

# A GENERALIZATION OF STANLEY'S MONSTER RECIPROCITY THEOREM

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ABSTRACT. By studying the reciprocity property of linear Diophantine systems in light of Malcev-Neumann series, we present in this paper a new approach to and a generalization of Stanley's monster reciprocity theorem. A formula for the "error term" is given in the case when the system does not have the reciprocity property. We also give a short proof of Stanley's reciprocity theorem for linear homogeneous Diophantine systems.

**Keywords:** Reciprocity property, linear Diophantine system, Laurent series, Malcev-Neumann series

## 1. INTRODUCTION

Let  $A$  be an  $r$  by  $n$  matrix with integer entries, and let  $\mathbf{b}$  be an  $r$ -vector in  $\mathbb{Z}^r$ . Many combinatorial problems turn out to be equivalent to finding all nonnegative integral (column) vectors  $\alpha \in \mathbb{N}^n$  satisfying

$$(1.1) \quad A\alpha = \mathbf{b},$$

especially in the homogeneous case when  $\mathbf{b}$  equals  $\mathbf{0}$ , of which the solution space is a rational cone. Such problems are also known as solving a linear Diophantine system.

There are two closely related generating functions associated to (1.1):

$$E(\mathbf{x}; \mathbf{b}) = \sum_{(\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n} x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \bar{E}(\mathbf{x}; \mathbf{b}) = \sum_{(\alpha_1, \dots, \alpha_n) \in \mathbb{P}^n} x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

where the first sum ranges over all  $\alpha = (\alpha_1, \dots, \alpha_n)$  such that  $A\alpha = \mathbf{b}$ , and the second sum ranges over all positive integral  $\alpha$  such that  $A\alpha = -\mathbf{b}$ . We omit  $\mathbf{b}$  in the homogeneous case. The following well-known reciprocity theorem for homogeneous linear diophantine equations was given by Stanley as [3, Theorem 4.1].

**Theorem 1.1** (Reciprocity Theorem). *Let  $A$  be an  $r$  by  $n$  integral matrix of full rank  $r$ . If there is at least one  $\alpha \in \mathbb{P}^n$  such that  $A\alpha = \mathbf{0}$ , then we have as rational functions*

$$E(x_1, \dots, x_n) = (-1)^{n-r} \bar{E}(x_1^{-1}, \dots, x_n^{-1}).$$

Previous proofs of this theorem used decompositions into simplicial cones or lattice cones, or complicated algebraic technique. See [4, p. 214] and [5] for further information. We will give a short proof using a signed cone decomposition and induction.

In the general situation, the best known result (up to now) is the monster reciprocity theorem, which was given by Stanley [4] in 1974. The theorem will be stated later after new notation is introduced. It includes as special cases many combinatorial reciprocity theorems, such as the reciprocity theorem for homogeneous linear Diophantine system, that for Ehrhart polynomials, and that for P-partitions, etc. We will give a simple approach to this theorem. As applications, we give detailed, and short, implication of the reciprocal domain theorem [4, Proposition 8.3].

The new approach uses the idea of Malcev-Neumann series [2; 7; 9], which defines a total ordering on the group of monomials to clarify the series expansion of rational functions. We study the reciprocity property of an object that is more general, but less combinatorial, than that was studied in [4]. The new objects we are going to study are *Elliott-rational functions*, while the previous objects are Elliott-rational functions with a monomial numerator. By an Elliott-rational function, we mean the one that can be written as

$$F(\lambda_1, \dots, \lambda_r, \mathbf{x}) = \frac{p(\lambda_1, \dots, \lambda_r, \mathbf{x})}{\prod_{i=1}^m (y_i - z_i)},$$

where  $p$  is a polynomial and  $y_i$  and  $z_i$  are monomials.

In this larger set of objects, it is much easier to build up the reduction steps. Theorem 3.8, a general result that gives a reciprocity formula for Elliott-rational functions, turns out to be easy to prove. We shall use this result to formulate the monster reciprocity theorem (Theorem 4.2).

In Section 2, we introduce the basic idea of Malcev-Neumann series and reformulate the reciprocity of linear Diophantine system in terms of constant terms. In Section 3, we develop the reciprocity theorem for Elliott-rational functions. We apply our result in Section 4 to give the generalized monster reciprocity theorem. In section 5, we illustrate the monster reciprocity theorem by examples, and as an application, we give a simple derivation of Theorem 1.1. Section 6 includes an inductive (combinatorial) proof of Theorem 1.1.

## 2. RECIPROCITY IN TERMS OF CONSTANT TERMS

Solving a linear Diophantine system (LD-system for short) means finding all vectors  $\alpha \in \mathbb{N}^n$  that satisfy  $A\alpha = \mathbf{b}$ , where  $A$  is an  $r$  by  $n$  matrix with integral entries. More

precisely, we want to solve the following system of equations:

$$\begin{aligned}
 & a_{1,1}\alpha_1 + a_{1,2}\alpha_2 + \cdots + a_{1,n}\alpha_n = b_1 \\
 & a_{2,1}\alpha_1 + a_{2,2}\alpha_2 + \cdots + a_{2,n}\alpha_n = b_2 \\
 (2.1) \quad & \dots\dots\dots = \dots \\
 & a_{r,1}\alpha_1 + a_{r,2}\alpha_2 + \cdots + a_{r,n}\alpha_n = b_r.
 \end{aligned}$$

We assume the rank of  $A|\mathbf{b}$  equals the rank of  $A$ , for otherwise, the LD-system has no solution even in  $\mathbb{Q}$ .

Let  $C_i$  be the  $i$ th column vector of  $A$ . Then the above system is the same as

$$C_1\alpha_1 + C_2\alpha_2 + \cdots + C_n\alpha_n = \mathbf{b}.$$

Now let  $E(\mathbf{b})$  and  $\bar{E}(\mathbf{b})$  be the sets of all such solutions in  $\mathbb{N}^n$  and  $\mathbb{P}^n$  respectively. It is interesting to study the following two associated generating functions of (2.1):

$$(2.2) \quad E(\mathbf{x}; \mathbf{b}) = \sum_{\alpha \in E(\mathbf{b})} \mathbf{x}^\alpha, \quad \bar{E}(\mathbf{x}; \mathbf{b}) = \sum_{\alpha \in \bar{E}(-\mathbf{b})} \mathbf{x}^\alpha$$

where  $\mathbf{x} = (x_1, \dots, x_n)$  and if  $\alpha = (\alpha_1, \dots, \alpha_n)$ , then  $\mathbf{x}^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ .

The above equation defines two rational functions in  $\mathbf{x}$ . If as rational functions  $E(\mathbf{x}; \mathbf{b}) = (-1)^{n-r} \bar{E}(\mathbf{x}^{-1}; \mathbf{b})$ , then we say that the system (2.1) has the *R-property* (short for *reciprocity property*).

We can compute  $E(\mathbf{x}; \mathbf{b})$  by replacing the  $r$  linear constraints with  $r$  new variables  $\lambda_1, \lambda_2, \dots, \lambda_r$  and then take the constant terms. Let  $\Lambda$  be  $(\lambda_1, \dots, \lambda_r)$ , and let  $\text{CT}_\Lambda F$  be the constant term of  $F$  in  $\Lambda$ . We have

$$\begin{aligned}
 E(\mathbf{x}; \mathbf{b}) &= \sum_{\alpha \in \mathbb{N}^n} \text{CT}_\Lambda \lambda_1^{a_{1,1}\alpha_1 + \cdots + a_{1,n}\alpha_n - b_1} \cdots \lambda_r^{a_{r,1}\alpha_1 + \cdots + a_{r,n}\alpha_n - b_r} \mathbf{x}^\alpha \\
 (2.3) \quad &= \text{CT}_\Lambda \frac{\lambda_1^{-b_1} \cdots \lambda_r^{-b_r}}{\prod_{i=1}^n (1 - \lambda_1^{a_{1,i}} \lambda_2^{a_{2,i}} \cdots \lambda_r^{a_{r,i}} x_i)} = \text{CT}_\Lambda \frac{\Lambda^{-\mathbf{b}}}{\prod_{i=1}^n (1 - \Lambda^{C_i} x_i)},
 \end{aligned}$$

with the working ring  $\mathbb{C}[\Lambda, \Lambda^{-1}][[\mathbf{x}]]$ , where  $\Lambda^{-1}$  means  $(\lambda_1^{-1}, \dots, \lambda_r^{-1})$ . The above conversion can be trait back to MacMahon [1]. Similarly we get

$$(2.4) \quad \bar{E}(\mathbf{x}; \mathbf{b}) = \text{CT}_\Lambda \frac{\Lambda^{\mathbf{b}} \prod_{i=1}^n \Lambda^{C_i} x_i}{\prod_{i=1}^n (1 - \Lambda^{C_i} x_i)}.$$

We define  $\mathcal{E}(\Lambda, \mathbf{x}; \mathbf{b})$  and  $\bar{\mathcal{E}}(\Lambda, \mathbf{x}; \mathbf{b})$  to be the crude generating functions of  $E(\mathbf{x}, \mathbf{b})$  and  $\bar{E}(\mathbf{x}; \mathbf{b})$  as

$$(2.5) \quad \mathcal{E}(\Lambda, \mathbf{x}; \mathbf{b}) = \frac{\Lambda^{-\mathbf{b}}}{\prod_{i=1}^n (1 - \Lambda^{C_i} x_i)}, \quad \bar{\mathcal{E}}(\Lambda, \mathbf{x}; \mathbf{b}) = \frac{\Lambda^{\mathbf{b}} \prod_{i=1}^n \Lambda^{C_i} x_i}{\prod_{i=1}^n (1 - \Lambda^{C_i} x_i)},$$

and observe that as rational functions

$$\bar{\mathcal{E}}(\Lambda^{-1}, \mathbf{x}^{-1}; \mathbf{b}) = (-1)^n \mathcal{E}(\Lambda, \mathbf{x}; \mathbf{b}).$$

However, the series expansion of the two sides of the above equation are different. The change of variables by  $\Lambda \rightarrow \Lambda^{-1}$ , which corresponds to multiplying each row of (2.1) by  $-1$ , will not make a difference when taking constant terms. Therefore, the system has the R-property if and only if as rational functions

$$\underset{\Lambda}{\text{CT}} \mathcal{E}(\Lambda, \mathbf{x}) = (-1)^r \underset{\Lambda}{\text{CT}'} \mathcal{E}(\Lambda, \mathbf{x}),$$

where we expand  $\mathcal{E}(\Lambda, \mathbf{x})$  on the LHS at  $\mathbf{x} = \mathbf{0}$ , while on the RHS at  $\mathbf{x} = \infty$ .

