On the Hamiltonian index

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April 23, 2006

The **line graph** of $G = (V(G), E(G))$ has $E(G)$ as its vertex set, and two vertices are adjacent in $L(G)$ if and only if the corresponding edges are adjacent (share an end vertex) in G.

Haray and Nash-Williams gave a relation between the hamiltonicity of line graphs and the eulerian subgraphs of its original graphs.

Theorem 1. *(Harary and Nash-Williams, Can. Math. Bull. 1965) Let G be a graph with at least three edges. Then $L(G)$ is hamiltonian if and only if G has an eulerian subgraph H such that $d_G(e, H) = 0$ for any edge $e \in E(G)$.*

For any two subgraphs H_1 and H_2 of G, define the **distance** $d_G(H_1, H_2)$ between H_1 and H_2 to be the minimum of the distances $d_G(v_1, v_2)$ over all pairs with $v_1 \in V(H_1)$ and $v_2 \in V(H_2)$.

Corollary 2. *Let G be a graph with at least 3 edges. If G has a spanning eulerian subgraph, then $L(G)$ is hamiltonian.*

Theorem 1 and Corollary 2 shows that after applying the line graph operation iteratively a finite number of times, the resulting graph will become hamiltonian. Two natural questions then can be raised.

(1) For which graph is this the case?

(2) If this is the case for a graph, what is the smallest number of iterations that will yield a hamiltonian graph?

The m-**iterated line graph** $L^m(G)$ is defined recursively by $L^0(G) = G, L^m(G) = L(L^{m-1}(G))$, where $L^1(G)$ denotes $L(G)$. The **hamiltonian index** of a graph G, denoted by $h(G)$, is the smallest integer m such that $L^m(G)$ is hamiltonian.

Chartrand (Trans. Amer. Math. Soc., 1968) showed that if a connected graph G is not a path, then the hamiltonian index of G exists.

There have already appeared many upper bounds on $h(G)$ in literature. The following are the existing bounds that are rather easy to describe.

Theorem 3. *(Lai, Discrete Mathematics, 1988) Let G be a connected simple graph that is not a path, and let l be the length of a longest branch of G which is not contained in a 3-cycle. Then $h(G) \leq l + 1$.*

Theorem 4. *(Saražin, Discrete Mathematics, 1994) Let G be a connected simple graph on n vertices other than a path. Then $h(G) \leq n - \Delta(G)$. Here $\Delta(G)$ is the maximal degree of G.*

These known bounds are based on the characterization of graphs with hamiltonian line graphs obtained in (Harary and Nash-Williams, Can. Math. Bull. 1965),i.e.,Theorem 1.

Theorem 1 is a good tool for investigating cyclic properties of line graphs. However, when one uses it to investigate the (hamiltonian) cycles in the n-iterated line graph of a graph, closed trails in its $(n-1)$-iterated line graph should be considered. Since it is not convenient to examine $(n-1)$-iterated line graph when $n \geq 2$, this leads to a natural question: for any integer $n \geq 2$, does there exist a characterization of those graphs for which $L^n(G)$ is hamiltonian? This was also mentioned by Capobianco and Molluzzo in 1978. Liu and Xiong gave a affirmative answer for it.

Let G be a graph. For each integer $i \geq 0$, define $V_i(G) = \{v \in V(G) : d_G(v) = i\}$ and $W(G) = V(G) \setminus V_2(G)$. **A branch** in G is a non-trivial path whose end vertices are in $W(G)$ and whose internal vertices, if any, have degree 2 in G (and thus are not in $W(G)$). If a branch has length 1, then it has no internal vertices. We denote by $B(G)$ the set of branches of G and by $B_1(G)$ the subset of $B(G)$ in which every branch has an end vertex in $V_1(G)$. For any subgraph H of G, we denote by $B_H(G)$ the set of branches of G whose edges are all in H.

Liu and Xiong characterized the graphs for which the n-iterated line graph is hamiltonian, for any integer $n \geq 2$.

