Analytic Combinatorics: An Introduction

Jason Zhicheng Gao
School of Mathematics and Statistics,
Carleton University, Canada, and
Center for Combinatorics, LPMC,
Nankai University, Tianjin, China

Generating Functions

a_n: # combinatorial structures of size n.

Its ordinary generating function (OGF) is

$$A(z) = \sum_n a_n z^n.$$

Its exponential generating function (OGF) is

$$\hat{A}(z) = \sum_n a_n z^n / n!.$$

Notation: $[z^n] \sum f_n z^n = f_n$.

Analytic Combinatorics

Combinatorial methods: $\{a_n\} \Rightarrow A(z), \hat{A}(z).$

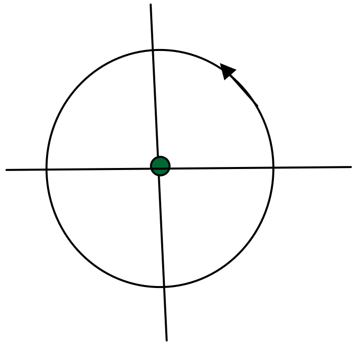
Analytic methods: $A(z), \hat{A}(z) \Rightarrow \{a_n\}.$

See Flajolet and Sedgewick's book

``Analytic Combinatorics"

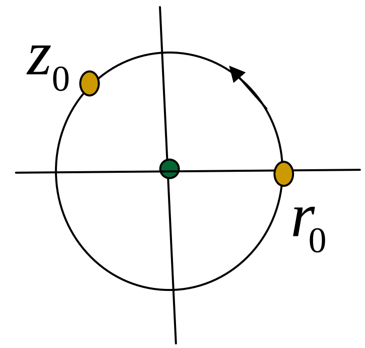
http://algo.inria.fr/flajolet/Publications/books.html

Cauchy's Formula: $f_n = \frac{1}{2\pi i} \oint_{\Gamma} \frac{F(z)}{z^{n+1}} dz.$



Radius of convergence of a power series

Every power series $F(z) = \sum f_n z^n$



has a radius of convergence r such that

$\sum f_n z^n$ converges when $|z| < r$, and

it diverges when $|z| > r$.

$Fact: r = \min\{|z_0| : z_0$ is a singularity of $F(z)\}$.

For any $0 < r_0 < r$, let Γ be the circle of radius r_0,

centered at 0, and let M=max $\{|F(z)| : z \in \Gamma\}$.

Then $|f_n| \leq M r_0^{-n}$.

Example 1: Derangements

Let a_n be the number permutations of n elements with no fixed points. Define $a_0 = 1$. Then

$$\hat{A}(z) = \sum_{n\geq 0} \frac{a_n}{n!} z^n = \exp(z^2/2 + z^3/3 + \cdots) = \frac{e^{-z}}{1-z}.$$

$$\frac{a_n}{n!} = [z^n]\frac{e^{-z}}{1-z} = [z^n]\left(\frac{e^{-1}}{1-z} + \frac{e^{-z} - e^{-1}}{1-z}\right) = e^{-1} + [z^n]\frac{e^{-z} - e^{-1}}{1-z}$$

We note that $\dfrac{e^{-z} - e^{-1}}{1-z}$ is analytic everywhere, and hence

$$[z^n]\frac{e^{-z} - e^{-1}}{1-z} = O(r_0^{-n}), \text{ for any } r_0 > 1.$$

Notation for asymptotics

In particularly $|a_n/n! - 1/e| = O(2006^{-n})$.

Notation. We write

$a_n = O(b_n)$ if $|a_n/b_n|$ is bounded,

$a_n = o(b_n)$ if $a_n/b_n \to 0$ as $n \to \infty$,

$a_n \sim b_n$ if $a_n/b_n \to 1$ as $n \to \infty$.

Hence we have

$a_n/n! \sim 1/e$, or $a_n \sim n!/e$.

Darboux's Lemma

Let g(z) be analytic in some disk $|z| < 1 + \delta$,

and suppose $g(1) \neq 0$.

Then, for any $\alpha \notin \{0, -1, -2, \ldots.\}$,

$$[z^n]g(z)(1-z)^{-\alpha} \sim g(1)[z^n](1-z)^{-\alpha}$$

$$\sim \frac{g(1)}{\Gamma(\alpha)} n^{\alpha-1}.$$

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$$

Example 2. Labeled 2-regular graphs

The exponential generating function for labeled 2-regular graphs is given by

$$\hat{A}(z) = (1-z)^{-1/2} e^{-z/2-z^2/4}.$$

Hence

$$[z^n]\hat{A}(z) \sim e^{-1/2-1/4}[z^n](1-z)^{-1/2}$$

$$\sim \frac{e^{-3/4}}{\Gamma(1/2)} n^{-1/2} = \frac{e^{-3/4}}{\sqrt{\pi}} n^{-1/2}.$$

Example 3: Triangulations of the Projective Plane

The ordinary generating function for the number of rooted n-vertex triangulations of the projective plane is given by

$$A(z) = \frac{1}{3}(1-t)^3\left(1-3t-(1-t)\sqrt{1-4t}\right)$$

$$-\frac{t^3(1-5t+12t^2-14t^3+4t^4+3t^5)}{(1-2t)(1-t)^3},$$ where

$t = t(z) = z + \cdots$ is the power series satisfying $z = t(1-t)^3$.