As we shall see later, the different expansions appearing in the above equation is easily explained in the context of Malcev-Neumann series.

The group of monomials in  $\Lambda$  and  $\mathbf{x}$  can be given a total ordering “ $\preceq^\rho$ ” that is compatible with its group structure; i.e., for any monomials  $A, B$  and  $C$ ,  $A \preceq B$  implies  $AC \preceq^\rho BC$ . This is equivalent to a total ordering  $\leq^\rho$  on the additive group  $\mathbb{Z}^{n+r}$ . An important such ordering  $\leq$  is the reverse lexicographical ordering on  $\mathbb{Z}^{n+r}$ . Then a *Malcev-Neumann series* (or *MN-series* for short) with respect to  $\preceq^\rho$  is a formal series on  $\Lambda$  and  $\mathbf{x}$  with a well-ordered *support*: the set of monomials corresponds to the nonzero terms. Recall that a *well-ordered set* is a totally ordered set such that every nonempty subset has a minimum.

For our purpose,  $\rho$  will denote an injective endomorphism of  $\mathbb{Z}^{n+r}$  (a nonsingular integral matrix), and  $\leq^\rho$  will be the induced total ordering defined by  $a \leq^\rho b$  if and only if  $\rho(a) \leq \rho(b)$ . We denote by  $\mathbb{C}^\rho \langle\langle \Lambda, \mathbf{x} \rangle\rangle$  the corresponding field of MN-series with respect to  $\rho$ . The field of iterated Laurent series  $\mathbb{C} \langle\langle \Lambda, \mathbf{x} \rangle\rangle$ , where  $\rho$  is the identity map and is omitted, has been studied in [8; 9]. For a more general setting of MN-series, the readers are referred to [7; 9] or [2, Chapter 13].

The series expansion of MN-series will be explained in more details in the next section. Let us review some properties of MN-series [7] to see that such fields are suitable for dealing with different kinds of series expansions of rational functions.

For any total ordering  $\leq^\rho$ ,  $\mathbb{C}^\rho \langle\langle \Lambda, \mathbf{x} \rangle\rangle$  is a field. In particular,  $\mathbb{C} \langle\langle \Lambda, \mathbf{x} \rangle\rangle$  is the field of iterated Laurent series [8].

The field  $\mathbb{C}(\Lambda, \mathbf{x})$  of rational functions is naturally embedded into  $\mathbb{C}^\rho \langle\langle \Lambda, \mathbf{x} \rangle\rangle$  for any  $\rho$ . This follows from the field structure of  $\mathbb{C}^\rho \langle\langle \Lambda, \mathbf{x} \rangle\rangle$  and the fact that every polynomial has a finite support.

Every rational function  $F(\Lambda, \mathbf{x})$  has a unique expansion in  $\mathbb{C}^\rho \langle\langle \Lambda, \mathbf{x} \rangle\rangle$ . The expansions of  $F$  for different  $\rho$  are usually different. For instance, the expansion of  $1/(x - y)$  in  $K \langle\langle x, y \rangle\rangle$  is

$$\frac{1}{x - y} = \frac{1}{x} \cdot \frac{1}{1 - y/x} = \frac{1}{x} \sum_{k \geq 0} y^k / x^k,$$

but the expansion in  $K \langle\langle y, x \rangle\rangle$  is

$$\frac{1}{x - y} = \frac{1}{-y} \cdot \frac{1}{1 - x/y} = \frac{1}{-y} \sum_{k \geq 0} x^k / y^k.$$

Note that we can write  $K\langle\langle y, x \rangle\rangle$  as  $K^\rho\langle\langle x, y \rangle\rangle$  where  $\rho$  is defined by the matrix  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ , or by abuse of notation,  $\rho(x) = y$  and  $\rho(y) = x$ .

Recall also that every subset of a well-ordered set is well-ordered. Thus the following operators  $\text{CT}_\lambda$ ,  $\text{PT}_\lambda$ , and  $\text{NT}_\lambda$  are well-defined for MN-series.

$$\text{CT}_\lambda \sum_{k \in \mathbb{Z}} b_k \lambda^k = b_0, \quad \text{PT}_\lambda \sum_{k \in \mathbb{Z}} b_k \lambda^k = \sum_{k \geq 0} b_k \lambda^k, \quad \text{and} \quad \text{NT}_\lambda \sum_{k \in \mathbb{Z}} b_k \lambda^k = \sum_{k < 0} b_k \lambda^k.$$

Obviously, for an MN-series  $F(\lambda)$ ,  $\text{CT}_\lambda F(\lambda) = \text{PT}_\lambda F(\lambda)|_{\lambda=0}$ . The constant term operators are commutative so that taking the constant term in a set of variables is defined by iteration.

Now it is easy to see that Theorem 1.1 is a consequence of the following proposition.

**Proposition 2.1.** *Suppose that  $\bar{E}$  is nonempty. Then*

$$(2.6) \quad \text{CT}_\Lambda \mathcal{E}(\mathbf{x}; \mathbf{0}) = (-1)^{\text{rank}(A)} \text{CT}_\Lambda^\rho \mathcal{E}(\mathbf{x}; \mathbf{0}),$$

where  $\rho$  is the endomorphism defined by  $\rho(x_i) = x_i^{-1}$  and  $\rho(\lambda_i) = \lambda_i$ .

On the other hand, it is easy to deal with the case of  $\text{rank}(A) < r$ . So Theorem 1.1 is equivalent to Proposition 2.1, whose proof will be given in section 6.

The following lemma asserts that elementary row operation will not change the solution space of an LD-system. We give it here to show that all the work can be done algebraically.

**Lemma 2.2** ([7], Corollary 3.18). *Suppose  $\mathbf{y}$  is another set of variables. If  $\Phi \in K[\mathbf{x}, \mathbf{x}^{-1}]\langle\langle \mathbf{y} \rangle\rangle$ , then for  $f_i = x_1^{b_{i1}} \cdots x_n^{b_{in}}$  with  $\det(b_{ij})_{1 \leq i, j \leq n} \neq 0$ ,*

$$\text{CT}_{\mathbf{x}} \Phi(f_1, \dots, f_n) = \text{CT}_{\mathbf{x}} \Phi(x_1, \dots, x_n).$$

### 3. RECIPROCITY OF ELLIOTT-RATIONAL FUNCTIONS

It is convenient for our purpose to denote by  $K$  the field  $\mathbb{C}(\mathbf{x})$ . The field of rational functions  $\mathbb{C}(\Lambda, \mathbf{x})$  can be identified with  $K(\Lambda)$ . Usually we are taking constant terms in the ring  $\mathbb{C}[\Lambda, \Lambda^{-1}][[\mathbf{x}]]$ . This ring can be embedded into  $\mathbb{C}^\rho\langle\langle \Lambda, \mathbf{x} \rangle\rangle$ , as long as  $\preceq^\rho$  is compatible with the relation  $x_i \succcurlyeq^\rho \lambda_j$  for all  $i$  and  $j$ , where  $A \succcurlyeq^\rho B$  means that  $A \succ^\rho B^k$  for any positive integer  $k$ .

The case  $r = 1$  is illustrative for our understanding of the series expansion for MN-series, and in this particular case, we need not restrict ourselves to Elliott-rational functions. Let us consider the following problem.

*Problem:* Given a rational function  $Q(\lambda)$  (short for  $Q(\lambda, \mathbf{x})$ ) of  $\lambda$  and  $\mathbf{x}$ , compute  $\text{PT}_\lambda^\rho Q(\lambda, \mathbf{x})$ , where the notation  $\text{PT}_\lambda^\rho$  indicates that  $Q(\lambda)$  is treated as an element of  $\mathbb{C}^\rho\langle\langle \lambda, \mathbf{x} \rangle\rangle$ , and we use similar notations for the CT and NT operators.

To deal with this problem, we shall understand that  $Q(\lambda)$  is not only an element of  $K(\lambda)$ , but also an element of  $\mathbb{C}^\rho \langle\langle \lambda, \mathbf{x} \rangle\rangle$ . As an element of  $K(\lambda)$ ,  $Q(\lambda)$  can be written as  $p(\lambda)/q(\lambda)$ , where  $p(\lambda)$  and  $q(\lambda)$  are both in  $K[\lambda]$ . As an element of  $\mathbb{C}^\rho \langle\langle \lambda, \mathbf{x} \rangle\rangle$ , the denominator  $q(\lambda)$  plays an important role.

Recall that  $\mathbb{C}^\rho \langle\langle \lambda, \mathbf{x} \rangle\rangle$  is equipped with a total ordering  $\preceq^\rho$  on its group of monomials and that its elements have well-ordered supports. Thus for a nonzero element  $\eta$ , we can define its *order*  $\text{ord } \eta$  to be  $\min \text{supp}(\eta)$ , and its *initial term* to be the term with the least order. The order of 0 is treated as  $\infty$ . Let us write  $q(\lambda) = \sum_{i=0}^d a_i \lambda^i$ , with  $a_i \in \mathbb{C}(\mathbf{x})$  and  $a_d \neq 0$ . To expand  $Q(\lambda)$  into a series in  $\mathbb{C}^\rho \langle\langle \lambda, \mathbf{x} \rangle\rangle$ , we need to find the  $\lambda$ -*initial term*  $a_j \lambda^j$ , i.e., the  $j$  such that  $\text{ord}(a_j \lambda^j) \prec^\rho \text{ord}(a_i \lambda^i)$  for all  $i \neq j$ . This can be achieved because of the different powers in  $\lambda$ . Then

$$\frac{1}{q(\lambda)} = \frac{1}{a_j \lambda^j} \frac{1}{1 + \sum_{i \neq j} a_i / a_j \lambda^{i-j}} = \frac{1}{a_j \lambda^j} \sum_{k \geq 0} (-1)^k \left( \sum_{i \neq j} a_i / a_j \lambda^{i-j} \right)^k.$$

This expansion is justified by the composition law [9, Theorem 2.2].

