdots F_n} j(g_1, \dots, g_n).$$

Since Jacobian number is always an integer, it is clear now that we can write

$$LJ(F_1, \dots, F_n) = j(F_1, \dots, F_n) + \text{higher order terms.}$$

To show that  $j(F_1, \dots, F_n)$  is the  $x$ -initial term, we need to show that all the other term that are independent of  $x$  cancel. (Note that we do not have this trouble when all the coefficients belong to  $K$ .) This is equivalent to saying that

$$\text{CT}_x LJ(F_1, \dots, F_n) = j(F_1, \dots, F_n),$$

which follows from Lemma 3.14. □

**Example 3.16.** Consider the field  $K\langle\langle x, t \rangle\rangle$ . Let  $F = x^2 + xt + x^3t$ . Then the initial  $x$ -term of  $F$  is  $x^2$ . Now let us see what happens to the log Jacobian  $LJ(F|x)$  of  $F$  with respect to  $x$ .

$$\begin{aligned} LJ(F|x) &= \frac{x}{F} \frac{\partial F}{\partial x} = \frac{x(2x + t + 3x^2t)}{x^2(1 + t/x + xt)} \\ &= (2 + t/x + 3xt) \sum_{k \geq 0} (-1)^k (t/x + xt)^k \end{aligned}$$

It is not clear that 2 is the unique term in the expansion of  $\text{CT}_x LJ(F|x)$ , but all the other terms cancel. We continue to check as the following.

$$\begin{aligned} \text{CT}_x LJ(F|x) &= \text{CT}_x (2 + t/x + 3xt) \sum_{k \geq 0} (-1)^k (t/x + xt)^k \\ &= 2 \sum_{k \geq 0} \binom{2k}{k} t^{2k} - t \sum_{k \geq 0} \binom{2k+1}{k} t^{2k+1} - 3t \sum_{k \geq 0} \binom{2k+1}{k+1} t^{2k+1} \\ &= 2 + \sum_{k \geq 1} \left( 2 \binom{2k}{k} - 4 \binom{2k-1}{k} \right) t^{2k}. \end{aligned}$$

Now it is easy to see that the terms, other than 2, not containing  $x$  in the expansion of the log Jacobian really cancel.

From Theorem 3.6 and Lemma 3.11, we see directly the following result.

**Corollary 3.17.** *If  $F_i$  are all  $x$ -monomials in  $K_w[\mathcal{G} \oplus \mathcal{H}]$ , and  $\Phi \in K_w^{\mathbf{F}}[\mathcal{G} \oplus \mathcal{H}]$ , which indicates that  $j(F_1, \dots, F_n)$  is nonzero, then*

$$\text{CT}_{\mathbf{x}} \Phi(F_1, \dots, F_n) = \text{CT}_{F_1, \dots, F_n} \Phi(F_1, \dots, F_n).$$

This is saying that change of variables by monomials will not change the constant terms.

In the case that all  $F_i$  are monomials in  $K[\mathbf{x}, \mathbf{x}^{-1}]$  with  $j(\mathbf{F}) \neq 0$ ,  $\Phi$  is in  $K[\mathbf{x}, \mathbf{x}^{-1}]$  if and only if  $\Phi(F_1, \dots, F_n)$  is. We always have

$$\text{CT}_{F_1, \dots, F_n} \Phi(F_1, \dots, F_n) = \text{CT}_{x_1, \dots, x_n} \Phi(x_1, \dots, x_n).$$

More generally, we have the following result.