Theorem 5. *(Liu and Xiong, Discrete Math., 2002) Let G be a connected graph that is not a 2-cycle and let $n \geq 2$ be an integer. Then $h(G) \leq n$ if and only if $EU_n(G) \neq \emptyset$ where $EU_n(G)$ denotes the set of those subgraphs H of G which satisfy the following conditions:*

(i) *any vertex of H has even degree in H;*

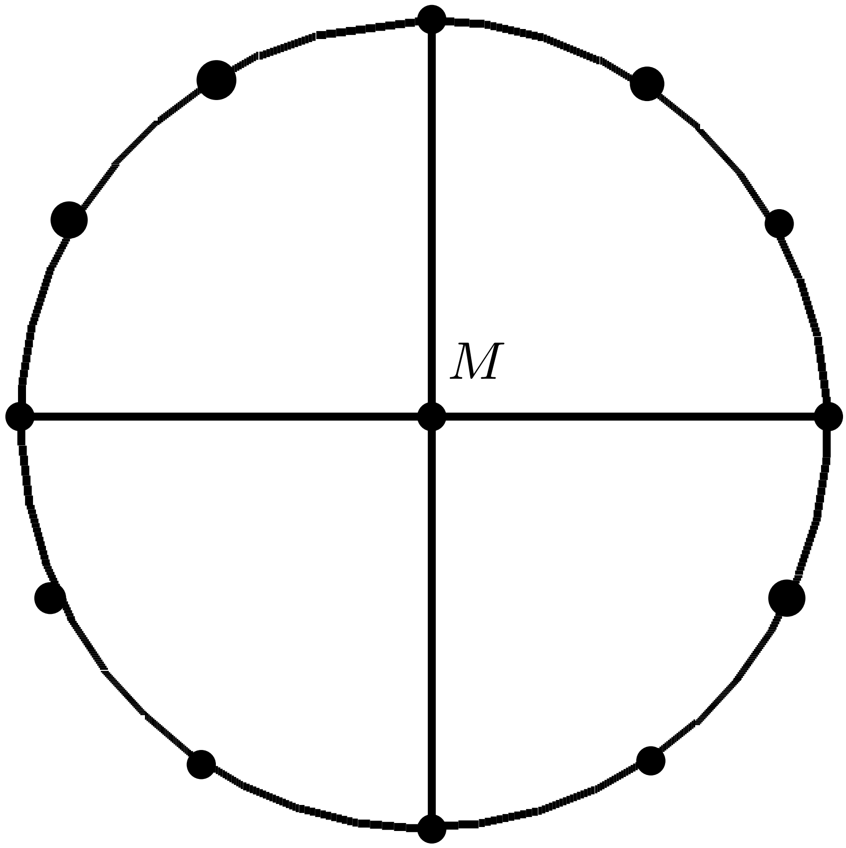
(ii) $V_0(H) \subseteq \bigcup_{i=3}^{\Delta(G)} V_i(G) \subseteq V(H)$;

(iii) $d_G(H_1, H - H_1) \leq n - 1$ *for any subgraph* H_1 *of* H;

(iv) $|E(b)| \leq n + 1$ *for any branch* b *in* $B(G) \setminus B_H(G)$;

(v) $|E(b)| \leq n$ *for any branch in* $B_1(G)$.

Note that the condition that $n \geq 2$ is necessary. See the following figure.



a graph G with $EU_1(G) = \emptyset$ but $L(G)$ is hamiltonian

Using Theorem 5, Xiong improved Theorem 4 as follows since $dia(G) - 1 \leq n - \Delta(G)$.

Theorem 6. *(Xiong, Graphs and Combinatorics, 2001) Let G be a connected graph that is not a path. Then $h(G) \leq dia(G) - 1$, where $dia(G)$ denotes the diameter of G.*