Using Implicit Function Theorem, one can find the dominant singularity of $t(z)$ to be $r = 27/256$,

Example 3: Triangulations of the Projective Plane

Expansion of $t(z)$ and $A(z)$ at $r = 27/256$:

$$t(z) = \sum f_i (1 - z/r)^{i/2},$$

$$A(z) = \sum g_i (1 - z/r)^{i/4},$$

with $g_1 = -(81/1024)24^{1/4}$.

$$[z^n]A(z) \sim \frac{g_1}{\Gamma(-1/4)} n^{-5/4} \left(256/27\right)^n.$$

Flajolet-Odlyzko's Transfer Theorems

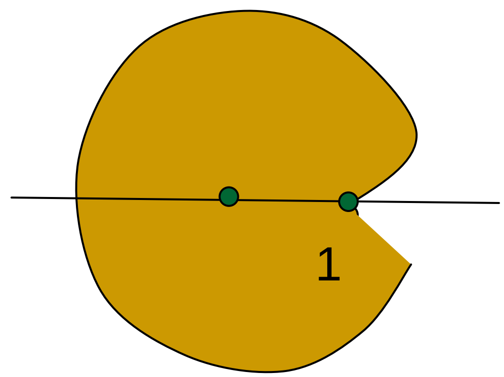
$$A(z) = O(B(z)) \quad (\text{as } z \to 1) \quad \Rightarrow \quad a_n = O(b_n),$$

$$A(z) = o(B(z)) \quad (\text{as } z \to 1) \quad \Rightarrow \quad a_n = o(b_n),$$

$$A(z) \sim B(z) \quad (\text{as } z \to 1) \quad \Rightarrow \quad a_n \sim b_n.$$

$B(z)$ are typically of the form

$$(1-z)^{-\alpha}\left(\log\frac{1}{1-z}\right)^{\beta}.$$



$A(z)$ is analytic in the region

$$\Delta = \{z : |z| < 1+\delta, z \neq 1, \theta \leq \arg(z-1) \leq 2\pi - \theta\}$$

Flajolet-Odlyzko's Transfer Theorems

If $\alpha \notin \{0,-1,-2,\ldots\}$, then

$$[z^n](1-z)^{-\alpha}\left(\log\frac{1}{1-z}\right)^{\beta} \sim \frac{1}{\Gamma(\alpha)} n^{\alpha-1}(\log n)^{\beta}.$$

For $\alpha=-k$, we have

$$[z^n](1-z)^{k}\left(\log\frac{1}{1-z}\right)^{\beta} \sim \beta(-1)^k k! n^{-k-1}(\log n)^{\beta-1}.$$

Example 4. Irreducible Polynomials over a Finite Field

Let I_n be the number of irreducible monic polynomials of degree n over F_q. Define $I_0 = 1$ and $I(z) = \sum_{n \geq 0} I_n z^n$.

It can be shown that $I(z) \sim \log\left(\frac{1}{1-qz}\right)$, $z \to 1/q$,

and $1/q$ is the only singularity of $I(z)$ in the disk

$|z| < 1/\sqrt{q}$. Hence $I_n \sim [z^n]\log\left(\frac{1}{1-qz}\right) = q^n/n$.

Example 5. Children's Rounds

The exponential generating function for the number of ``children's rounnds" is given by

$$R(z) = (1-z)^{-z} = \frac{1}{1-z} e^{(1-z)\log(1-z)}$$

$$= \frac{1}{1-z} + \log(1-z) + O\left(\frac{1}{1-z}\log^2(1-z)\right), z \to 1.$$

$$[z^n]R(z) = 1 - 1/n + O((1/n^2)\log n).$$

Hayman's Theorem (1956)

If $f(z) = \sum_{n \geq 0} f_n z^n$ is ``H-admissible", then

$$f_n \sim \frac{1}{\sqrt{2\pi b(r)}} f(r) r^{-n}, \text{ where}$$

$$a(r) := r \frac{d}{dr} \log(f(r)) = n, \text{ and}$$

$$b(r) = r \frac{d}{dr} a(r).$$

Hayman-admissible Functions

Facts about H-admissible functions:

1. if $p(z)$ is a polynomial such that the Taylor series of $\exp(p(z))$ has positive coefficients for all sufficiently large n, then $\exp(p(z))$ is H-admissible.
2. if $f(z)$ is H-admissible then $\exp(f(z))$ is H-admissible.
3. if $f(z)$ and $g(z)$ are H-admissible then $f(z)g(z)$ is H-admissible.

$\exp(z), \exp(z + z^2/2),$ and $\exp(\exp(z))$ are all H-admissible.

Example 6: Stirling's formula

For $f(z) = \exp(z) = \sum z^n / n!$,

we have $\log(f(z)) = z$.