**Corollary 3.18.** *Suppose  $\mathbf{y}$  is another set of variables. If  $\Phi \in K[\mathbf{x}, \mathbf{x}^{-1}]\langle\langle \mathbf{y} \rangle\rangle$ , and if  $F_i$  are all monomials in  $\mathbf{x}$  with  $j(\mathbf{F}) \neq 0$ , then*

$$\text{CT}_{\mathbf{x}} \Phi(F_1, \dots, F_n) = \text{CT}_{\mathbf{x}} \Phi(x_1, \dots, x_n).$$

**Example 3.19.** Evaluate the following constant term in  $\mathbb{C}\langle\langle x, y, t \rangle\rangle$ .

$$\text{CT}_{x, y} -x^3 e^{\frac{t}{xy}} (3xy - 2t) \left( x^3 y e^{\frac{t}{xy}} - tx - ty \right)^{-1} (x - y)^{-1} \left( -1 + x^3 e^{\frac{t}{xy}} \right)^{-1}. \quad (3.7)$$

This is an example that is hard to evaluate without using our residue theorem.

*Solution.* Let  $F = x^2 y e^{\frac{t}{xy}}$ ,  $G = xy^2 e^{\frac{t}{xy}}$ . It is easy to compute the log Jacobian and the Jacobian number. We have

$$LJ(F, G|x, y) = 3 - \frac{2t}{xy}, \text{ and } j(F, G|x, y) = 3.$$

We can check that (3.7) can be written as

$$\text{CT}_{x, y} \frac{F^3 G}{(F^2 - (F + G)t)(F - G)(G - F^2)} LJ(F, G|x, y).$$

Thus by the residue theorem, the above constant term equals

$$\text{CT}_{F, G} \frac{3F^3 G}{(F^2 - (F + G)t)(F - G)(G - F^2)} = \text{CT}_{F, G} \frac{3}{\left(1 - \frac{(F+G)t}{F^2}\right)\left(1 - \frac{G}{F}\right)\left(1 - \frac{F^2}{G}\right)}, \quad (3.8)$$

where on the right hand side of (3.8), we can check that 1 is the initial term of each factor in the denominator.

At this stage, we can use the series expansion to obtain the constant term. But we will evaluate it by the computational rule 4.

Starting from the left hand-side of (3.8), we first take the constant term in  $G$ . We can solve for  $G$  in the denominator since all these three factors are linear in  $G$ . Only

one root,  $F^2$ , has higher order than  $G$ . Thus we can apply rule 4 and get

$$\begin{aligned} \text{CT}_{F,G} \frac{3F^3G}{(F^2 - (F+G)t)(F-G)(G-F^2)} &= \text{CT}_F \frac{3F^3}{(F^2 - (F+F^2)t)(F-F^2)} \\ &= \text{CT}_F \frac{3F}{(F - (1+F)t)(1-F)} \\ &= \frac{3}{(1-t)(1 - \frac{t}{1-t})}, \end{aligned}$$

where in the last step, we applied rule 4 again. One can check that the two roots of the denominator for  $F$  are  $t/(1-t)$  and 1, and that only the former root has higher order than  $F$ .

After simplification, we finally get

$$\text{CT}_{x,y} -x^3 e^{\frac{t}{xy}} (3xy - 2t) \left( x^3 y e^{\frac{t}{xy}} - tx - ty \right)^{-1} (x-y)^{-1} \left( -1 + x^3 e^{\frac{t}{xy}} \right)^{-1} = \frac{3}{1-2t}.$$

□

#### 4. ANOTHER VIEW OF LAGRANGE'S INVERSION FORMULA

Let  $F_1, \dots, F_n$  be power series in variables  $x_1, \dots, x_n$  of the form  $F_i = x_i +$  “higher degree terms”, with indeterminate coefficients for each  $i$ . It is known, e.g., [10, Proposition 5, p. 219], that  $\mathbf{F} = (F_1, \dots, F_n)$  has a unique compositional inverse, i.e., there exists  $\mathbf{G} = (G_1, \dots, G_n)$  where each  $G_i$  is a power series in  $x_1, \dots, x_n$  such that  $F_i(G_1, \dots, G_n) = x_i$  and  $G_i(F_1, \dots, F_n) = x_i$  for all  $i$ . The above is known as the non-diagonal case.

