

PROOF OF A CONJECTURE ON THE SLIT PLANE PROBLEM

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ABSTRACT. Let $a_{i,j}(n)$ denote the number of walks in n steps from $(0,0)$ to (i,j) , with steps $(\pm 1,0)$ and $(0,\pm 1)$, never touching a point $(-k,0)$ with $k \geq 0$ after the starting point. Bousquet-Mélou and Schaeffer conjectured a closed form for the number $a_{-i,i}(2n)$ when $i \geq 1$. In this paper, we prove their conjecture, and give a formula for $a_{-i,i}(2n)$ for $i \leq -1$.

Keywords: Walks on the slit plane.

1. INTRODUCTION AND THEOREMS

The problem of walks on the slit plane was first studied by M. Bousquet-Mélou and G. Schaeffer in [1]. See also [2].

Let $a_{i,j}(n)$ denote the number of walks in n steps from $(0,0)$ to (i,j) , with steps $(\pm 1,0)$ and $(0,\pm 1)$, never touching a point $(-k,0)$ with $k \geq 0$ after the starting point. These are called *walks on the slit plane*.

Let \bar{x} denote x^{-1} and \bar{y} denote y^{-1} . In [1], the authors showed (Theorem 1) that

$$S(x, y; t) = \sum_{n \geq 0} \sum_{i, j \in \mathbb{Z}} a_{i,j}(n) x^i y^j t^n = \frac{(1 - 2t(1 + \bar{x}) + \sqrt{1 - 4t})^{1/2} (1 + 2t(1 - \bar{x}) + \sqrt{1 + 4t})^{1/2}}{2(1 - t(x + \bar{x} + y + \bar{y}))}, \quad (1.1)$$

where $S(x, y; t)$ is the complete generating function for walks on the slit plane.

The authors also conjectured a closed form for $a_{-i,i}(2n)$ for $i \geq 1$. By reflecting in the x -axis, we see that $a_{-i,i}(2n) = a_{-i,-i}(2n)$, the closed form of which is given as (1.2) in the following theorem.

Theorem 1.1. *For $i \geq 1$ and $n \geq i$, we have*

$$a_{-i,-i}(2n) = \frac{i}{2n} \binom{2i}{i} \binom{n+i}{2i} \frac{\binom{4n}{2n}}{\binom{2n+2i}{2i}}, \quad (1.2)$$

$$a_{i,i}(2n) = a_{-i,-i} + 4^n \frac{i}{n} \binom{2i}{i} \binom{2n}{n-i}. \quad (1.3)$$

We will prove this theorem in the next section. Theorem 1.4 below is a basic tool to prove the conjecture.

There are two key steps in proving the conjecture that might be worth mentioning: one is using Theorem 1.4 to obtain the generating function (2.2) that involves $a_{i,i}(2n)$ for all integers i ; the other is guessing the formula (1.3).

Let R be a commutative ring with unit, and $R[x, \bar{x}][[t]]$ the ring of formal power series in t with coefficients Laurent polynomials in x . An element of $R[x, \bar{x}][[t]]$ is written as $f(x; t)$, to emphasize that $f(x; t)$ is regarded as a power series in t .

If $f(x; t) \in R[x, \bar{x}][[t]]$, then it can be written as

$$f(x; t) = \sum_{n \geq 0} \sum_{i \in \mathbb{Z}} f_{i,n} x^i t^n.$$

Let $\text{CT}_x f(x; t)$ denote the constant term of $f(x; t)$ in x , i.e.,

$$\text{CT}_x f(x; t) = \sum_{n \geq 0} f_{0,n} t^n.$$

Note that if we are working in $R[x, \bar{x}][[t]]$, and if $u \in tR[[t]]$, then $\frac{x}{x-u}$ has to be interpreted as

$$\frac{x}{x-u} = \frac{1}{1-u/x} = \sum_{n \geq 0} u^n / x^n.$$

So

$$\text{CT}_x \frac{x}{x-u} x^k = \begin{cases} u^k & \text{if } k \geq 0, \\ 0 & \text{if } k < 0. \end{cases}$$

By linearity, we have the following:

Lemma 1.2. *Let $Q(x; t) \in R[x][[t]]$ be a formal power series in t , with coefficients in the polynomial ring $R[x]$. If $u = u(t) \in tR[[t]]$ is a formal power series in t with constant term 0, then*

$$\text{CT}_x \frac{x}{x-u} Q(x; t) = Q(u; t). \quad (1.4)$$

The following lemma is a well-known result. See, e.g., [3, Theorem 4.2].