It is now clear that we have the following three situations.

- (1) If  $j$  equals 0, then for any polynomial  $p(\lambda)$ ,  $p(\lambda)/q(\lambda)$  contains only nonnegative powers in  $\lambda$ . In this case, we say that  $1/q(\lambda)$  is  $\text{PT}^\rho$  in  $\lambda$ .
- (2) If  $j$  equals  $d$ , then for any polynomial  $p(\lambda)$  of degree in  $\lambda$  less than  $d$ ,  $p(\lambda)/q(\lambda)$  contains only negative powers in  $\lambda$ . In this case, we say that  $1/q(\lambda)$  is  $\text{NT}^\rho$  in  $\lambda$ .
- (3) If  $j$  equals neither 0, nor  $d$ , then  $1/q(\lambda)$  contains both positive and negative powers in  $\lambda$ . Thus  $1/q(\lambda)$  is neither  $\text{PT}^\rho$  nor  $\text{NT}^\rho$  in  $\lambda$ .

**Lemma 3.1.** *Let  $q_1$  and  $q_2$  be polynomials in  $\lambda$ . Then for any total ordering  $\leq^\rho$*

- *Both  $1/q_1(\lambda)$  and  $1/q_2(\lambda)$  are  $\text{PT}^\rho$  in  $\lambda$  if and only if  $1/(q_1 q_2)$  is.*
- *Both  $1/q_1(\lambda)$  and  $1/q_2(\lambda)$  are  $\text{NT}^\rho$  in  $\lambda$  if and only if  $1/(q_1 q_2)$  is.*
- *For all the other cases,  $1/(q_1 q_2)$  is neither  $\text{PT}^\rho$  in  $\lambda$  nor  $\text{NT}^\rho$  in  $\lambda$ .*

*Proof.* We prove the first case for PT as follows. The other cases are similar. Write

$$q_1 = \sum_{i=0}^{d_1} a_i \lambda^i, \quad q_2 = \sum_{i=0}^{d_2} b_i \lambda^i, \quad \text{and } q_1 q_2 = \sum_{i=0}^{d_1+d_2} c_i \lambda^i.$$

Suppose that  $a_{j_1} \lambda^{j_1}$  and  $b_{j_2} \lambda^{j_2}$  are the  $\lambda$ -initial term of  $q_1$  and  $q_2$  respectively. Now if we expand the product  $q_1 q_2$  but do not collect terms, then  $a_{j_1} b_{j_2} \lambda^{j_1+j_2}$  is the unique term with the least order. So the order of  $c_{j_1+j_2} \lambda^{j_1+j_2}$  has to equal the order of  $a_{j_1} b_{j_2} \lambda^{j_1+j_2}$ . This implies that the  $\lambda$ -initial term of  $q_1 q_2$  is  $c_{j_1+j_2} \lambda^{j_1+j_2}$ . The assertion for PT in the lemma hence follows from the fact that  $j_1 + j_2 = 0 \Leftrightarrow j_1 = 0$  and  $j_2 = 0$ . (Remember that  $j_1, j_2 \geq 0$ ).  $\square$

A direct consequence of the above lemma is the following corollary.

**Corollary 3.2.** *If  $1/q_1(\lambda)$  is  $\text{PT}^\rho$  in  $\lambda$  and  $1/q_2(\lambda)$  is  $\text{NT}^\rho$  in  $\lambda$ , then  $q_1(\lambda)$  and  $q_2(\lambda)$  cannot have a nontrivial common divisor in  $K[\lambda]$ , i.e., they are relatively prime.*

**Definition 3.3.** If  $q(\lambda)$  can be factored as  $q_1(\lambda)q_2(\lambda)$  such that  $1/q_1(\lambda)$  is  $\text{PT}^\rho$  in  $\lambda$  and  $1/q_2(\lambda)$  is  $\text{NT}^\rho$  in  $\lambda$ , then we say that  $q(\lambda)$  is  $\rho$ -factorable, and  $q(\lambda) = q_1(\lambda)q_2(\lambda)$  is a  $\rho$ -factorization. Such factorization is unique (if it exists) up to a constant in  $K$ .

**Theorem 3.4.** *Let  $p(\lambda), q(\lambda) \in K[\lambda]$ . If  $q(\lambda)$  is  $\rho$ -factorable, then  $\text{CT}_\lambda^\rho p(\lambda)/q(\lambda)$  is in  $K$ , i.e., is rational.*

*Proof.* Suppose  $q(\lambda) = q_1(\lambda)q_2(\lambda)$  is such a  $\rho$ -factorization. Since  $1/q_1(\lambda)$  is  $\text{PT}^\rho$  in  $\lambda$  and  $1/q_2(\lambda)$  is  $\text{NT}^\rho$  in  $\lambda$ ,  $q_1(\lambda)$  and  $q_2(\lambda)$  are relatively prime in  $K[\lambda]$  by Corollary 3.2. Thus we have the unique partial fraction expansion in  $K(\lambda)$ :

$$(3.1) \quad \frac{p(\lambda)}{q(\lambda)} = p_0(\lambda) + \frac{p_1(\lambda)}{q_1(\lambda)} + \frac{p_2(\lambda)}{q_2(\lambda)},$$

where  $p_i$  are polynomials in  $\lambda$  for  $i = 0, 1, 2$  and  $\deg p_i(\lambda) < \deg q_i(\lambda)$  for  $i = 1, 2$ . Since when expanded as series in  $\mathbb{C}^\rho \langle\langle \lambda, \mathbf{x} \rangle\rangle$ ,  $p_0(\lambda)$  and  $p_1(\lambda)/q_1(\lambda)$  contains only nonnegative powers in  $\lambda$ , and  $p_2(\lambda)/q_2(\lambda)$  contains only negative powers in  $\lambda$ , we have

$$\text{PT}_\lambda^\rho \frac{p(\lambda)}{q(\lambda)} = p_0(\lambda) + \frac{p_1(\lambda)}{q_1(\lambda)}.$$

Thus  $\text{CT}_\lambda^\rho = p_0(0) + p_1(0)/q_1(0)$  is in  $\mathbb{C}(\mathbf{x})$ . □

This theorem generalizes a result of Hadamard [6, Proposition 4.2.5], which says that the Hadamard product of two rational power series is rational. This statement can be easily seen from the following observation: Let  $f = \sum_{k \geq 0} f_k x^k$  and  $g = \sum_{k \geq 0} g_k x^k$ . Then the Hadamard product of  $f$  and  $g$  is

$$\sum_{k \geq 0} f_k g_k x^k = \text{CT}_\lambda f(\lambda)g(x/\lambda),$$

where we are taking the constant term in  $\mathbb{C} \langle\langle \lambda, x \rangle\rangle$  for the RHS of the above equation.

For any total ordering  $\preceq^\rho$  on the monomials of  $K(\lambda)$ , we let  $\preceq^{\bar{\rho}}$  be the total ordering such that  $m_1 \prec^{\bar{\rho}} m_2$  if and only if  $m_1 \succ^\rho m_2$  for all monomials  $m_1$  and  $m_2$ .

Then we have a sort of reciprocity invariant, for which we need three more notations. We use the notation  $\text{CT}_{\lambda=0} F(\lambda)$  to indicate that  $F(\lambda)$  is treated as an element in  $K((\lambda))$  and  $\text{CT}_{\lambda=\infty} F(\lambda)$  to indicate that  $F(\lambda)$  is treated as an element in  $K((\lambda^{-1}))$ . We define

$$(3.2) \quad \mathcal{I}_\lambda F(\lambda) = \text{CT}_{\lambda=0} F(\lambda) + \text{CT}_{\lambda=\infty} F(\lambda).$$

**Theorem 3.5.** *Suppose that  $p(\lambda), q(\lambda) \in K[\lambda]$ , with  $q(\lambda)$  being  $\rho$ -factorable. Then the following is always true as rational functions in  $K$ :*

$$(3.3) \quad \text{CT}_\lambda^\rho \frac{p(\lambda)}{q(\lambda)} + \text{CT}_\lambda^{\bar{\rho}} \frac{p(\lambda)}{q(\lambda)} = \mathcal{I}_\lambda \frac{p(\lambda)}{q(\lambda)}.$$

Theorem 3.5 gives an invariant of a rational function when taking the constant term in  $\lambda$ . This invariant is independent of the choice of the total ordering  $\leq^\rho$  when applicable. This fact is the key in our new approach to the monster reciprocity theorem in Section 4.

*Proof of Theorem 3.5.* Write  $q(\lambda)$  as  $q_1(\lambda)q_2(\lambda)\lambda^s$ , such that  $1/q_1(\lambda)$  is  $\text{PT}^\rho$  in  $\lambda$ ,  $1/q_2(\lambda)$  is  $\text{NT}^\rho$ , and  $q_1(0)q_2(0) \neq 0$ . Clearly  $s \geq 0$  and the partial fraction decomposition of  $p(\lambda)/q(\lambda)$  can be written as

$$\frac{p(\lambda)}{q(\lambda)} = \frac{p_{-1}(\lambda)}{\lambda^s} + p_0(\lambda) + \frac{p_1(\lambda)}{q_1(\lambda)} + \frac{p_2(\lambda)}{q_2(\lambda)},$$

where  $\deg p_{-1} < s$ ,  $\deg p_1 < \deg q_1$ ,  $\deg p_2 < \deg q_2$ , and  $p_0$  is a polynomial.

Now we are going to apply different operators on this partial fraction decomposition. Applying  $\text{CT}_\lambda^\rho$  to  $p(\lambda)/q(\lambda)$  gives us  $p_0(0) + p_1(0)/q_1(0)$ , and applying  $\text{CT}_\lambda^{\bar{\rho}}$  to  $p(\lambda)/q(\lambda)$  gives us  $p_0(0) + p_2(0)/q_2(0)$ . Therefore

$$\text{CT}_\lambda^\rho \frac{p(\lambda)}{q(\lambda)} + \text{CT}_\lambda^{\bar{\rho}} \frac{p(\lambda)}{q(\lambda)} = 2p_0(0) + \frac{p_1(0)}{q_1(0)} + \frac{p_2(0)}{q_2(0)}.$$

Applying  $\text{CT}_{\lambda=0}$  to  $p(\lambda)/q(\lambda)$  gives us  $p_0(0) + p_1(0)/q_1(0) + p_2(0)/q_2(0)$ , and applying  $\text{CT}_{\lambda=\infty}$  to  $p(\lambda)/q(\lambda)$  gives us  $p_0(0)$ . Thus the theorem follows.  $\square$

**Remark 3.6.** In the proof of Theorem 3.5, we see that  $\bar{\rho}$  can be replaced with  $\sigma$  if  $\sigma$  switches the PT and NT properties of  $1/q_1(\lambda)$  and  $1/q_2(\lambda)$  with respect to  $\rho$ .