Lagrange inversion gives a formula of  $G$ 's in terms of the  $F$ 's. Such a formula is very useful in combinatorics. A good summary on this subject can be found in [4].

The diagonal (or Good's) Lagrange inversion formula deals with the diagonal case: when  $F_i$  divides  $x_i$  for every  $i$ , or equivalently,  $F_i = x_i H_i$ , where  $H_i \in K[[x_1, \dots, x_n]]$  with constant term 1. Good's formula can be easily derived by Jacobi's residue theorem, but we use our residue theorem:

Write  $F_i = x_i H_i$ , where  $H_i$  is in  $K[[x_1, \dots, x_n]]$  with constant term 1. Consider this in the field  $K\langle\langle x_1, x_2, \dots, x_n \rangle\rangle$ . Then  $x_i$  is the initial term of  $F_i$ , and the Jacobian number  $j(F_1, \dots, F_n)$  equals 1.

Change variables by  $y_i = F_i(\mathbf{x})$ , we will have  $x_i = G_i(\mathbf{y})$ . Then

$$[y_1^{k_1} \cdots y_n^{k_n}] G_i(\mathbf{y}) = \text{Res}_{\mathbf{y}} y_1^{-1-k_1} \cdots y_n^{-1-k_n} G_i(\mathbf{y}) \quad (4.1)$$

$$= \text{Res}_{\mathbf{x}} F_1^{-1-k_1} \cdots F_n^{-1-k_n} x_i J(\mathbf{F}), \quad (4.2)$$

where  $J(\mathbf{F})$  is the Jacobian of  $F_1, \dots, F_n$ .

A similar computation applies to the non-diagonal case. First let us see that we cannot apply the residue theorem in  $K\langle\langle x_1, \dots, x_n \rangle\rangle$ , because the Jacobian number

might equal 0. For example, if  $x_n$  does not divide  $F_n$ , then it is easily seen that the power of  $x_n$  in the initial term of  $F_i$  is zero for all  $i$ . So the Jacobian number of  $F_1, \dots, F_n$  is 0.

This difficulty can be overcome by introducing a new variable  $t$ . After we get a suitable formula, replace  $t$  by 1. The result obtained this way is equivalent to the homogeneous expansion introduced in [1].

The working field is  $K\langle\langle x_1, x_2, \dots, x_n, t \rangle\rangle$ . In stead of dealing with  $F_1, \dots, F_n$  directly, we consider the compositional inverse of the system  $y_i = F_i(x_1 t, \dots, x_n t)$ . Clearly if there is a solution, we shall have  $x_i t = G_i(\mathbf{y})$ . Then by setting  $t = 1$ , we will get the desired result.

Since the initial term of  $F_i(x_1 t, \dots, x_n t)$  is  $x_i t$ , the Jacobian number is 1. It is also easy to see that  $J(\mathbf{F}(t\mathbf{x})) = t^n J(\mathbf{F})|_{\mathbf{x}=t\mathbf{x}}$ . So we have the same formula as in (4.1), but interpreted differently. Setting  $t = 1$  in the final result is valid, since the power in  $t$  equals the sum of powers in the  $x_i$ 's. This is equivalent to the homogeneous expansion.

Let  $\Phi \in K[[y_1, \dots, y_n]]$ . We have

$$[y_1^{k_1} \cdots y_n^{k_n}] \Phi(\mathbf{G}) = \operatorname{Res}_{\mathbf{x}} F_1^{-1-k_1} \cdots F_n^{-1-k_n} \Phi(\mathbf{x}) J(\mathbf{F}). \quad (4.3)$$

Multiplying both sides of the above equation by  $y_1^{k_1} \cdots y_n^{k_n}$ , and summing on all nonnegative integers  $k_1, k_2, \dots, k_n$ , we get

$$\Phi(\mathbf{G}(\mathbf{y})) = \operatorname{Res}_{\mathbf{x}} \frac{1}{F_1 - y_1} \cdots \frac{1}{F_n - y_n} J(\mathbf{F}) \Phi(\mathbf{x}), \quad (4.4)$$

which is true as power series in the  $y_i$ 's.