Theorem 7. *(Xiong, Graphs and Combinatorics, 2001) Let G be a connected simple graph of order $n \geq 61$ that are all not pathes and $\bar{G}$ the complement of G. Then*

$$\min\{h(G), h(\bar{G})\} \leq 1$$

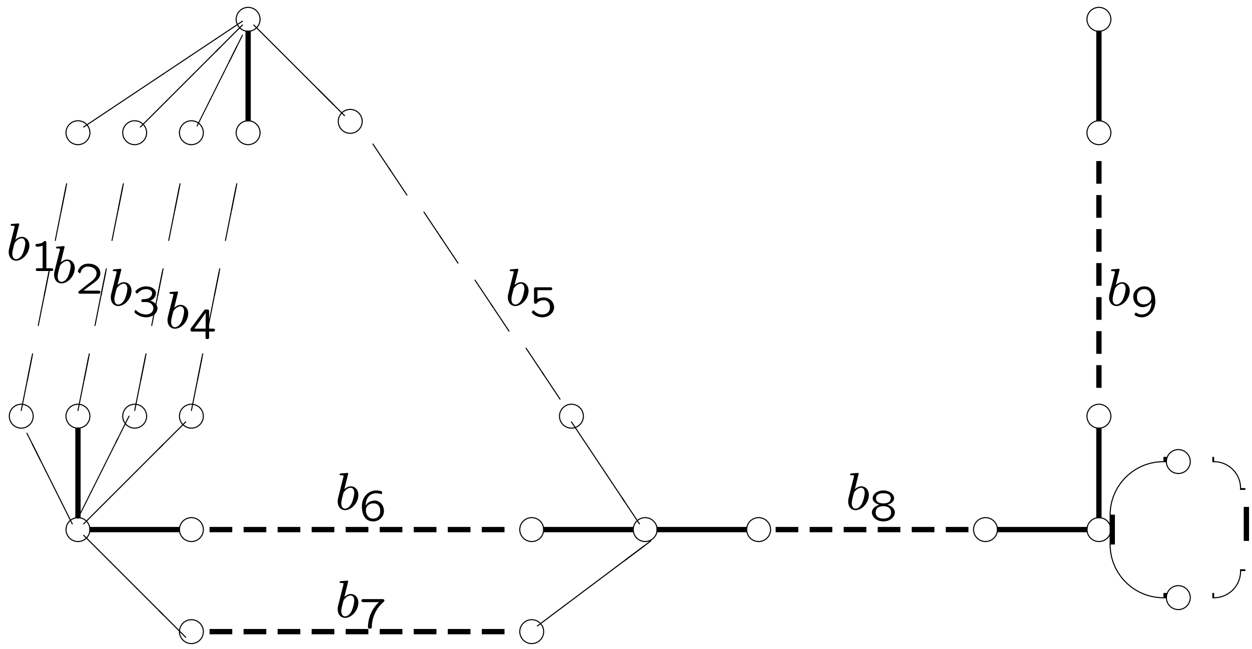
and If $\min\{h(G), h(\bar{G})\} = 1$, *then*

$$\max\{h(G), h(\bar{G})\} \leq (n-1)/2$$

and all bound are sharp.

For any subset S of $B(G)$, we denote by $G-S$ the subgraph obtained from $G[E(G)\backslash E(S)]$ by deleting all internal vertices of degree 2 in any branch of S. A subset S of $B(G)$ is called a **branch cut** if $G-S$ has more components than G. A minimal branch cut is called a **branch-bond**. If G is connected, then a branch cut S of G is a minimal subset of $B(G)$ such that $G-S$ is disconnected. It is easily shown that, for a connected graph G, a subset S of $B(G)$ is a branch-bond if and only if $G-S$ has exactly two components. We denote by $BB(G)$ the set of branch-bonds of G. The **length** of a branch-bond $S \in BB(G)$, denoted by $l(S)$, is the length of a shortest branch in it. Define $BB_2(G) = \{S \in BB(G) : |S| = 1$ and both endvertices of $b \in S$ have degree ≥ 3 in $G\}$ and $BB_3(G) = \{S \in BB(G) : |S| \geq 3$ and $|S|$ is odd$\}$. For convenience, we denote $BB_1(G) = B_1(G)$.