Lemma 1.3. *If $G(x, t) \in R[[x, t]]$, then there is a unique $X = X(t)$ in $tR[[t]]$ such that $X - tG(X, t) = 0$.*

Theorem 1.4. *Let $G(x, t), F(x, t) \in R[[x, t]]$, and let $X = X(t)$ be the unique element in $tR[[t]]$ such that $X - tG(X, t) = 0$. Then*

$$\text{CT}_x \frac{x}{x-tG(x, t)} F(x, t) = \left. \frac{F(X, t)}{1 - t \frac{\partial}{\partial x} G(x, t)} \right|_{x=X}. \quad (1.5)$$

Proof. Write $G(x, t) = \sum_{n \geq 0} a_n(t) x^n$. Then

$$\frac{x - tG(x, t)}{x - X} = \frac{x - tG(x, t) - (X - tG(X, t))}{x - X} = 1 - t \sum_{n \geq 0} a_n(t) (x^{n-1} + x^{n-2}X + \cdots + X^{n-1}),$$

which is an element in $R[[x, t]]$ with constant term 1. Setting $x = X$, we get

$$\left. \frac{x - tG(x, t)}{x - X} \right|_{x=X} = 1 - t \left. \frac{\partial}{\partial x} G(x, t) \right|_{x=X}.$$

By Lemma 1.2, we have

$$\begin{aligned} \text{CT}_x \frac{x}{x-tG(x, t)} F(x, t) &= \text{CT}_x \frac{x}{x-X} \left(\frac{x-tG(x, t)}{x-X} \right)^{-1} F(x, t) \\ &= \left(\frac{x-tG(x, t)}{x-X} \right)^{-1} F(x, t) \Big|_{x=X} \\ &= \left. \frac{F(X, t)}{1 - t \frac{\partial}{\partial x} G(x, t)} \right|_{x=X}. \end{aligned}$$

■

Theorem 1.4 is a generalization of Lagrange's inversion formula. If we set F and G to be independent of t , we can easily derive Lagrange's inversion formula. See [4, Theorem 5.4.2]. This topic will be explored further in [5].

2. THE PROOF OF THE CONJECTURE

Let

$$C(t) = \sum_{n \geq 0} C_n t^n = \frac{1 - \sqrt{1 - 4t}}{2t}$$

be the Catalan generating function, and let

$$u = tC(t)C(-t) = \frac{\sqrt{1 + 4t} - 1}{\sqrt{1 - 4t} + 1}.$$

Much of the computation here involves rational functions of u . We shall use the following facts from [1].

$$\begin{aligned} \mathbb{C}(u) &= \mathbb{C}(t, \sqrt{1 - 4t}, \sqrt{1 + 4t}), \\ C(t) &= \frac{1 + u^2}{1 - u}, \quad C(-t) = \frac{1 + u^2}{1 + u}, \quad C(4t^2) = \frac{(1 + u^2)^2}{(1 - u^2)^2}. \end{aligned}$$

We shall prove Theorem 1.1 by computing the diagonal generating function $F(y; t)$. More precisely, let

$$F(y; t) = \sum_{n \geq 0} \sum_{i \in \mathbb{Z}} a_{i,i} (2n) y^i t^{2n}.$$

Since $S(x, y\bar{x}; t)$ belongs to $\mathbb{C}[x, y, \bar{x}, \bar{y}][[t]]$, it is easy to check that

$$F(y; t) = \text{CT}_x S(x, y\bar{x}; t) = \text{CT}_x S(\bar{x}, xy; t). \quad (2.1)$$

Lemma 2.1.

$$F(y; t) = \frac{\left[\frac{1 + \sqrt{1 - 4t}}{2} - t \left(1 + \frac{1 - \sqrt{1 - 4t^2(1+y)^2/y}}{2t(1+y)} \right) \right]^{1/2} \left[\frac{1 + \sqrt{1 + 4t}}{2} + t \left(1 - \frac{1 - \sqrt{1 - 4t^2(1+y)^2/y}}{2t(1+y)} \right) \right]^{1/2}}{\sqrt{1 - 4t^2(1+y)^2/y}}. \quad (2.2)$$

Proof. Using (2.1) and (1.1), we get

$$\begin{aligned} F(y; t) &= \text{CT}_x S(\bar{x}, xy; t) \\ &= \text{CT}_x \frac{1}{2(1 - t(x + \bar{x} + xy + \bar{x}\bar{y}))} (1 - 2t(1 + x) + \sqrt{1 - 4t})^{1/2} (1 + 2t(1 - x) + \sqrt{1 + 4t})^{1/2} \\ &= \text{CT}_x \frac{x}{2(x - t(x^2 + 1 + x^2y + \bar{y}))} (1 - 2t(1 + x) + \sqrt{1 - 4t})^{1/2} (1 + 2t(1 - x) + \sqrt{1 + 4t})^{1/2}. \end{aligned}$$