As an element of  $K[\lambda]$ ,  $q(\lambda)$  can be factored into the product of irreducible polynomials. Let  $q(\lambda) = q_1(\lambda) \cdots q_k(\lambda)$  be such a factorization. By Lemma 3.1  $q(\lambda)$  is  $\rho$ -factorable if and only if every  $1/q_i$  is either  $\text{PT}^\rho$  or  $\text{NT}^\rho$ . When this is the case, the  $\rho$ -factorization can be obtained by collecting similar terms.

Elliott-rational functions are  $\rho$ -factorable for any  $\rho$ . Such a function  $F$  can be written as follows:

$$(3.4) \quad F = \frac{p(\lambda)}{(\lambda^{j_1} - a_1) \cdots (\lambda^{j_n} - a_n)(\lambda^{k_1} - b_1) \cdots (\lambda^{k_m} - b_m)},$$

where  $p(\lambda)$  is a polynomial in  $\lambda$ ,  $j_i$  and  $k_i$  are positive integers,  $m$  and  $n$  are nonnegative integers, and  $a_i$  and  $b_i$  are monomials independent of  $\lambda$ . For a particular  $\rho$ , we require that  $1/(\lambda^{j_i} - a_i)$  is  $\text{NT}^\rho$  in  $\lambda$ , and  $1/(\lambda^{k_i} - b_i)$  is  $\text{PT}^\rho$  in  $\lambda$ . Note that  $a_1$  can be 0. “The method of Elliott” [1, p. 111–114] shows that  $\text{CT}_\lambda^\rho F$  is always Elliott-rational.

A rational function of  $\lambda$  is *proper* in  $\lambda$  if the degree in  $\lambda$  of its numerator is less than that of its denominator.

**Corollary 3.7.** *Let  $F(\lambda)$  be of the form (3.4). If  $F(0) = 0$ , and  $F(\lambda)$  is proper in  $\lambda$ , then for any  $\rho$ , we have a reciprocity formula*

$$\text{CT}_{\lambda}^{\rho} F(\lambda) = -\text{CT}_{\lambda}^{\bar{\rho}} F(\lambda),$$

where both sides are regarded as elements in  $K$ .

More generally, a rational function  $F$  is said to have the  $R$ -property with respect to  $\rho$  if

$$(3.5) \quad \text{CT}_{\Lambda}^{\rho} F = (-1)^d \text{CT}_{\Lambda}^{\bar{\rho}} F.$$

for some integer  $d$ . Here we restrict our interest to the case when  $d$  equals  $r$ , the number of  $\lambda$ 's. We have the following reciprocity formula for Elliott-rational functions.

**Theorem 3.8.** *Let  $F(\Lambda, \mathbf{x})$  be an Elliott-rational function and let  $\leq^{\rho}$  be a total ordering on  $\mathbb{Z}^{n+r}$  that is compatible with its additive group structure. Then*

$$(3.6) \quad \text{CT}_{\Lambda}^{\bar{\rho}} F = (-1)^r \text{CT}_{\Lambda}^{\rho} F + \sum_{i=0}^{r-1} (-1)^i \text{CT}_{\lambda_r, \dots, \lambda_{i+2}}^{\rho} \mathcal{I}_{\lambda_{i+1}} \text{CT}_{\lambda_i, \dots, \lambda_1}^{\bar{\rho}} F,$$

where  $\text{CT}_{\lambda_i, \dots, \lambda_1}^{\rho}$  is the identity operator for  $i = 0$  and similar for  $\text{CT}_{\lambda_r, \dots, \lambda_{i+2}}^{\bar{\rho}}$  when  $i = r - 1$ .

*Proof.* Since we are always taking constant terms in  $\lambda_i$ , we omit the  $\lambda$  for convenience. We compute the following in two different ways.

$$(3.7) \quad \sum_{i=0}^{r-1} (-1)^i \text{CT}_{r, \dots, i+1}^{\bar{\rho}} \text{CT}_{i, \dots, 1}^{\rho} F + (-1)^i \text{CT}_{r, \dots, i+2}^{\bar{\rho}} \text{CT}_{i+1, \dots, 1}^{\rho} F.$$

Using Theorem 3.5, we can rewrite (3.7) as

$$\sum_{i=0}^{r-1} (-1)^i \text{CT}_{r, \dots, i+2}^{\bar{\rho}} \mathcal{I}_{i+1} \text{CT}_{i, \dots, 1}^{\rho} F.$$

On the other hand, most of the terms in (3.7) cancel with each other. The only terms left are given by

$$\text{CT}_{r, \dots, 2, 1}^{\bar{\rho}} F + (-1)^{r-1} \text{CT}_{r, \dots, 2, 1}^{\rho} F.$$

The proposition then follows. □

Theorem 3.8 gives the error term of the reciprocity formula. A different error term representation was given in [5] in terms of cohomology. However, the computation of this error term saved only a little work for general  $r$ . Our formula for the error term is true for any fixed order of  $\lambda_1, \dots, \lambda_r$ , and any fixed order of  $x_1, \dots, x_n$ . This suggests that some simplifications might exist and a better formula is possible. We have not succeeded in finding a such formula.

Since simple equivalent condition for  $F$  to have the R-property is unlikely, we search for a sufficient condition. Corollary 3.9 and Proposition 3.10 below play important roles in our formulating the monster theorem.

A rational function  $F$  is said to have the *I-property* with respect to  $\rho$  if for  $i = 1, 2, \dots, r$ , we have

$$(3.8) \quad \mathcal{I}_{\lambda_i} \text{CT}_{\lambda_{i-1}}^\rho \cdots \text{CT}_{\lambda_1}^\rho F = 0.$$

**Corollary 3.9.** *If an Elliott-rational function has the I-property, then it has the R-property.*

This result is a direct consequence of Theorem 3.8. The special case of this corollary when the Elliott-rational function has a monomial numerator was shown by a complicated computation in [4, Lemma 9.2].

In the case  $r = 1$ , Theorem 3.8 gives the equivalence between the I-property and the R-property. Moreover, we have, as shown below, a nice equivalent condition [4, Proposition 10.3] for the R-property that contains no algebraic expression.

**Proposition 3.10** ([4]). *Let  $\mathcal{E}(\mathbf{x}; b)$  be the crude generating function associated to an LD-system consisting of a single equation  $A\alpha = a_1\alpha_1 + \cdots + a_n\alpha_n = b$ :*

$$\mathcal{E}(\mathbf{x}; b) = \frac{\lambda^{-b}}{(1 - \lambda^{a_1}x_1) \cdots (1 - \lambda^{a_n}x_n)}.$$

*Then the following four conditions are equivalent for any  $\rho$ :*

- (1)  $\mathcal{E}(\mathbf{x}; b)$  has the R-property.
- (2)  $\mathcal{E}(\mathbf{x}; b)$  has the I-property.
- (3)  $\text{CT}_{\lambda=0} \mathcal{E}(\mathbf{x}; b) = 0$  and  $\text{CT}_{\lambda=\infty} \mathcal{E}(\mathbf{x}; b) = 0$ .
- (4) *The following two conditions are both satisfied:*
  - (a) *There does not exist a  $\beta \in \mathbb{Z}$  with  $A\beta = b$  such that  $\beta_e < 0$  if  $a_e > 0$  and  $\beta_e \geq 0$  if  $a_e < 0$ .*
  - (b) *There does not exist a  $\gamma \in \mathbb{Z}$  with  $A\gamma = b$  such that  $\gamma_e \geq 0$  if  $a_e > 0$  and  $\gamma_e < 0$  if  $a_e < 0$ .*

The proof of this proposition, which is not given in full detail here, proceeds by showing that  $\text{CT}_{\lambda=0} \mathcal{E}(\mathbf{x}; b)$  and  $\text{CT}_{\lambda=\infty} \mathcal{E}(\mathbf{x}; b)$  have no common terms when expanded as Laurent series. The reader is referred to [4, Proposition 10.3] for details.

#### 4. THE MONSTER RECIPROCITY THEOREM

Consider an LD-system  $A\alpha = \mathbf{b}$  as in (2.1). The crude generating function  $\mathcal{E}(\Lambda, \mathbf{x}; \mathbf{b})$  is an Elliott-rational function with a monomial numerator. We say such a function has the *matrix form* since we are going to represent it by a matrix. The problem is to find a simple sufficient condition for  $\mathcal{E}(\Lambda, \mathbf{x}; \mathbf{b})$  to have the R-property. A homology version

solution can be found in [5]. The best known result was Stanley's monster reciprocity theorem [4, Theorem 10.2], which says that the LD-system has the R-property if certain linear combinations of its equations have the R-property. We present here a simple approach to this problem.

The central idea of our approach to this problem, as in [4], is to apply Corollary 3.9 and Proposition 3.10. If the following checking procedure returns a true, then  $\mathcal{E}(\Lambda, \mathbf{x}; \mathbf{b})$  has the I-property and hence the R-property with respect to  $\rho$ . Note that the converse of this statement is false.

The checking procedure for  $\mathcal{E}(\Lambda, \mathbf{x}; \mathbf{b})$ :

- (1) Let  $T_1 = \mathcal{E}(\Lambda, \mathbf{x}; \mathbf{b})$ . If  $\mathcal{I}_{\lambda_1} T_1(\Lambda) \neq 0$  then RETURN false.
- (2) Write  $\text{CT}_{\lambda_1} \rho T_1(\Lambda)$  as a sum of matrix forms in an efficient way. For every matrix form  $T_2$ , if  $\mathcal{I}_{\lambda_2} T_2 \neq 0$ , then RETURN false.
- (3) Repeat the above step for every matrix form  $T_2$  with respect to  $\lambda_2$ , and then for every  $T_3$  with respect to  $\lambda_3, \dots$ , until we have checked if  $I_{\lambda_r} T_r(\lambda_r) \neq 0$ . If no false is returned, then RETURN true.