See the following figure for an example.



$$BB_1(G) = \{\{b_9\}\}, BB_2(G) = \{\{b_8\}\},$$
$$BB_3(G) = \{\{b_1, b_2, b_3, b_4, b_5\}, \{b_5, b_6, b_7\}\}$$

The definition of $BB_i(G)$ illustrated

For $i \in \{1, 2, 3\}$, define

$$h_i(G) = \begin{cases} \max\{l(S) : S \in BB_i(G)\}, & \text{if } BB_i(G) \text{ is not empty} \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 8. *(Chartrand and Wall, Studia, Sci. Math. Hung. 1973) If T is a tree which is not a path, then*

$$h(T) = max\{h_1(T), h_2(T) + 1\}.$$

The following lower bound for $h(G)$ involves odd branch-bonds.

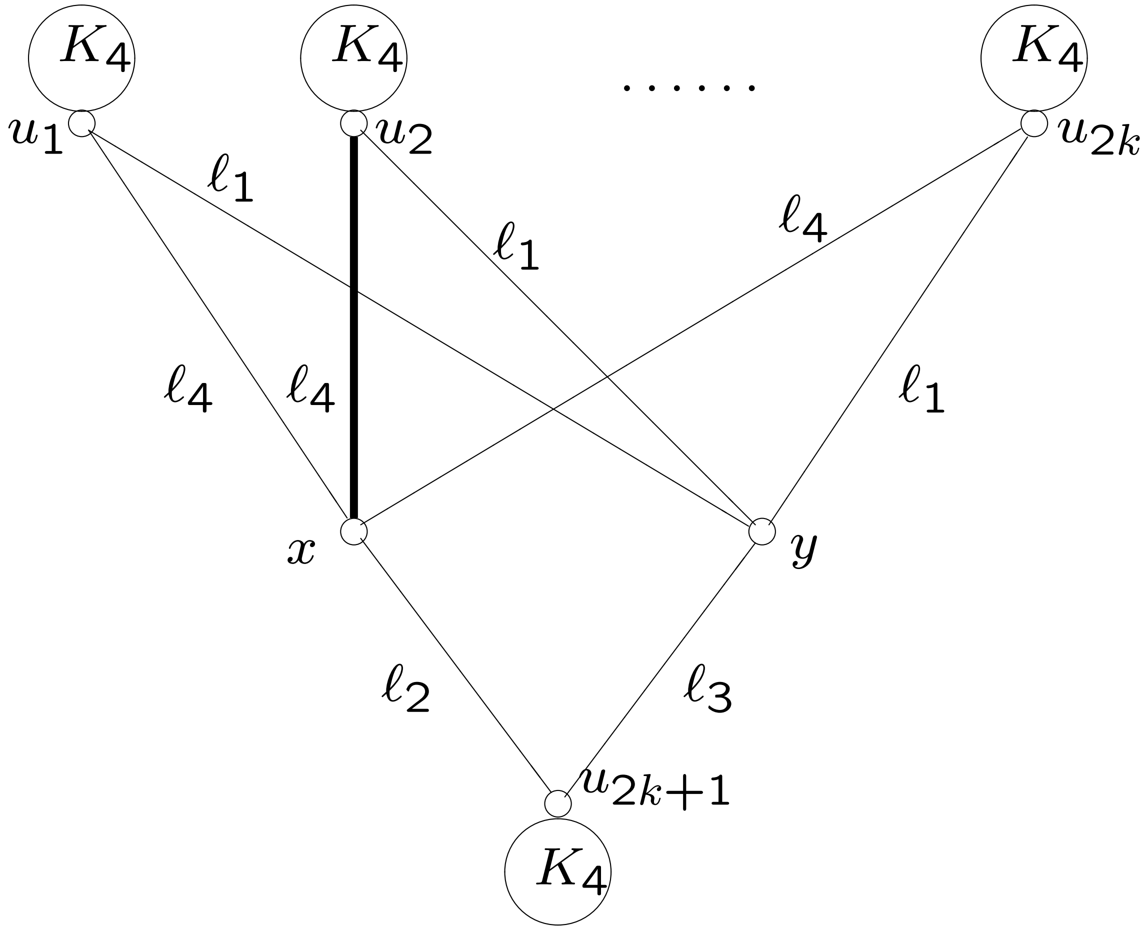
Theorem 9. *(H.J. Broersma, Xueliang Li and Liming Xiong, Discrete Math., 2004) Let G be a connected graph with $h(G) \geq 1$. Then*

$$h(G) \geq max\{h_1(G), h_2(G) + 1, h_3(G) - 1\} \quad (1)$$

A upper bound for $h(G)$ is the following.