Applying Theorem 1.4 with $R = \mathbb{C}[y, \bar{y}]$, this becomes

$$\frac{1}{2(1 - t(2X + 2Xy))} (1 - 2t(1 + X) + \sqrt{1 - 4t})^{1/2} (1 + 2t(1 - X) + \sqrt{1 + 4t})^{1/2},$$

where $X = X(t)$ is the unique solution in $tR[[t]]$ such that $X = t(X^2 + 1 + \bar{y} + X^2y)$. We can solve for X by the quadratic formula:

$$X = \frac{1 - \sqrt{1 - 4t^2(1+y)^2/y}}{2t(1+y)}.$$

Equation (2.2) then follows. ■

It is clear that for any $G(y; t) \in R[y, \bar{y}][[t]]$, there is a unique decomposition $G(y; t) = G_+(y; t) + G_0(t) + G_-(\bar{y}; t)$, such that $G_+(y; t), G_-(\bar{y}; t) \in yR[y][[t]]$ and $G_0(t) \in R[[t]]$.

Our task now is to find this decomposition of $F(y; t)$. There is no general theory to do this. For this particular $F(y; t)$, thanks to the work of Bousquet-Mélou and Schaeffer, we can guess the formulas for F_+ and F_- and prove them.

The variable s defined by the following is useful:

$$s = tC(4t^2) = \frac{u}{1-u^2} \text{ and } t = \frac{s}{1+4s^2}. \quad (2.3)$$

Note that s is also $S_{0,1}(t)$, the generating function of walks on the slit plane that end at $(0, 1)$. See [1, p. 11].

Lemma 2.2. *We have the decomposition*

$$F(y; t) = F_+(y, t) + 1 + F_-(\bar{y}, t),$$

where

$$F_+(y, t) = F_-(y, t) + \frac{1}{2}((1 - 4s^2y)^{-1/2} - 1), \quad (2.4)$$

$$F_-(y, t) = \frac{(1-u^2)s^2yC(s^2y)}{1+u^2C^2(s^2y)s^2y} \frac{1}{\sqrt{1-4s^2y}}. \quad (2.5)$$

Proof. Let

$$T(y; t) = \frac{(1-u^2)s^2yC(s^2y)}{1+u^2C^2(s^2y)s^2y} \frac{1}{\sqrt{1-4s^2y}} + \frac{1}{2}((1-4s^2y)^{-1/2} - 1) + 1 + \frac{(1-u^2)s^2\bar{y}C(s^2\bar{y})}{1+u^2C^2(s^2\bar{y})s^2\bar{y}} \frac{1}{\sqrt{1-4s^2\bar{y}}}.$$

From Lemma 2.1, we have

$$F(y; t) = \frac{\left[\frac{1+\sqrt{1-4t}}{2} - t \left(1 + \frac{1-\sqrt{1-4t^2(1+y)^2/y}}{2t(1+y)} \right) \right]^{1/2} \left[\frac{1+\sqrt{1+4t}}{2} + t \left(1 - \frac{1-\sqrt{1-4t^2(1+y)^2/y}}{2t(1+y)} \right) \right]^{1/2}}{\sqrt{1-4t^2(1+y)^2/y}}.$$

Therefore, it suffices to show that $T(y; t) = F(y; t)$. Since it is easy to see that $T(y; 0) = F(y; 0) = 1$, the proof will be completed by showing that $T^2(y; t) - F^2(y; t) = 0$.

Using the variable u , we can get rid of the radicals $\sqrt{1-4t}$ and $\sqrt{1+4t}$ by the following:

$$\sqrt{1-4t} = \frac{1-2u-u^2}{1+u^2}, \text{ and } \sqrt{1+4t} = \frac{1+2u-u^2}{1+u^2}.$$

The radicals left are $D = \sqrt{1-4s^2y}$, $E = \sqrt{1-4s^2\bar{y}}$, and $\sqrt{1-4t^2(1+y)^2/y}$, which is easily checked to be equal to DE .

Rewriting $T^2 - F^2$ in terms of u, D, E , we get a rational function of u, D, E . For $i = 1, 2$ (the degrees in D and E are both 4), replacing D^{2i} by $(1-4s^2y)^i$, D^{2i+1} by $(1-4s^2y)^i D$, E^{2i} by $(1-4s^2\bar{y})^i$, and E^{2i+1} by $(1-4s^2\bar{y})^i E$, we find that the expression reduces to 0. \blacksquare

Now we need to show the following.