The basic tool in finding these  $T_i$ 's is partial fraction decomposition of rational functions. Using the constant term operators seems neater than using residue operators as in [4].

Our task is to find a simple equivalent condition for the checking procedure to return a true. Such a condition will be our monster reciprocity theorem. In order to do so, we represent a matrix form  $T$  as an augmented matrix. In fact, we can keep track everything by adding a row of monomials in the  $x$ 's on the top and a column of monomials in the  $\lambda$ 's to the left of an LD-system. Therefore, the checking procedure will be done by matrix operations. Note that using matrix operations is one important aspect of the monster reciprocity theorem.

We use the following identification:

$$T = \frac{y_{n+1} \Lambda^{-\mathbf{b}}}{\prod_{i=1}^n (1 - \Lambda^{C_i} y_i)} \equiv \left[ \begin{array}{ccc|c} y_1 & \cdots & y_n & y_{n+1} \\ C_1 & \cdots & C_n & \mathbf{b} \end{array} \right],$$

where  $y_i$  are monomials in  $\mathbf{x}$ , and  $C_i$  are column vectors. It would be clearer if we add  $\lambda_i$  to the left of the  $i$ th row, but this is unnecessary after applying Lemma 2.2 and requiring that the  $i$ th row (with  $i \geq 2$ ) of  $T_s$  is indexed by  $\lambda_{s+i-1}$ .

The row operations we are going to perform will never involve the top row. The column operations, when acting on the first row, are treated as multiplications instead of additions for the obvious reason. We allow fractional entries and fractional powers. Roots of unity might appear, but will not be a trouble.

Three special matrix operations will be useful. We define  $T \leftarrow C \langle i \rangle$  to be the matrix obtained from  $T$  by adding  $-a_{1,j}/a_{1,i}$  times the  $i$ th column to the  $j$ th column for all  $j \neq i$ . This operation is exactly Gaussian column elimination by taking the  $(2, i)$ th

entry of  $T$  as the pivot. Similarly we define the Gaussian row elimination  $T \leftarrow R\langle i \rangle$ . The third operation  $T \leftarrow D\langle i \rangle$  is defined to be the matrix obtained from  $T$  by deleting the second row and the  $i$ -th column.

Combination of the operations will also be used from left to right. For instance,  $T \leftarrow CR\langle i \rangle := T \leftarrow C\langle i \rangle \leftarrow R\langle i \rangle$ . Since row operations commute with column operations, we have  $T \leftarrow CR\langle i \rangle = T \leftarrow RC\langle i \rangle$ . It is easy to verify the following.

$$T \leftarrow CRD\langle i \rangle = T \leftarrow RCD\langle i \rangle = T \leftarrow CD\langle i \rangle.$$

For example, if  $T$  is given by

$$(4.1) \quad T = \frac{\lambda_1^{-b} \lambda_2^{-c}}{(1 - \lambda_1^3 x_1 / \lambda_2)(1 - \lambda_2 x_2 / \lambda_1)(1 - x_3 / \lambda_1^2 \lambda_2)} \equiv \left[ \begin{array}{ccc|c} x_1 & x_2 & x_3 & 1 \\ 3 & -1 & -2 & b \\ -1 & 1 & -1 & c \end{array} \right],$$

then

$$T \leftarrow C\langle 1 \rangle = \left[ \begin{array}{ccc|c} x_1 & x_2 x_1^{\frac{1}{3}} & x_3 x_1^{\frac{2}{3}} & x_1^{-\frac{b}{3}} \\ 3 & 0 & 0 & 0 \\ -1 & \frac{2}{3} & -\frac{5}{3} & c + \frac{b}{3} \end{array} \right], \quad T \leftarrow R\langle 1 \rangle = \left[ \begin{array}{ccc|c} x_1 & x_2 & x_3 & 1 \\ 3 & -1 & -2 & b \\ 0 & \frac{2}{3} & -\frac{5}{3} & c + \frac{b}{3} \end{array} \right],$$

and

$$T \leftarrow CD\langle 1 \rangle = T \leftarrow CRD\langle 1 \rangle = \left[ \begin{array}{cc|c} x_2 x_1^{\frac{1}{3}} & x_3 x_1^{\frac{2}{3}} & x_1^{-\frac{b}{3}} \\ \frac{2}{3} & -\frac{5}{3} & c + \frac{b}{3} \end{array} \right].$$

The above three operations are generalized to sequences of integers. For instance,  $T \leftarrow R\langle i_1, \dots, i_p \rangle$  is the matrix obtained from  $T$  by applying Gaussian row elimination by first taking the  $(2, i_1)$ th entry of  $T$  as the pivot, then taking the  $(3, i_2)$ th entry as the pivot, and so on. However, the elimination cannot go backwards. For instance, we are not allowed to eliminate the nonzero entries in the second row when taking the  $(3, i_2)$ th entry as the pivot.

More precisely, pick out the  $i_1, \dots, i_p$ th columns of  $T_1$ , and rearrange them as follows:

$$T_1(i_1, \dots, i_p) := \left[ \begin{array}{cccc} x_{i_1} & x_{i_2} & \cdots & x_{i_p} \\ a_{1,i_1} & a_{1,i_2} & \cdots & a_{1,i_p} \\ a_{2,i_1} & a_{2,i_2} & \cdots & a_{2,i_p} \\ \vdots & \vdots & \vdots & \vdots \\ a_{r,i_1} & a_{r,i_2} & \cdots & a_{r,i_p} \end{array} \right].$$

If all the pivots encountered are nonzero, then when ignoring the top row

$$(4.2) \quad T_1(i_1, \dots, i_p) \leftarrow R\langle i_1, \dots, i_p \rangle = \begin{bmatrix} x_{i_1} & x_{i_2} & \cdots & x_{i_p} \\ a_{1,i_1} & a_{1,i_2} & \cdots & a_{1,i_p} \\ 0 & a'_{2,i_2} & \cdots & a'_{2,i_p} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a'_{p,i_p} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$$

will be an upper triangular square matrix followed by a zero matrix, and

$$(4.3) \quad T_1(i_1, \dots, i_p) \leftarrow RC\langle i_1, \dots, i_p \rangle = \begin{bmatrix} x_{i_1} & y_{i_2} & \cdots & y_{i_p} \\ a_{1,i_1} & 0 & \cdots & 0 \\ 0 & a'_{2,i_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a'_{p,i_p} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$$

will be a diagonal square matrix followed by a zero matrix. Since the matrix operations we have performed do not change the determinants, the  $a'_{s,i_s}$  can be inductively computed by the formulas  $a'_{1,i_1} = a_{1,i_1}$ , and  $\prod_{j=1}^s a'_{j,i_j} = \det(a_{k,i_l})_{1 \leq k, l \leq s}$ .

We denote by  $\mathcal{M}(z_1, \dots, z_k)$  is a generic monomial in  $z_1, \dots, z_k$  whose exact expression is not needed.

Though we can formulate the monster reciprocity theorem for any  $\rho$ , the result seems nicer if we assume that  $\rho$  satisfies the following condition:

$$\forall y = \mathcal{M}(\mathbf{x}), \quad y \succ^\rho \lambda_s \Rightarrow y\mathcal{M}(\lambda_{s+1}, \lambda_{s+2}, \dots) \succ^\rho \lambda_s, \quad (*)$$

For example, condition  $(*)$  holds for any injective  $\rho$  such that  $\rho(x_i)$  is a monomial in  $x$  and  $\rho(\lambda_i)$  is a monomial in  $\Lambda$ . We will explain two such  $\rho$  in detail in the next section.

**Definition 4.1.** With notation as in (4.3), we define  $(i_1, \dots, i_p)$  of distinct entries ranging from 1 to  $n$  to be a *contribution sequence* of length  $p$  with respect to  $\rho$  if  $y_{i_s}^{\text{sign}(-a'_{s,i_s})} \prec^\rho \lambda_s$  for all  $s$ . The empty sequence is a contribution sequence of length 0.

The name contribution sequence is in correspondence with the ‘‘pole sequence’’ in [4]. The condition in this definition will be replaced with simple ones for two special  $\rho$  in the next section.

**Theorem 4.2** (Generalized Monster Reciprocity Theorem). *Let  $T$  be a matrix form corresponding to an  $r$  by  $n$  matrix of full rank, and let  $\preceq^\rho$  be a total ordering on the group of monomials in  $\Lambda$  and  $\mathbf{x}$  satisfying condition  $(*)$ . If for every contribution sequence  $(i_1, \dots, i_p)$  of  $T$  with  $p < r$ , the second row of  $T \leftarrow RD\langle i_1, \dots, i_p \rangle$  has the  $R$ -property, then  $T$  has the  $R$ -property with respect to  $\rho$ .*

*Proof of Theorem 4.2.* For given  $T$  and  $\preceq^\rho$ . The checking procedure will return a true if and only if every  $T_p$  encountered has the property that  $\mathcal{I}_{\lambda_p} T_p = 0$ , which is the same as the condition that the second row of  $T_p$  has the R-property by Proposition 3.10.

We claim that  $T_p$  must be *similar* to the following form for some contribution sequence  $(i_1, \dots, i_{p-1})$ :

$$(4.4) \quad T_1 \leftarrow CD\langle i_1, \dots, i_{p-1} \rangle = T_1 \leftarrow RCD\langle i_1, \dots, i_{p-1} \rangle.$$

The term similar will be explained later. Assuming  $T_p$  be given by (4.4), we can complete the proof of the theorem as follows. We observe that the  $C$  operations after the  $R$  operations do not affect the  $(p+1)$ st row (and below) of  $T_1$ . See (4.2). Therefore the second row of  $T_p$  is the same as the second row of  $T_1 \leftarrow RD\langle i_1, \dots, i_{p-1} \rangle$ .

We prove the claim by induction on  $p$ . The claim is trivial for  $p = 1$ . Now assume the claim is true for  $p = s$  and we need to show that the claim is true for  $p = s + 1$ .