Theorem 10. *(H.J. Broersma, Xueliang Li and Liming Xiong, Discrete Math., 2004) Let G be a connected graph that is not a path. Then*

$$h(G) \leq max\{h_1(G), h_2(G) + 1, h_3(G) + 1\} \quad (2)$$



A graph G_0 with the value from $h_3(G_0) - 1$ to $h_3(G_0) + 1$

$$
\begin{aligned}
\mathsf{h}(\mathsf{G}_0) &= \max\{ \mathsf{l}_3 - 1, l_2 + 1\} \\
&= \begin{cases} l_3 - 1, & \text{if} \quad l_2 \leq l_3 - 2 \\ l_3, & \text{if} \quad l_2 = l_3 - 1 \\ l_3 + 1, & \text{if} \quad l_2 = l_3. \end{cases}
\end{aligned}
$$

A *block* of a graph G is maximal connected subgraph which contains no cut vertex of itself. A block of G is called *acyclic block* if it is a single edge of G and a *cyclic block* otherwise.

Theorem 11. *(Saražin, Discrete Math.,1993) If every cyclic block of G is hamiltonian, then $h(G) = \max\{h_1(G), h_2(G) + 1\}$.*

For every cyclic block B of G, we construct a *split block* SB from B as follows:
(i) split each vertex $x \in V_2(B) \cap (\cup_{i=3}^{\Delta(G)} V_i(G))$ into a triangle $x_1x_2x_3$ in SB;
(ii) replace the two edge ux and vx (say) in $E(B)$ by ux_1 and vx_2 in $E(SB)$.

Theorem 12. *(Liu and Xiong, Discrete Math., 2002) Let* G *be a connected graph and let* $SB_1, SB_2, \cdots, SB_t$ *be all split blocks of* G*. Then*

$$h(G) = \max\{h_1(G), h_2(G)+1, h(SB_1), \cdots, h(SB_t)\}$$

Corollary 13. *Let* G *be a connected graph and let* $SB_1, SB_2, \cdots, SB_t$ *be all the split blocks of* G*. Then* $h(G) = \max\{h_1(G), h_2(G) + 1\}$ *if and only if* $h(SB_i) \leq \max\{h_1(G), h_2(G)+1\}$ *for each* i*.*

Theorems 8 and 11 are consequences of Corollary 13.

Remark It is not difficult to determine $h_1(G)$, $h_2(G)$ of a graph. By Theorem 12, we can determine the hamiltonian index of a graph by determining the hamiltonian indices of its blocks. Since each split block is 2-connected, we only need to consider graphs with the connectivity at least 2. In fact, if the connectivity of a graph is at least 3, then the hamiltonian index is at most 2. So we can assume the connectivity of a graph is 2 when we consider the hamiltonian index.

For $\{b_1, b_2, \cdots, b_m\} \subseteq B(G)$ with $|E(b_i)| \geq 2$ for each i, the contraction of G, denoted by $G//\{b_1, b_2, \cdots, b_m\}$, is defined to a graph which is obtained from G by contracting an edge of b_i,i.e., replacing b_i by a new branch of length $|E(b_i)| - 1$ for each i.

Theorem 14. *(Liu and Xiong, Discrete Math., 2002) Let G be a connected graph and let $b_1, b_2, \cdots, b_m$ be all branches of length at least 2 in G. If $h(G) \geq 4$, then*

$$h(G) = h(G//\{b_1, b_2, \cdots, b_m\}) + 1.$$

It is well known that the complexity of determining wether a graph G is hamiltonian, i.e.,$h(G) = 0$ is NP-complete. Bertossi (Inform. Process. Lett. 1981) showed that the complexity of determining wether $h(G) \leq 1$ is also NP-complete. However we give the following.