Lemma 2.3.

$$F_-(y, t) = \sum_{n \geq 0} \sum_{i \geq 1} b_i(2n) t^n y^i, \quad (2.6)$$

where

$$b_i(2n) = \frac{i}{2n} \binom{2i}{i} \binom{n+i}{2i} \frac{\binom{4n}{2n}}{\binom{2n+2i}{2i}}. \quad (2.7)$$

We will give two proofs of this lemma. The first one starts from a formula in [1]. Since the proof of the formula is not given in [1], the first proof is not complete. We include it here as an example of computing the generating function by Theorem 1.4. The second proof is self-contained, and is simpler.

Let

$$f(y, t) = \sum_{n \geq 1} \sum_{i \geq 1} b_i(2n) t^n y^i. \quad (2.8)$$

We need to show that $F_-(y, t) = f(y, t)$.

First Proof of Lemma 2.3. It was stated in [1] that

$$\sum_{n \geq 0} b_i(2n) t^n = \frac{(-1)^i}{(1-u^2)^{2i-1}} \sum_{k=i}^{2i-1} \binom{2i-1}{k} (-1)^k u^{2k}. \quad (2.9)$$

Let s be as in (2.3). Using the following fact

$$\binom{n}{k} = \text{CT}_\alpha \frac{1}{\alpha^k} (1+\alpha)^n,$$

we can compute $f(y, t)$ by Theorem 1.4:

$$\begin{aligned} f(y, t) &= \sum_{i \geq 1} \frac{(-1)^i}{(1-u^2)^{2i-1}} \sum_{k=i}^{2i-1} \binom{2i-1}{k} (-1)^k u^{2k} y^i \\ &= \sum_{i \geq 1} \frac{(1-u^2)(-1)^i}{(1-u^2)^{2i}} \sum_{r=0}^{i-1} \binom{2i-1}{i+r} (-1)^{i+r} u^{2i+2r} y^i, \text{ where } r = k - i \\ &= (1-u^2) \sum_{r \geq 0} (-1)^r u^{2r} \sum_{i \geq r+1} \binom{2i-1}{i-1-r} \frac{u^{2i}}{(1-u^2)^{2i}} y^i \\ &= (1-u^2) \sum_{r \geq 0} (-u^2)^r \sum_{i \geq r+1} \text{CT}_\alpha (1+\alpha)^{2i-1} \left(\frac{1}{\alpha}\right)^{i-1-r} (s^2 y)^i \\ &= \text{CT}_\alpha (1-u^2) \sum_{r \geq 0} \frac{\alpha}{1+\alpha} (-u^2)^r \alpha^r \sum_{i \geq r+1} \frac{(1+\alpha)^{2i}}{\alpha^i} (s^2 y)^i \\ &= \text{CT}_\alpha \frac{\alpha}{1+\alpha} (1-u^2) \sum_{r \geq 0} (-u^2 \alpha)^r \left(\frac{(1+\alpha)^2}{\alpha} s^2 y\right)^{r+1} \frac{1}{1 - \frac{(1+\alpha)^2}{\alpha} s^2 y} \\ &= \text{CT}_\alpha (1-u^2)(1+\alpha) s^2 y \frac{1}{1 + u^2(1+\alpha)^2 s^2 y} \cdot \frac{1}{1 - \frac{(1+\alpha)^2}{\alpha} s^2 y}. \end{aligned}$$

Now

$$(1-u^2)(1+\alpha) s^2 y \frac{1}{1 + u^2(1+\alpha)^2 s^2 y}$$

is a power series in t with coefficients in $\mathbb{C}[y][\alpha]$, and

$$\frac{1}{1 - \frac{(1+\alpha)^2}{\alpha} s^2 y} = \frac{\alpha}{\alpha - (1+\alpha)^2 s^2 y}.$$

Solving the denominator for α , we get two solutions:

$$\frac{1 - 2s^2 y + \sqrt{1 - 4s^2 y}}{2s^2 y} \text{ and } \frac{1 - 2s^2 y - \sqrt{1 - 4s^2 y}}{2s^2 y}.$$

Only the latter is a power series in t with constant term 0, which can also be written as $A = C(s^2y) - 1$.