It's natural to ask if we can get this formula directly from the residue theorem. The answer is yes. The argument is given as follows.

The working field is  $K\langle\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle\rangle$ . We make the change of variables by  $z_i = F_i - y_i$ . Then  $x_i = G_i(\mathbf{y} + \mathbf{z})$ , and the initial term of  $F_i - y_i$  is  $x_i$ , for  $y_i$  has higher order. Thus the Jacobian number is 1, and the Jacobian determinant still equals to  $J(\mathbf{F})$ . Applying the residue theorem, we get

$$\operatorname{Res}_{\mathbf{x}} \frac{1}{F_1 - y_1} \cdots \frac{1}{F_n - y_n} J(\mathbf{F}) \Phi(\mathbf{x}) = \operatorname{Res}_{\mathbf{z}} \frac{1}{z_1 z_2 \cdots z_n} \Phi(\mathbf{G}(\mathbf{y} + \mathbf{z})).$$

Since  $\Phi(G(\mathbf{y} + \mathbf{z}))$  is in  $K[[\mathbf{y}, \mathbf{z}]]$ . The final result is obtained by setting  $\mathbf{z} = \mathbf{0}$  in  $\Phi(G(\mathbf{y} + \mathbf{z}))$ .

Note that  $J(\mathbf{F}) \in K[[\mathbf{x}]]$  has constant term 1. Therefore  $J(\mathbf{F})^{-1} \Phi(\mathbf{x})$  is also in  $K[[\mathbf{x}]]$ . Hence we can reformulate (4.4) as

$$\operatorname{Res}_{\mathbf{x}} \frac{1}{F_1 - y_1} \cdots \frac{1}{F_n - y_n} \Phi(\mathbf{x}) = \Phi(\mathbf{x}) J(\mathbf{F})^{-1}|_{\mathbf{x}=\mathbf{G}}.$$

5. ABOUT DYSON'S CONJECTURE

Our residue theorem can be used to prove a conjecture of Dyson.

**Theorem 5.1.** *Let  $a_1, \dots, a_n$  be  $n$  nonnegative integers. Then the following equation holds as Laurent polynomials in  $\mathbf{z}$ .*

$$\text{CT}_{\mathbf{z}} \prod_{1 \leq i \neq j \leq n} \left(1 - \frac{z_i}{z_j}\right)^{a_j} = \frac{(a_1 + a_2 + \dots + a_n)!}{a_1! a_2! \dots a_n!}. \quad (5.1)$$

For  $n = 3$  this assertion is equivalent to the familiar Dixon identity:

$$\sum_j (-1)^j \binom{a+b}{a+j} \binom{b+c}{b+j} \binom{c+a}{c+j} = \frac{(a+b+c)!}{a! b! c!}. \quad (5.2)$$

Theorem 5.1 was first proved by Wilson [16] and Gunson [8] independently. A similar proof was given in [2]. These proofs use integrals of analytic functions. A simple induction proof was found by Good [6]. We are going to give a proof by using the residue theorem for Malcev-Neumann series.

Let  $\mathbf{z}$  be the vector  $(z_1, z_2, \dots, z_n)$ . If  $\mathbf{z}$  appears in the computation, we use  $\mathbf{z}$  for the product  $\mathbf{z}^{\mathbf{1}} = z_1 z_2 \dots z_n$ . We use similar notation for  $\mathbf{u}$ .