By choosing appropriate positive integer  $N$  and letting  $\lambda = \lambda_s^{1/N}$ , (note that  $T_{s+1}$  will be independent of the choice of  $N$ ), we can assume that

$$T_s(\lambda_s) = T'_s(\lambda) = \frac{\lambda^{-b} \tilde{y}_{m+1}}{\prod_{i=1}^m (1 - \lambda^{a_i} \tilde{y}_i)} = \left[ \begin{array}{ccc|c} y_1 & \cdots & y_m & y_{m+1} \\ a_1 & \cdots & a_m & b \\ * & \cdots & * & * \end{array} \right],$$

where  $a_k$  and  $b$  are integers,  $\tilde{y}_k = y_k \mathcal{M}(\lambda_{s+1}, \lambda_{s+2}, \dots)$ , and the  $*$ 's are column of integers that we do not care. Dividing the second row of  $T'_s$  by  $N$  will give us the the second row of  $T_s$ . Since we have deleted  $s - 1$  columns,  $m$  equals  $n - s + 1$ .

We observe that  $y_k = x_{k'} \mathcal{M}(x_{i_1}, \dots, x_{i_{s-1}})$  for  $k \leq m$ , where  $k' - k$  equals the number of  $j$ 's such that  $k' > i_j$ . Therefore  $y_1, \dots, y_m$  are independent of each other. It is now straightforward to check that the partial fraction decomposition of  $T'_s(\lambda)$  with respect to  $\lambda$  is:

$$T'_s(\lambda) = L(\lambda) + \sum_{k=1}^m \sum_{j=1}^{|a_k|} \frac{-\zeta_{k,j} \tilde{y}_k^{-1/a_k}}{\lambda - \zeta_{k,j} \tilde{y}_k^{-1/a_k}} \cdot \frac{(\zeta_{k,j} \tilde{y}_k^{-1/a_k})^{-b} \tilde{y}_{m+1}}{a_k \prod_{i=1, i \neq k}^m (1 - (\zeta_{k,j} \tilde{y}_k^{-1/a_k})^{a_i} \tilde{y}_i)},$$

where  $L(\lambda)$  is a Laurent polynomial in  $\lambda$ , and  $\zeta_{k,j}$  ranges over all  $a_k$ th roots of unity.

By Proposition 3.10,  $\mathcal{I}_{\lambda_s} T_s = 0$  implies that  $\text{CT}_{\lambda_s = \infty} T_s = 0$  and hence  $\text{CT}_{\lambda_s}^\rho L(\lambda) = 0$ . Together with the fact that for any  $u$  independent of  $\lambda$ ,

$$\text{CT}_{\lambda_s}^\rho \frac{1}{\lambda - u} = \begin{cases} 0, & \text{if } u \succ^\rho \lambda, \\ (-u)^{-1}, & \text{if } u \prec^\rho \lambda, \end{cases}$$

we have

$$\text{CT}_{\lambda_s}^\rho T_s = \sum_k \sum_{j=1}^{|a_k|} \frac{(\zeta_{k,j} \tilde{y}_k^{-1/a_k})^{-b} \tilde{y}_{m+1}}{a_k \prod_{i=1, i \neq k}^m (1 - (\zeta_{k,j} \tilde{y}_k^{-1/a_k})^{a_i} \tilde{y}_i)},$$

where the sum ranges over all  $k$  such that  $\zeta_{k,j} \tilde{y}_k^{-1/a_k} \prec^\rho \lambda = \lambda_s^{1/N}$ , which, by condition  $(*)$ , is equivalent to  $y_i^{\text{sign}(-a_i)} \prec^\rho \lambda_s$ . For such  $k$ , we can check that  $(i_1, \dots, i_{s-1}, k')$  is a contribution sequence.

Let

$$\mathcal{T}_k(x_{k'}^{-1/a_k}) = \frac{(\tilde{y}_k^{-1/a_k})^{-b} \tilde{y}_{m+1}}{\prod_{i=1, i \neq k}^m (1 - (\tilde{y}_k^{-1/a_k})^{a_i} \tilde{y}_i)} = T_s \leftarrow CD\langle k \rangle,$$

where we emphasize  $\mathcal{T}_k$  as a function of  $x_{k'}^{-1/a_k}$ . By delaying the deletion procedure, we can check that

$$\mathcal{T}_k(x_{k'}^{-1/a_k}) = T_1 \leftarrow CD\langle i_1, \dots, i_{s-1}, k' \rangle.$$

Then  $\text{CT}_{\lambda_s} T_s$  is a sum of  $T_{s+1}$ 's, each have the form

$$\mathcal{T}_k^{(j)} = \frac{1}{a_k} \mathcal{T}_k(\zeta_{k,j} x_{k'}^{-1/a_k})$$

for some  $j$  with  $1 \leq j \leq |a_k|$  and  $k$  with  $(i_1, \dots, i_{s-1}, k')$  being a contribution sequence.

We say that  $\mathcal{T}_k^{(j)}$  is similar to  $\mathcal{T}_k$  and it is clear that  $\mathcal{T}_k^{(j)}$  has the I-property if and only if  $\mathcal{T}_k$  has. This completes the proof of the claim. □

**Remark 4.3.** If  $T$  satisfies the condition in Theorem 4.2, then for  $p \leq r$ , it follows from the proof that  $\text{CT}_{\lambda_1, \dots, \lambda_p}^\rho T$  can be expressed as a sum of group terms indexed by contribution sequences  $(i_1, \dots, i_p)$  of  $T$ , with the corresponding group being a sum of terms similar to  $T_1 \leftarrow \langle i_1, \dots, i_p \rangle$ . In particular,  $\text{CT}_\Lambda^\rho$  can be expressed as at most a sum of  $n(n-1) \cdots (n-r+1)$  groups, since there are at most  $n(n-1) \cdots (n-r+1)$  contribution sequences. A fast way to compute the sum for each group and an effective way to reduce the number of contribution sequences will result in an efficient algorithm for computing  $E(\mathbf{x}, \mathbf{b})$ .

## 5. EXAMPLES AND APPLICATIONS

To apply Theorem 4.2 to a particular  $LD$ -system, we need to choose a working field  $\mathbb{C}^\rho \langle\langle \Lambda, \mathbf{x} \rangle\rangle$  to work with. The choice of  $\rho$  is not unique, but we will concentrate on two special cases that always work. One is the case that  $\rho$  is the identity; the other is equivalent to that in [4]. In both cases, we can simplify the condition in finding the contribution sequence.

Case 1: Let  $\mathbb{C} \langle\langle \Lambda, \mathbf{x} \rangle\rangle$  be the working field. The condition  $y_{i_s}^{\text{sign}(-a'_{s,i_s})} \prec \lambda_s$  in Definition 4.1 can be replaced with  $\text{sign}(a'_{s,i_s} k_l) > 0$ , where if we write  $y_{i_s} = x_1^{k_1} \cdots x_n^{k_n}$ , then  $l$  is the largest such that  $k_l \neq 0$ . In practice, we put the sign of  $k_l$  at the upper front of  $y_j$ .

**Example 5.1.** Let  $(\mathbf{E}, (b, c))$  the following LD-system:

$$\begin{aligned} 3\alpha_1 - \alpha_2 - 2\alpha_3 &= b, \\ -\alpha_1 + \alpha_2 - \alpha_3 &= c. \end{aligned}$$

Then the crude generating function  $T = T_1$  of this LD-system is given by (4.1). Using Maple, we find that  $(\mathbf{E}, (b, c))$  has the R-property for all  $(b, c)$  plotted by  $\bullet$ , and has I-property for all  $(b, c)$  plotted by  $\circ$  in the following Figure 1, where we tested all  $-12 \leq b, c \leq 12$ . Thus the R-property does not implies the I-property.

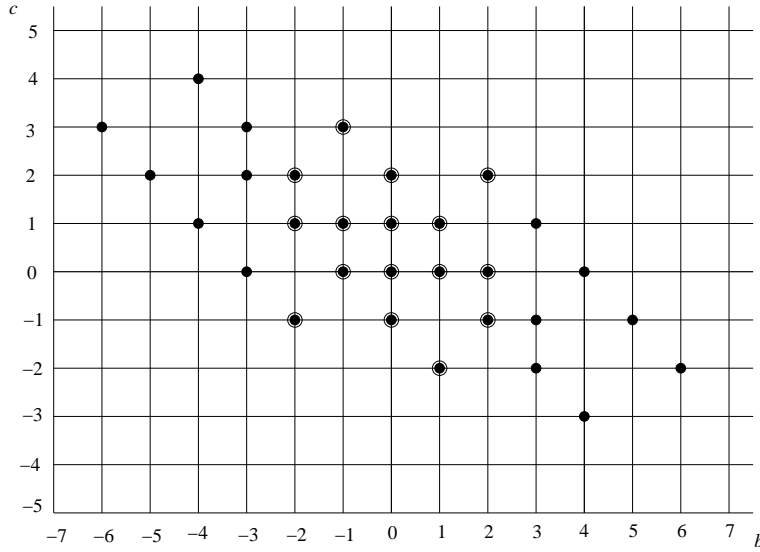


FIGURE 1. The R-property and I-property for  $(\mathbf{E}, (b, c))$ .

Let  $\mathbb{C}\langle\langle \Lambda, \mathbf{x} \rangle\rangle$  be the working field. We want to apply Theorem 4.2 to find such pairs.

Since the second row of  $T_1$  has only one positive entries, only (1) is a contribution sequence of length 1. So after eliminating  $\lambda_1$ , we get a sum of three terms similar to  $T_2$  given by

$$T_2 = T_1 \leftarrow CD\langle 1 \rangle = \left[ \begin{array}{ccc|c} \otimes & +x_2x_1^{\frac{1}{3}} & +x_3x_1^{\frac{2}{3}} & x_1^{-\frac{b}{3}} \\ \otimes & \frac{2}{3} & -\frac{5}{3} & c + \frac{b}{3} \end{array} \right],$$

where we kept the first column to keep track the original column numbers. Now it is easy to see that the only contribution sequence of length 2 is (1, 2), though we do not need it.