Thus we can apply Theorem 1.4 to get

$$\begin{aligned} f(y, t) &= \text{CT}_\alpha \frac{\alpha}{\alpha - (1 + \alpha)^2 s^2 y} (1 - u^2)(1 + \alpha) s^2 y \frac{1}{1 + u^2(1 + \alpha)^2 s^2 y} \\ &= (1 - u)^2 s^2 y (1 + A) \frac{1}{1 + u^2(1 + A)^2 s^2 y} \frac{1}{1 - 2s^2 y(A + 1)} \\ &= (1 - u^2) s^2 y C(s^2 y) \frac{1}{1 + u^2 C^2(s^2 y) s^2 y} \frac{1}{\sqrt{1 - 4s^2 y}}, \end{aligned}$$

which completes the proof. ■

The second proof derives a different form of $F_-(y; t)$.

Second Proof of Lemma 2.3. We begin with finding the generating function of $2nb_i(n)$, which equals $t \frac{\partial}{\partial t} f(y, t)$.

We claim that

$$\sum_{n \geq 0} \binom{n+i}{2i} \frac{\binom{4n}{2n}}{\binom{2n+2i}{2i}} t^{2n} = \frac{\sqrt{1+4s^2} s^{2i}}{1-4s^2}, \quad (2.10)$$

where the relation between t and s is given in (2.3).

It is easy to check that

$$\binom{n+i}{2i} \frac{\binom{4n}{2n}}{\binom{2n+2i}{2i}} = \binom{2n-1/2}{n-i} 4^{n-i}.$$

In the well-known formula

$$\frac{C(x)^k}{\sqrt{1-4x}} = \sum_{n \geq 0} \binom{2n+k}{n} x^n,$$

by setting $x = 4t^2$, and $k = 2i - 1/2$, we get

$$\sum_{n \geq 0} \binom{n+i}{2i} \frac{\binom{4n}{2n}}{\binom{2n+2i}{2i}} t^{2n} = t^{2i} \frac{C(4t^2)^{2i-1/2}}{\sqrt{1-16t^2}}.$$

Using (2.3) to write the above in terms of s , we get (2.10).

Now we have

$$t \frac{\partial}{\partial t} f(y; t) = \sum_{i \geq 1} \sum_{n \geq 0} i \binom{2i}{i} \binom{n+i}{2i} \frac{\binom{4n}{2n}}{\binom{2n+2i}{2i}} t^{2n} y^i = \frac{2s^2 y}{(1-4s^2 y)^{3/2}} \frac{\sqrt{1+4s^2}}{1-4s^2}.$$

Hence

$$\begin{aligned} f(y; t) &= \int \frac{2s^2 y}{(1-4s^2 y)^{3/2}} \frac{\sqrt{1+4s^2}}{1-4s^2} \frac{dt}{t} \\ &= \int \frac{2s^2 y}{(1-4s^2 y)^{3/2}} \frac{\sqrt{1+4s^2}}{1-4s^2} \frac{1-4s^2}{s(1+4s^2)} ds \\ &= \frac{y\sqrt{1+4s^2}}{2(1+y)\sqrt{1-4s^2 y}} + \text{constant}, \end{aligned}$$

where the constant is independent of t . By setting $t = 0$, and hence $s = 0$, we get $f(y; 0) = \frac{y}{2(1+y)} + \text{constant}$.

Recalling equation (2.8), we see that $f(y; 0) = 0$. Thus the constant equals $-\frac{y}{2(1+y)}$. This gives another form of $f(y; t)$:

$$f(y; t) = \frac{y\sqrt{1+4s^2}}{2(1+y)\sqrt{1-4s^2y}} - \frac{y}{2(1+y)} = \frac{y(1+u^2)}{2(1+y)(1-u^2)\sqrt{1-4s^2y}} - \frac{y}{2(1+y)},$$

which is easily checked to be equal to $F_-(y; t)$ as given in (2.5). ■

Proof of Theorem 1.1. We gave a formula for the generating function

$$F(y; t) = \sum_{n \geq 0} \sum_{i \in \mathbb{Z}} a_{i,i}(2n) y^i t^{2n}$$

in Lemma 2.1. In Lemma 2.2, we showed that

$$F_-(y, t) = \sum_{n \geq 0} \sum_{i > 0} a_{-i,-i}(2n) y^i t^{2n}$$

has a formula as given in (2.5). The proof of (1.2) is thus accomplished by Lemma 2.3.

For equation (1.3), once we get the formula (2.4), it is an easy exercise to show that

$$\frac{1}{2}((1-4s^2y)^{-1/2} - 1) = \sum_{n \geq 1} \sum_{i \geq 1} 4^n \frac{i}{n} \binom{2i}{i} \binom{2n}{n-i} y^i t^{2n}.$$

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