Let  $\Delta(\mathbf{z}) = \Delta(z_1, \dots, z_n) = \prod_{i < j} (z_i - z_j) = \det(z_i^{n-j})$  be the Vandermonde determinant in  $\mathbf{z}$ , and let  $\Delta_j(\mathbf{z}) = \Delta(z_1, \dots, \hat{z}_j, \dots, z_n)$ , where  $\hat{z}_j$  means to omit  $z_j$ . We introduce new variables  $u_j = (-1)^{j-1} z_j^{n-1} \Delta_j(\mathbf{z})$ . Then they satisfy the equations

$$\begin{aligned} \Delta(\mathbf{z}) &= \sum_{j=1}^n (-1)^{j-1} z_j^{n-1} \Delta_j(\mathbf{z}) = u_1 + u_2 + \dots + u_n, \\ u_1 \dots u_n &= \prod_{j=1}^n (-1)^{j-1} z_j^{n-1} \Delta_j(\mathbf{z}) = (-1)^{\binom{n}{2}} \mathbf{z}^{n-1} (\Delta(\mathbf{z}))^{n-2}. \end{aligned}$$

We also have

$$\prod_{i=1, i \neq j}^n \left(1 - \frac{z_i}{z_j}\right) = (-1)^{j-1} \frac{\Delta(\mathbf{z})}{z_j^{n-1} \Delta_j(\mathbf{z})} = \frac{u_1 + u_2 + \dots + u_n}{u_j}.$$

Thus equation (5.3) is equivalent to

$$\text{CT}_{\mathbf{z}} \frac{(u_1 + u_2 + \dots + u_n)^{a_1 + a_2 + \dots + a_n}}{u_1^{a_1} \dots u_n^{a_n}} = \frac{(a_1 + a_2 + \dots + a_n)!}{a_1! a_2! \dots a_n!},$$

which is a direct consequence of the multinomial theorem and the following proposition.

**Proposition 5.2.** *For any series  $\Phi(\mathbf{z}) \in K^{\mathbf{u}} \langle\langle \mathbf{z} \rangle\rangle$ , we have*

$$\text{CT}_{\mathbf{z}} \Phi(u_1, \dots, u_n) = \text{CT}_{\mathbf{u}} \Phi(u_1, \dots, u_n).$$

**Corollary 5.3.** *Let  $a_1, \dots, a_n$  be  $n$  integers. Then in the field  $\mathbb{C}\langle\langle z_1 \cdots z_n \rangle\rangle$ , we have*

$$\begin{aligned} \text{CT}_{\mathbf{z}} \prod_{1 \leq i \neq j \leq n} \left(1 - \frac{z_i}{z_j}\right)^{a_j} \\ = \begin{cases} \frac{(a_1 + a_2 + \cdots + a_n)!}{a_1! a_2! \cdots a_n!}, & \text{if } a_i \geq 0; \\ (-1)^{\sum_{i=2}^n a_i} \frac{(-a_1 - 1)!}{(-1 - \sum_{i=1}^n a_i)! a_2! \cdots a_n!}, & \text{if } a_2, \dots, a_n \geq 0 \text{ and } \sum_{i=1}^n a_i < 0; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

In fact, we can prove a more general formula. Let  $r$  be an integer and let  $u_j^{(r)} = (-1)^{j-1} z_j^r \Delta_j(\mathbf{z})$ . Then  $u_1^{(r)} + \cdots + u_n^{(r)}$  equals  $h_{r-n+1}(z_1, z_2, \dots, z_n) \Delta(\mathbf{z})$  for  $r \geq n-1$  and equals 0 for  $0 \leq r \leq n-2$ . We have the following generalization.