Conjecture 15. *(Liu and Xiong, Discrete Math., 2002) The complexity of determining* $h(G)(\geq 2)$ *is polynomial.*

In order to solve Conjecture 15, by Theorem 14, we only need to check the complexity of determining whether the hamiltonian index is 2 or 3 since the hamiltonian index at least 4 can reduce to 2 or 3 by polynomial time.

The following consequences of Theorems 9 and 10 are easily obtained.

Corollary 16. *(Catlin et al. J. Graph Theory, 1990) Let G be a connected graph that is neither a path nor a 2-cycle. Then*

$$h(G) \leq \max_{\{u,v\}\subseteq W(G)} \min_{P} X(P) + 1,$$

where $X(P)$ denotes the length $|E(b)|$ of the longest branch b in $B_P(G)$ and P is a subgraph induced by all branches in G whose end vertices are u and v.

Proof. Let S be a branch-bond in $BB(G)$ with $l(S) = max\{h_1(G), h_2(G)+1, h_3(G)+1\}$. Then any path of G between two vertices u and v in two components of $G - S$, respectively, must have a branch in S. Hence

$$\max\{h_1(G), h_2(G) + 1, h_3(G) + 1\}$$
$$\leq \max_{\{u,v\}\subseteq W(G)} \min_P X(P) + 1.$$

This relation and Theorem 10 give Corollary 16. □

Theorem 8 (Chartrand and Wall, Studia, Sci. Math. Hung. 1973) If T is a tree which is not a path, then

$$h(T) = max\{h_1(T), h_2(T) + 1\}.$$

Proof of Theorem 8. If T is a tree, then $h_3(T) = 0$. Hence by Theorems 9 and 10, we obtain Corollary 8. $\square$

Corollary 17. *(Balakrishnan and Paulraja, Journal of Combinatorics, Information & System Sciences, 1985) Let G be a connected graph with at least four edges. If the only 2-degree cut sets of G are the singleton subsets which are neighbors of end vertices of G, then $h(G) \leq 2$.*

Proof. One can easily check that $h_1(G) \leq 2, h_2(G) \leq 1$ and $h_3(G) \leq 1$. Hence this corollary follows from Theorem 10. □

Corollary 18. *(Lesniak-Foster and Williamson Can. Math. Bull. 20 (1977)) Let G be a connected graph with at least four edges. If every vertex of degree two is adjacent to an end vertex, then $h(G) \leq 2$.*

Proof. From the condition of this corollary, we know $h_1(G) \leq 2, h_2(G) \leq 1$ and $h_3(G) \leq 1$. Hence this corollary follows from Theorem 10. $\square$

Corollary 19. *(Chartrand and Wall, Studia, Sci. Math. Hung. 8 (1973)) Let G be a connected graph other than a path. If $\delta(G) \geq 3$, then $h(G) \leq 2$.*

Proof. This is obvious. $\square$

Corollary 20. *Let G be a 2-connected graph. Then*

$$h_3(G) - 1 \leq h(G) \leq h_3(G) + 1.$$

Question 21. *How to characterize those graphs G with $h(G) = h_3(G)-1, h_3(G), h_3(G)+1$, respectively?*

Question 22. *How to use $h_1(G), h_3(G)$ to characterize those simple graphs G with $h(G) = n - \Delta(G)$?*

Obviously $h_3(G) - 1 \leq h(G) \leq h_3(G) + 1$ for any 2-connected graph G. Hence we propose the following

Question 23. *How to use $h_3(G)$ to characterize those simple 2-connected graphs G with $h(G) = n - \Delta(G)$?*

The following result is an attempt to answer Question 23.

Theorem 24. *If G is a simple 2-connected graph with $h(G) = n - \Delta(G)$, then $h(G) \leq h_3(G) + 1 \leq 3$.*

Proof. If $h_3(G) \geq 3$, then $\Delta(G) \leq n-(3(h_3(G)-2)+2)$. Hence $h_3(G) \leq (n - \Delta(G) + 4)/3$ and $\Delta(G) \leq n - 5$. This implies that $h_3(G) \leq (n - \Delta(G) + 4)/3 < n - \Delta(G) - 1$. By Theorem 38, $h(G) \leq h_3(G) + 1 < n - \Delta(G)$, a contradiction. So $h_3(G) \leq 2$. Hence $h(G) \leq h_3(G) + 1 \leq 3$. $\square$

If we improve the connectivity of a graph, then we can obtain a stronger result than Theorem 4.