Hence $a(r) = r,\ b(r) = ra'(r) = r$.

Solving $a(r) = n$, we obtain $r = n$.

Hence $1/n! \sim \dfrac{1}{\sqrt{2n\pi}} e^n n^{-n}$,

or $n! \sim \sqrt{2n\pi}(n/e)^n$.

Example 7: The number of involutions

The exponential generating for the number of involutions

is given by $f(z) = \exp(z + z^2/2)$.

We have $\log(f(z)) = z + z^2/2$,

$a(r) = r(1+r),\ b(r) = ra'(r) = r(1+2r)$.

Solving $r(1+r) = n$, we obtain

$r = n^{1/2}(1+1/(4n))^{1/2} - 1/2$

$= n^{1/2} - 1/2 + (1/8)n^{-1/2} - (1/128)n^{-3/2} + O(n^{-2})$.

Hence the number of involutions of n elements is

$$\sim \frac{1}{\sqrt{2}} n^{n/2} \exp\left(-n/2 + \sqrt{n} - 1/4\right),$$

Example 8: Bell numbers

The exponential generating function for the total number of partitions of {1,2,..,n} is given by

$$\hat{A}(z) = \exp(\exp(z) - 1) = e^{-1}\exp(\exp(z)).$$

Let $f(z) = \exp(\exp(z))$. We have $\log(f(z)) = \exp(z)$,

$a(r) = r\exp(r),\ b(r) = ra'(r) = r(1+r)\exp(r).$

Solving $r\exp(r) = n,$ we obtain

$$r = \log n - \log r = \log n - \log\log n + O(\log\log n/\log n).$$

$$(1/n)\log a_n = \log n - \log\log n - 1 + O(\log\log n/\log n).$$

Multivariate generating functions, moments, and distribution.

Let $a_{n,k}$ be the number of monic polynomials of degree n over F_q which have exactly k irreducible factors.

Define $A(z,w) = \sum_{n,k} a_{n,k} z^n w^k$. Note $A(z,1) = 1/(1-qz)$.

Then the average number of irreducible factors in a random monic polynomial of degree n is $\dfrac{\sum_k k a_{n,k}}{\sum_k a_{n,k}} = \dfrac{[z^n] A_w(z,1)}{[z^n] A(z,1)}$.

The variance and other higher moments can also be calculated by taking higher order partial derivatives w.r.t. w.

The number of irreducible factors of polynomials over a finite field

$$A(z,w) = \exp\left(\sum_{k\geq 1} I(z^k)w^k/k\right), I(z) \sim \log\frac{1}{1-qz},$$

$$A_w(z,1) = \exp\left(\sum_{k\geq 1} I(z^k)/k\right)\sum_{k\geq 1} I(z^k) \sim \frac{1}{1-qz}\log\frac{1}{1-qz}.$$

$$[z^n]A_w(z,1) \sim [z^n]\frac{1}{1-qz}\log\frac{1}{1-qz} \sim q^n \log n.$$

Hence the average number of irreducible factors is

$\sim q^n \log n / q^n = \log n.$

One can also show $A_{ww}(z,1) \sim \dfrac{1}{1-qz}\left(\log\dfrac{1}{1-qz}\right)^2.$

Distribution of irreducible factors of monic polynomials over F_q

The probability generating function for the sequence $\{a_{n,k}\}$

is $p_n(w) = \sum_k a_{n,k} w^k / \sum_k a_{n,k} = [z^n]A(z,w)/[z^n]A(z,1).$

Note $A(z,w) \sim (1-qz)^{-w}$, and hence

$$p_n(w) \sim \frac{1}{\Gamma(w)} n^{w-1} = \frac{1}{\Gamma(w)} e^{(w-1)\log n}.$$

It follows that $\left\{ \frac{a_{n,k}}{\sum_k a_{n,k}} \right\}$ is asymptotically normal with

mean $\log n$ and variance $\log n.$

A general central limit theorem

Suppose the probability generating function for a sequence $\{a_{n,k}\}$ satisfies, with $s = \log\ w$,

$p_n(w) \sim e^{U(s)\beta_n + V(s)}$ uniformly in some disk |s|<δ.

Suppose $\beta_n \to \infty, U''(0) \neq 0$. Then

$\left\{ \frac{a_{n,k}}{\sum_k a_{n,k}} \right\}$ is asymptotically normal with

mean $U'(0)\beta_n$ and variance $U''(0)\beta_n$.

Some References

Bender (1974), Asymptotic methods in Enumeration,
16 SIAM Review 485-515.

Wilf (1990), ``Generatingfunctionology''.

Odlyzko (1995), Asymptotic enumeration methods, in
``Handbook of Combinatorics'' Vol II. 1063-1229.

Flajolet and Sedgewick (2006) ``Analytic Combinatorics'',
http://algo.inria.fr/flajolet/Publications/books.html

Pemantle and Wilson (2006) Twenty combinatorial examples
of asymptotics derived from multivariate generating functions.

Mckay and Wormald (1990) Asymptotic enumeration by degree
sequence of graphs with high degree,Europ. J. Combin.11,565-580.