Therefore, Theorem 4.2 tells us that the LD-system has the R-property if the following two equations have the R-property:

$$\begin{aligned} 3\alpha_1 - \alpha_2 - 2\alpha_3 &= b, \\ 2\alpha_2 - 5\alpha_3 &= 3c + b. \end{aligned}$$

Using Maple, we find all such  $(b, c)$  as plotted by  $\circ$  in Figure 1:

**Example 5.2.** Consider the equivalent LD-system  $(\mathbf{E}'(-c, b))$ :

$$\begin{aligned}\alpha_1 - \alpha_2 + \alpha_3 &= -c, \\ 3\alpha_1 - \alpha_2 - 2\alpha_3 &= b,\end{aligned}$$

where we multiplied both sides of the second equation by  $-1$  and switched the two equations.

We need to find  $(b, c)$  for  $S$  to have the R-property, where

$$S = \left[ \begin{array}{ccc|c} x_1 & x_2 & x_3 & 1 \\ 1 & -1 & 1 & -c \\ 3 & -1 & -2 & b \end{array} \right].$$

This time we have two contribution sequences of length 1: (1) and (3). Therefore, Theorem 4.2 tells us that the LD-system has the R-property if the following three equations have the R-property, where the second and third equation are from  $S \leftarrow \langle 1 \rangle$  and  $S \leftarrow \langle 3 \rangle$ :

$$\begin{aligned}\alpha_1 & -\alpha_2 & +\alpha_3 & = -c, \\ & 2\alpha_2 & -5\alpha_3 & = b + 3c, \\ 5\alpha_1 & -3\alpha_2 & & = b - 2c.\end{aligned}$$

Using Maple, we obtain the same pairs  $(b, c)$  as in the previous one, i.e., those plotted by  $\circ$  in Figure 1. All these three equations are needed to apply Theorem 4.2. The following coincidence is worth mentioning: if we only consider the second and the third equation, we will get all  $(b, c)$  plotted by  $\bullet$  in Figure 1, i.e., those  $(b, c)$  such that  $(\mathbf{E}', (b, c))$  has the R-property. Since the first equation comes from the empty contribution sequence, we come back to check the previous example, which is obviously not the case.

Case 2: Let  $\rho$  be the injective homomorphism into  $\mathbb{C}\langle\langle \mathbf{x}, \Lambda, t \rangle\rangle$  by  $\rho(x_i) = x_i t$  and  $\rho(\lambda_i) = \lambda_{r-i+1}$ . Then the condition in Definition 4.1 can be replaced with  $\text{sign}(da'_{s,i_s}) > 0$ , where  $d$  is the total degree of  $y_{i_s}$  in the  $x$ 's. Since we only need to keep track of the total degree of the  $x$ 's, the  $x_i$  in the top row of  $T$  can be replaced with 1. The monster reciprocity theorem obtained this way is similar to that of [4], in which the computation used integration along the circles  $|\lambda_i| = 1 - \epsilon_i$  with  $1 \gg \epsilon_1 \gg \epsilon_2 \gg \dots$ , where  $\gg$  means "much greater", and the  $x_i$  is taken to satisfy  $|x_i| = \delta < 1$  for some positive real number  $\delta$ . In fact, the condition as in Definition 4.1 was completely written in terms of determinants.

Detailed example for this case, which will not be given here, can be found in [4, p. 245].

Now let us consider Linear homogeneous Diophantine system (LHD-system for short). We shall use Theorem 4.2 to derive the following theorem, which implies the reciprocal domain theorem [4, Proposition 8.3] including Theorem 1.1.

**Theorem 5.3.** *Suppose  $T$  is a matrix form corresponding to an LHD-system of full rank. Then for any  $\rho$  satisfying  $(*)$ ,  $T$  has the R-property if and only if either  $\text{CT}_\Lambda^\rho T = \text{CT}_\Lambda^{\bar{\rho}} T = 0$  or  $\text{CT}_\Lambda^\rho T \neq 0$  and  $\text{CT}_\Lambda^{\bar{\rho}} T \neq 0$ .*

If we let  $\rho$  be the identity map, then we get Theorem 1.1. If we let  $\rho(x_i) = x_i$  for  $i = 1, \dots, p$  and  $\rho(x_i) = x_i^{-1}$  for  $i = p + 1, \dots, n$  for  $p$  with  $1 < p < n$ , and  $\rho(\lambda_i) = \lambda_i$  for all  $i$ , then we will get the reciprocal domain theorem [4, Proposition 8.3].

*Proof of Theorem 5.3.* If  $T$  has the R-property, then

$$\text{CT}_\Lambda^\rho T = (-1)^r \text{CT}_\Lambda^{\bar{\rho}} T.$$

The implication thus follows. Now we show the converse is true.

Obviously we can suppose  $r > 0$ ,  $\text{CT}_\Lambda^\rho T \neq 0$  and  $\text{CT}_\Lambda^{\bar{\rho}} T \neq 0$ . We first show that the second row of  $T$  has the R-property. Since  $T$  corresponds to an LHD-system, we can write

$$T = \frac{1}{\prod_{i=1}^n (1 - \tilde{y}_i \lambda_1^{a_i})},$$

where  $\tilde{y}_i$  is a monomial independent of  $\lambda_1$ . If some of the  $a_i$  are positive and some of the  $a_i$  are negative, then Corollary 3.7 applies and the second row of  $T$  has the R-property. Otherwise, one of  $\text{CT}_{\lambda_1=0} T$  and  $\text{CT}_{\lambda_1=\infty} T$  will be 0 and the other will be nonzero. (Note that since the LHD-system has full rank, the case that  $a_i = 0$  for all  $i$  will not happen.) The statement then follows by Proposition 3.10.

Now by Lemma 2.2, if  $T'$  is obtained from  $T$  by elementary row operations, then  $\text{CT}_\Lambda^\rho T' \neq 0$  and  $\text{CT}_\Lambda^{\bar{\rho}} T' \neq 0$ . Therefore, the second row of  $T'$  has the R-property. This means every linear combination of the equations of  $T$  has the R-property. Thus the theorem follows from Theorem 4.2.  $\square$

**Remark 5.4.** The proof of the theorem, together with Remark 4.3, in fact shows the following statement: If  $\text{CT}_\Lambda^\rho T \neq 0$  and  $\text{CT}_\Lambda^{\bar{\rho}} T \neq 0$ , then  $\text{CT}_{\lambda_1, \dots, \lambda_p}^\rho T$  is proper in all  $\lambda_i$  for  $i > p$ . On the other hand, a simple proof of the statement will lead to a simple proof of Theorem 1.1. If we restrict ourself in  $\mathbb{C}[\Lambda, \Lambda^{-1}][[\mathbf{x}]]$ , the best known proof of Theorem 1.1 should be that given by the author in [9], which is included in the next section.

The above remark suggest a way to reduce the number of contribution sequences of an LHD-system: Following the notation as in Remark 4.3, since every  $T_p$  has the R-property for  $\lambda_p$ , we have a choice to choose all those terms with contribution or all those terms (with a minus sign) without contribution. The author is managing to develop a computer program implementing these techniques.

6. LINEAR HOMOGENEOUS DIOPHANTINE SYSTEMS

We are concentrating on linear homogeneous Diophantine systems (LHD-systems for short), i.e.,  $A\alpha = \mathbf{0}$ . Recall that  $C_i$  is the  $i$ th column vector of  $A$ . We omit the  $\mathbf{0}$  so that  $E$  and  $\bar{E}$  are the sets of all solutions of  $A\alpha = \mathbf{0}$  in  $\mathbb{N}^n$  and  $\mathbb{P}^n$  respectively, and similar for other notations. Since the proof closely related the linear system and its associate generating functions, we restate them as follows.

$$(6.1) \quad E(\mathbf{x}) = \text{CT}_{\Lambda} \mathcal{E}(\mathbf{x}), \quad \bar{E}(\mathbf{x}) = (-1)^n \text{CT}_{\Lambda} \mathcal{E}(\mathbf{x}^{-1}).$$

We are going to prove Proposition 2.1, i.e., to show that if  $\bar{E}$  is nonempty, then

$$(6.2) \quad \text{CT}_{\Lambda} \mathcal{E}(\mathbf{x}) = (-1)^{\text{rank}(A)} \text{CT}_{\Lambda} \rho \mathcal{E}(\mathbf{x}),$$

where we are taking constant term of MN-series and  $\rho(x_i) = x_i^{-1}$  for all  $i$ .

We shall see that all of the work is done algebraically. First, let us see some facts. Exchanging column  $i$  and  $j$  corresponds to exchanging  $x_i$  and  $x_j$ . Row operations, which will not change the solutions of  $A\alpha = 0$ , are equivalent to multiplying  $A$  on the left by an invertible matrix. This fact can be obtained by applying Lemma 2.2.

Let us see the simple case of  $r = 1$ . In this case,  $\mathcal{E}(\mathbf{x})$  has the form:

$$\mathcal{E}(\mathbf{x}) = \prod_{i=1}^n \frac{1}{1 - \lambda^{a_i} x_i}.$$

The condition that  $\bar{E}$  is nonempty is equivalent to saying that some of  $a_i$  have to be positive and some of  $a_i$  have to be negative. Thus when written in the normal form of a rational function in  $\lambda$ ,  $\mathcal{E}(\mathbf{x})$  is proper and its numerator divides  $\lambda$ . So Proposition 2.1 follows from Corollary 3.7.

The general case does not seem to work along this line because of two problems. One is how to use the conditions that  $\bar{E}$  is nonempty, and the other is how to connect to the rank of  $A$ . The proof we are going to give uses induction and Elliott's reduction identity [1, p. 111–114], which is easy to check and is not given here.