Theorem 25. *(Xiong, Preprint, 2005) Let* G *be a 2-connected simple graph. Then*

$$h(G) \leq \lfloor \frac{n - \Delta(G)}{3} \rfloor$$

and the bound is sharp.

The following result shows that, with one exceptional case, adding an edge to a graph cannot increase its hamiltonian index.

Theorem 26. *(Broersma, Ryjáček and Xiong, J. Graph Theory, 2005) Let* G *be a connected graph with at least three edges that is not a path. Then for any two vertices* $a, b \in V(G)$ *with* $d_G(a) + d_G(b) \geq 3$*, either* $h(G) = 1$ *and* $h(G + ab) = 2$ *or* $h(G) \geq h(G + ab)$*. Moreover, if* $dist_G(a, b) = 2$*, then*

$$h(G) \geq h(G + ab).$$

The following concept was introduced by Ryjáček. Let G be a claw-free graph and let $cl(G)$ be a graph obtained from G by recursively performing the local completion operation at locally connected vertices with noncomplete neighborhood, as long as this is possible. The graph $cl(G)$ is called the *closure* of the graph G. The following theorem summarizes basic properties of the closure operation.

Theorem 27. *(Ryjáček, J Combin. Theory, 1997) Let G be a claw-free graph. Then*

(i) $cl(G)$ is uniquely determined,

(ii) $c(cl(G)) = c(G)$,

(iii) $cl(G)$ is the line graph of a triangle-free graph.

Corollary 28. *(Ryjáček, J Combin. Theory, 1997) Let G be a claw-free graph. Then G is hamiltonian if and only if $cl(G)$ is hamiltonian.*

Theorem 27 and Corollary 28 say that the circumference and hamiltonicity are stable under the closure operation on claw-free graphs.

Theorem 29. *(Broersma, Ryjáček and Xiong, J. Graph Theory, 2005) Let G be a connected claw-free graph with at least three edges which is not a path. Then*

$$h(G) = h(cl(G)).$$

Let F be a graph and let $A \subset V(F)$. We say that the graph F is A-*contractible*, if for every even subset $X \subset A$ and for every partition A of X into two-element subsets the graph F^A has a DCT containing all vertices of A and all edges of $E(A)$. Note that this definition comprises the case where X is empty and $F^A = F$. Also, if F is A-contractible, then F is A'-contractible for any $A' \subset A$ (since every subset X of A' is a subset of A).

For a subgraph F of G, $G|_F$ denotes the graph obtained from G by identifying the vertices of F as a (new) vertex v_F, and by replacing the created loops by pendant edges (i.e. edges with one vertex of degree 1) attached to v_F. We say that the graph $G|_F$ is obtained from G by *contracting* the subgraph F (observe that $|E(G)| = |E(G|_F)|$).

Set $d_T(G) = \max\{\ |S| : S \subset E(G)$ and there is a closed trail $T \subset G$ such that every edge $e \in S$ has at least one vertex on $T\}$. The following result was proved by Ryjáček and Schelp.

Theorem 30. *(Ryjáček; Schelp, J. Graph Theory 2003) Let F be a connected graph and let $A \subset V(F)$. Then F is A-contractible if and only if*

$$d_T(G) = d_T(G|_F)$$

for every graph G such that $F \subset G$ and $A_G(F) = A$.

Corollary 31. *(Ryjáček; Schelp, J. Graph Theory 2003) Let G be a graph and let $F \subset G$ be an $A_G(F)$-contractible subgraph of G. Then G has a DCT if and only if $G|_F$ has a DCT.*

Note that $G|_F$ may contain multiple edges even if G is a simple graph. However, it is easy to observe that a multiple edge is a contractible subgraph and hence, by a series of subsequent contractions, it is always possible to reduce $G|_F$ to a certain simple graph G' with $d_T(G') = d_T(G|_F) = d_T(G)$.