**Theorem 5.4.** *If  $r$  is not equal to one of  $0, 1, \dots, n-2$ , or  $-(\binom{n-1}{2})$ , then for any series  $\Phi(\mathbf{z}) \in K^\rho \langle\langle \mathbf{z} \rangle\rangle$ , where  $\rho(z_i) = u_i^{(r)}$ , we have*

$$\text{CT}_{\mathbf{z}} \Phi(u_1^{(r)}, \dots, u_n^{(r)}) = \text{CT}_{\mathbf{u}^{(r)}} \Phi(u_1^{(r)}, \dots, u_n^{(r)}).$$

Note that Proposition 5.2 is the special case for  $r = n-1$  of Theorem 5.4. If we set  $r = n$ , the multinomial theorem yields the following:

**Corollary 5.5.** *Let  $a_1, \dots, a_n$  be  $n$  nonnegative integers. Then the following equation holds as Laurent polynomials in  $\mathbf{z}$ .*

$$\text{CT}_{\mathbf{z}} \frac{(z_1 + \cdots + z_n)^{a_1 + \cdots + a_n}}{z_1^{a_1} \cdots z_n^{a_n}} \prod_{1 \leq i \neq j \leq n} \left(1 - \frac{z_i}{z_j}\right)^{a_j} = \frac{(a_1 + a_2 + \cdots + a_n)!}{a_1! a_2! \cdots a_n!}. \quad (5.3)$$

By Theorem 3.6, Theorem 5.4 is equivalent to saying that the log Jacobian is a nonzero constant. To show this, we use the argument of Wilson [16].

**Lemma 5.6** (Lemma 4 [16]). *Let  $G(x_1, \dots, x_n)$  be a function of  $n$  variables such that*

- (1)  $G$  is a symmetric function of  $x_1, \dots, x_n$ .
- (2)  $G$  is a ratio of two polynomials in the  $x$ 's.
- (3)  $G$  is homogeneous of degree 0 in the  $x$ 's.
- (4) The denominator of  $G$  is  $\Delta(x_1, \dots, x_n)$ .

*Then  $G$  is a constant.*

*Proof of Theorem 5.4.* In order to compute the log Jacobian, we let

$$J = \det(J_{ij}) = \det \left( \frac{\partial \log u_i^{(r)}}{\partial \log z_j} \right).$$

Then  $J_{ii} = r$  and  $J_{ij} = \sum_{k \neq i} \frac{z_i}{z_k - z_j}$  for  $i \neq j$ . We first show that  $J$  is a constant by Lemma 5.6. It is easy to see that  $J$  satisfies the conditions 1, 2 and 3 in Lemma

5.6. Now we show that the denominator of  $J$  is  $\Delta(\mathbf{z})$ , so that we can claim that the Jacobian is a constant, and hence equals the Jacobian number.

Evidently  $J$  is the ratio of two polynomials in the  $\mathbf{z}$ 's, whose denominator is a product of factors  $z_i - z_j$  for some  $i \neq j$ . From the expression of  $J_{ij}$ , we see that  $z_i - z_j$  only appears in the  $i$ th or the  $j$ th column. Every 2 by 2 minor of the  $i$ th and  $j$ th columns are of the following form, in which we assume that  $k$  and  $l$  are not one of  $i$  and  $j$ .

$$\begin{vmatrix} J_{ki} & J_{kj} \\ J_{li} & J_{lj} \end{vmatrix} = \begin{vmatrix} \frac{z_k}{z_j - z_i} + \sum_{s \neq i, j} \frac{z_k}{z_s - z_i} & \frac{z_k}{z_i - z_j} + \sum_{s \neq i, j} \frac{z_k}{z_s - z_j} \\ \frac{z_l}{z_j - z_i} + \sum_{s \neq i, j} \frac{z_l}{z_s - z_i} & \frac{z_l}{z_i - z_j} + \sum_{s \neq i, j} \frac{z_l}{z_s - z_j} \end{vmatrix}.$$

In the above determinant, the terms containing  $(z_i - z_j)^2$  as the denominator cancel. Therefore, expanding the determinant according to the  $i$ th and  $j$ th column, we see that  $\Delta(\mathbf{z})$  is the denominator of  $J$ .