Clearly if  $a_{11}, \dots, a_{1,n}$  are all positive or are all negative, then  $\bar{E}$  is empty. So we can assume that  $a_{11} > 0$  and  $a_{12} < 0$ . Applying Elliott's reduction identity on  $\lambda_1$ , we get:

$$\mathcal{E}(\mathbf{x}) = \frac{1}{1 - \Lambda^{C_1+C_2} x_1 x_2} \left( \frac{1}{1 - \Lambda^{C_1} x_1} + \frac{1}{1 - \Lambda^{C_2} x_2} - 1 \right) \prod_{i \geq 3} \frac{1}{1 - \Lambda^{C_i} x_i}$$

Now expand  $\mathcal{E}(\mathbf{x})$  according to the middle term, and denote the resulting three summands by  $\mathcal{E}_1$ ,  $\mathcal{E}_2$ , and  $\mathcal{E}_3$  respectively. We have

$$(6.3) \quad \mathcal{E}(\mathbf{x}) = \mathcal{E}_1(x_1, x_1 x_2, x_3, \dots) + \mathcal{E}_2(x_1 x_2, x_2, x_3, \dots) - \mathcal{E}_3(x_1 x_2, x_3, \dots).$$

Then these  $\mathcal{E}_i$  are very similar to  $\mathcal{E}$ . Correspondingly, they are associated to matrices, and hence solution spaces that lie in  $\mathbb{N}^n$  and  $\mathbb{P}^n$ . More precisely,  $\mathcal{E}_i$ ,  $i = 1, 2, 3$ , are associated to  $A_1 = (C_1, C_1 + C_2, C_3, \dots, C_n)$ ,  $A_2 = (C_1 + C_2, C_2, C_3, \dots, C_n)$ , and  $A_3 = (C_1 + C_2, C_3, \dots, C_n)$  respectively. Thus  $E_i, E_i(\mathbf{x})$  and  $\bar{E}_i, \bar{E}_i(\mathbf{x})$  are defined correspondingly.

Now the matrix  $A_1$  is obtained from  $A$  by adding the second column to the first; the matrix  $A_2$  is obtained from  $A$  by adding the first column to the second. They are obtained from  $A$  through a column operation. So the rank of  $A_1$  and  $A_2$  are both equal to that of  $A$ . The rank of  $A_3$  might not equal the rank of  $A$ .

Applying  $\text{CT}_\Lambda$  and  $(-1)^n \text{CT}_\Lambda^\rho$  to (6.3) respectively, we get our key induction equations.

$$(6.4) \quad \begin{aligned} E(\mathbf{x}) &= E_1(x_1, x_1x_2, x_3, \dots) + E_2(x_1x_2, x_2, x_3, \dots) - E_3(x_1x_2, x_3, \dots), \\ \bar{E}(\mathbf{x}) &= \bar{E}_1(x_1, x_1x_2, x_3, \dots) + \bar{E}_2(x_1x_2, x_2, x_3, \dots) \\ (6.5) \quad &\quad + (-1)^{\text{rank}(A) - \text{rank}(A_3)} \bar{E}_3(x_1x_2, x_3, \dots). \end{aligned}$$

Looking more closely at these  $E_i$ , we can see that up to isomorphism,  $E_1$ ,  $E_2$ , and  $E_3$  are obtained from  $E$  by intersecting the half spaces  $\alpha_1 \geq \alpha_2$ ,  $\alpha_1 \leq \alpha_2$ , and the hyperplane  $\alpha_1 = \alpha_2$  respectively. For instance,  $(\alpha_1, \alpha_2, \dots)$  belongs to  $E$  with  $\alpha_1 \geq \alpha_2$  if and only if  $(\alpha_1 - \alpha_2, \alpha_2, \dots)$  belongs to  $E_1$ . Thus Elliott's reduction identity in fact corresponds to a signed decomposition of  $E$ . Equation (6.4) and (6.5) could be explained directly from geometry.

We need two more lemmas to give our proof of Proposition 2.1. We shall see that the condition on  $\bar{E}$  plays an important role.

If  $\bar{E}$  is nonempty, then  $\dim E = \dim \bar{E} = n - \text{rank}(A)$ . Clearly, the dimension of the solution space of  $A\alpha = 0$  is  $n - \text{rank}(A)$ . Let  $\gamma \in \bar{E}$ , and let  $\Upsilon_1, \dots, \Upsilon_{n - \text{rank}(A)}$  be a  $\mathbb{Z}$ -basis of the solution space in  $\mathbb{Z}^n$  with  $\Upsilon_1 = \gamma$ . Then for sufficiently large  $m$ ,  $m\gamma + \Upsilon_1, \dots, m\gamma + \Upsilon_{n - \text{rank}(A)}$  will be a linearly independent set in  $\bar{E}$ .

**Lemma 6.1.** *Suppose that  $\bar{E}$  is nonempty, and that  $\bar{E}_i$  is defined as above for  $i = 1, 2, 3$ . Then any two of the  $\bar{E}_i$  being nonempty implies that they are all nonempty.*

*Proof.* Suppose that  $\bar{E}_1$  and  $\bar{E}_2$  are nonempty. Then we have elements  $\beta$  and  $\gamma$  in  $\bar{E}$  such that  $\beta = (\beta_1, \beta_2, \dots)$  with  $\beta_1 > \beta_2$  and  $\gamma = (\gamma_1, \gamma_2, \dots)$  with  $\gamma_1 < \gamma_2$ . Then  $(\gamma_2 - \gamma_1)\beta + (\beta_1 - \beta_2)\gamma$  is in  $\bar{E}$  with the first two entries being equal. This means  $\bar{E}_3$  is nonempty.

Suppose that  $\bar{E}_1$  and  $\bar{E}_3$  are nonempty. Then we have elements  $\beta$  and  $\delta$  in  $\bar{E}$  such that  $\beta = (\beta_1, \beta_2, \dots)$  with  $\beta_1 > \beta_2$  and  $\delta = (\delta_1, \delta_2, \dots)$  with  $\delta_1 = \delta_2$ . Then for sufficiently large  $m$ ,  $m\delta - \beta$  is in  $\bar{E}$  with the first entry being smaller than the second. This means  $\bar{E}_2$  is nonempty.

The case that  $\bar{E}_2$  and  $\bar{E}_3$  are nonempty is similar to the previous case.  $\square$

**Lemma 6.2.** *If all of the  $\bar{E}_i$  are nonempty, then  $\text{rank}(A_3) = \text{rank}(A)$ .*

*Proof.* By hypothesis, it is clear that  $E$  is not contained in the hyperplane  $\alpha_1 = \alpha_2$ . Thus the intersection of  $E$  with the hyperplane has dimension  $\dim E - 1$ . So  $\dim E_3$  is also  $\dim E - 1$  and the rank of  $A_3$  equals  $n - 1 - \dim E_3 = \text{rank}(A)$ .  $\square$

*Proof of Proposition 2.1.* The base case, when  $A$  is the zero matrix, is trivial.

By exchanging rows, we can assume that not all of the entries in the first row of  $A$  are zero. Moreover, since the entries can not be all positive or negative, we can assume the first entry is positive and the second is negative by exchanging columns.

We use induction on  $S_1(A)$ , which is defined to be the sum of the absolute values of all the entries in the first row. Now the above argument applies, and it is easy to see that  $S_1(A_i) < S_1(A)$  for  $i = 1, 2, 3$ . Applying Lemma 6.1, we can reduce the seven cases of  $E_i$  being nonempty or not into the following four cases:

Case 1: only  $\bar{E}_1$  is nonempty. Let  $\beta$  in  $\bar{E}$  be such that  $\beta_1 > \beta_2$ . We claim that all  $\alpha$  with  $A\alpha = 0$  satisfy the condition  $\alpha_1 > \alpha_2$ , so that  $E_2(x_1x_2, x_2, x_3, \dots)$  equals  $E_3(x_1x_2, x_3, \dots)$ , and hence by induction we have

$$\begin{aligned} E(\mathbf{x}) &= E_1(x_1, x_1x_2, x_3, \dots) \\ &= (-1)^{\text{rank}(n-A_1)} \bar{E}_1(x_1^{-1}, x_1^{-1}x_2^{-1}, x_3^{-1}, \dots) = (-1)^{n-\text{rank}(A)} \bar{E}(\mathbf{x}^{-1}). \end{aligned}$$

If the claim does not hold, then  $\alpha_1 \leq \alpha_2$ . But for sufficiently large  $m$ ,  $m\beta - \alpha$  will produce an element in  $\bar{E}_2$  or  $\bar{E}_3$ , a contradiction.

Case 2: only  $\bar{E}_2$  is nonempty. This is similar to case 1.

Case 3: only  $\bar{E}_3$  is nonempty. This means that  $E$  is contained in the hyperplane  $\alpha_1 = \alpha_2$ . Thus

$$E_1(x_1, x_1x_2, x_3, \dots) = E_2(x_1x_2, x_2, x_3, \dots) = E_3(x_1x_2, x_3, \dots),$$

and we have

$$\text{rank}(A_3) = n - 1 - \dim(E_3) = n - \dim(E) - 1 = \text{rank}(A) - 1.$$

So

$$\begin{aligned} E(\mathbf{x}) &= E_3(x_1x_2, x_3, \dots) \\ &= (-1)^{n-1-\text{rank}(A_3)} \bar{E}_3(x_1^{-1}x_2^{-1}, x_3^{-1}, \dots) = (-1)^{n-\text{rank}(A)} \bar{E}(\mathbf{x}^{-1}). \end{aligned}$$

Case 4: all of  $\bar{E}_i$  are nonempty. By induction, we see that

$$E_i(\mathbf{x}) = (-1)^{n-\text{rank}(A_i)} \bar{E}_i(\mathbf{x}^{-1})$$

for  $i = 1, 2$ , and that

$$E_3(x_2, x_3, \dots) = (-1)^{n-1-\text{rank} A_3} \bar{E}(x_2^{-1}, x_3^{-1}, \dots).$$

From Lemma 6.2,  $\text{rank}(A_3) = \text{rank}(A)$ . Thus together with our key induction equations (6.4) and (6.5), we get

$$\begin{aligned} E(\mathbf{x}) &= E_1(x_1, x_1x_2, x_3, \dots) + E_2(x_1x_2, x_2, x_3, \dots) - E_3(x_1x_2, x_3, \dots) \\ &= (-1)^{n-\text{rank}(A)} (\bar{E}_1(x_1^{-1}, x_1^{-1}x_2^{-1}, x_3^{-1}, \dots) \\ &\quad + \bar{E}_2(x_1^{-1}x_2^{-1}, x_2^{-1}, x_3^{-1}, \dots) + \bar{E}_3(x_1^{-1}x_2^{-1}, x_3^{-1}, \dots)) \\ &= (-1)^{n-\text{rank } A} \bar{E}(\mathbf{x}). \end{aligned}$$

□

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