If G is a hamiltonian graph (i.e. $h(G) = 0$) and $F \subset G$ is a nontrivial subgraph of G, then $G|_F$ cannot be hamiltonian (since it has connectivity 1), and it is easy to observe that any hamiltonian cycle in G turns into a DCT in $G|_F$. Hence $h(G) = 0$ implies $h(G|_F) = 1$ for any nontrivial subgraph $F \subset G$. However, the following theorem shows that for $h(G) \geq 1$, i.e. for nonhamiltonian graphs, the hamiltonian index is stable under contraction of a contractible subgraph.

Theorem 32. *(Broersma, Ryjáček, Xiong, J. Graph Theory, 2005) Let G be a nonhamiltonian graph other than a path and F be an $A_G(F)$-contractible subgraph of G. Then*

$$h(G) = h(G|_F).$$

Theorem 33. *(Chvátal-Erdős, Discrete Math. 1972) If* $\kappa(G) \geq \alpha(G)$*, then* G *is hamiltonian.*

The following result extends above theorem.

Theorem 34. *(Lai, Xiong and Yan, Preprint, 2005) If* $\kappa(G) \geq \alpha(G) - t$*, then*

$$h(G) \leq \lfloor \frac{2t+2}{3} \rfloor,$$

where t *be an nonnegative integer and the bound is sharp.*

Related topic

Theorem 35. *(Ferrara and Gould, Preprint, 2002) Let G be a connected with at least three edges. Then for any $n \geq 2$, $L^n(G)$ has 2-factor if and only if $F_n(G) \neq \emptyset$ where $F_n(G)$ denotes the set of those subgraphs H of G that satisfy the following five conditions.*

(i') *any vertex of H has even degree in H;*

(ii') $V_0(H) \subseteq \bigcup_{i=3}^{\Delta(G)} V_i(G) \subseteq V(H)$;

(iii') $d_G(H_1, H - H_1) \leq n + 1$ *for any subgraph* H_1 *of* H;

(iv') $|E(b)| \leq n + 1$ *for any branch* b *in* $B(G) \setminus B_H(G)$;

(v') $|E(b)| \leq n$ *for any branch in* $B_1(G)$.

Observing that conditions (ii') and (iv') in the definition of $F_k(G)$ imply condition (iii') in the definition of $F_k(G)$, we obtain an equivalence of Theorem 35 as follows.

Theorem 36. *(Xiong, Preprint, 2004) Let G be a connected with at least three edges. Then for any $n \geq 2$, $L^n(G)$ has 2-factor if and only if $F_n(G) \neq \emptyset$ where $F_n(G)$ denotes the set of those subgraphs H of G that satisfy the following four conditions.*

(I) any vertex of H has even degree in H;

(II) $V_0(H) \subseteq \bigcup_{i=3}^{\Delta(G)} V_i(G) \subseteq V(H)$;

(III) $|E(b)| \leq n+1$ for any branch b in $B(G) \setminus B_H(G)$;

(IV) $|E(b)| \leq n$ for any branch in $B_1(G)$.

We let $f(G)$ denote the **2-factor index** of a graph, i.e., the minimum integer m with the m-iterated line graph that contains a 2-factor.

Theorem 37. *Let G be a graph that is not a path. Then $h(G) - 2 \leq f(G) \leq h(G)$.*

Define $BB_4(G) = \{S \in BB(G) \backslash BB_1(G) : S \text{ is odd}\}$.

$$\text{f}_i(G) = \max\{l(S) : S \in BB_i(G)\} \text{ for } i \in \{1, 4\},$$
$$\text{if } \text{BB}_i(G) \text{ is not empty }; 0, \text{ otherwise.}$$

In the following, we give a formula for $f(G)$.

Theorem 38. *(Xiong, Preprint, 2004) Let G be a connected graph that is not a path and*

$$k(G) = \max\{f_1(G), f_4(G) - 1\} \geq 2.$$

Then

$$f(G) = k(G).$$

Corollary 39. *The complexity of determining the $f(G)$ is polynomial if $k(G) \geq 2$.*

THANKS!!!