Now the initial term of  $z_i - z_j$  is  $z_i$  if  $i < j$ . We see that the initial term of  $u_1^{(r)}$  is  $z_1^r z_2^{n-2} z_3^{n-3} \dots z_{n-1}$ . Similarly we can get the initial term for  $u_j^{(r)}$ . The Jacobian number, denoted by  $j(r)$ , is thus the determinant

$$j(r) = \det \begin{pmatrix} r & n-2 & n-3 & \dots & 0 \\ n-2 & r & n-3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ n-2 & n-3 & n-4 & \dots & r \end{pmatrix},$$

where the displayed matrix has diagonal entries  $r$ , and other entries in each row are  $n-2, n-3, \dots, 0$ , respectively from left to right.

Since the row sum of each row is  $r + \binom{n-1}{2}$ ,  $j(-\binom{n-1}{2}) = 0$ . We claim that  $j(r) = 0$  when  $r = 0, 1, \dots, n-2$ . For in those cases,  $u_1^{(r)} + \dots + u_n^{(r)} = 0$ . This implies that the Jacobian is 0, and hence  $j(r) = 0$ . We can regard  $j(r)$  as a polynomial in  $r$  of degree  $n$ , and we have already got  $n$  zeros. So  $j(r) = r(r-1) \dots (r-n+2)(r + \binom{n-1}{2})$  up to a constant. This constant equals 1 through comparing the leading coefficient of  $r$ .

In particular,  $j(n-1) = \binom{n}{2}(n-1)! = \frac{n-1}{2}n!$ . Note that in [2], the constant was said to be  $\frac{n-3}{2}n!$ , which is not true. □

Another proof of Dyson's conjecture by our residue theorem is to use the change of variables by Wilson [16].

Let

$$v_j = \prod_{1 \leq i \leq n, i \neq j} (1 - z_j/z_i)^{-1}.$$

Then the initial term of  $v_j$  is  $z^{n-j} z_{j+1} \dots z_n$  up to a constant. Since the order of  $v_n$  is 0, we have to exclude  $v_n$  from the change of variables, for otherwise, the Jacobian number will be 0. In fact, we have the relation  $v_1 + v_2 + \dots + v_n = 1$ , which can be easily shown by Lemma 5.6.

Dyson's conjecture is equivalent to

$$\text{CT}_{\mathbf{z}} \prod_{j=1}^n v_j^{-a_j} = \frac{(a_1 + a_2 + \cdots + a_n)!}{a_1! a_2! \cdots a_n!} \quad (5.4)$$

*Another Proof of Dyson's Conjecture.* Using Lemma 5.6 and Wilson's argument, we can evaluate the following log Jacobian. (Or see [16] for details.)

$$\frac{\partial(\log v_1, \log v_2, \dots, \log v_{n-1})}{\partial(\log z_1, \log z_2, \dots, \log z_{n-1})} = (n-1)! v_n.$$

Then by the residue theorem

$$\text{CT}_{\mathbf{z}} \Phi(v_1, \dots, v_{n-1}, z_n) = \text{CT}_{v_1, \dots, v_{n-1}, z_n} (1 - v_1 - \cdots - v_{n-1})^{-1} \Phi(v_1, \dots, v_{n-1}, z_n).$$

In particular, (since the initial term of  $1 - v_1 - \cdots - v_{n-1}$  is 1), we have:

$$\begin{aligned} \text{CT}_{\mathbf{z}} \prod_{j=1}^n v_j^{-a_j} &= \text{CT}_{v_1, \dots, v_{n-1}, z_n} (1 - v_1 - \cdots - v_{n-1})^{-a_n - 1} \prod_{j=1}^{n-1} v_j^{-a_j} \\ &= [v_1^{a_1} \cdots v_{n-1}^{a_{n-1}}] \sum_{m \geq 0} \binom{a_n + m}{a_n} (v_1 + \cdots + v_{n-1})^m \\ &= \binom{a_n + a_1 + \cdots + a_{n-1}}{a_n} \binom{a_1 + \cdots + a_{n-1}}{a_1, \dots, a_{n-1}}. \end{aligned}$$

Equation (5.4) then follows.  $\square$

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