Zero-sum Problems in Abelian Groups

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Let G be a finite abelian group (written additively), and let $S = (a_1, \cdots, a_k)$ be a sequence of elements in G. We call S a zero-sum sequence if $\Sigma_{i=1}^{k} a_i = 0$.

Zero-sum problems are usual to study the conditions that ensure a sequence contains a nonempty zero-sum subsequence or a zero-sum subsequence of restricted size (of size $|G|$, $\exp(G)$, or of size not exceeding $\exp(G)$), or to describe the structure of long sequences which contains no nonempty zero-sum subsequence or contains no zero-sum subsequence of some special size. Many zero-sum problems are connected with factorizations in algebraic number theory, and some of them are related to graph theory, classical number theory and permutation matrices. The most famous zero-sum problems are Davenport's constant and the Erdős-Ginzburg-Ziv theorem, which are the start points of almost others zero-sum problems. The Davenport's constant was formulated by H. Davenport in 1966, and the Erdős-Ginzburg-Ziv theorem was established by P. Erdős, A. Ginzburg and A. Ziv in 1961. Later, from 1967 to

1976, H. B. Mann, J. E. Olson, P. van Emde Boas, and etc. developed zero-sum theory a great deal. Since 1990 to now, zero-sum theory has been developed very rapidly and widely. In 1996, Y. Caro made a survey paper on zero-sum results obtained mainly before 1993. However, since then a lot of zero-sum results has been obtained, many of them are great interesting and are more important.

Davenport's constant Define the Davenport's constant $D(G)$ of G to be the smallest integer d such that, every sequence of d elements in G contains a nonempty zero-sum subsequence.

Let C_n denote the cyclic group of n elements. Then

$$D(C_n) = n,$$

It is well know that G can be written in the unique form $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 | \cdots | n_r$, the r is called the *rank* of G, denote it by $r(G)$. Set $M(G) = 1 + \Sigma_{i=1}^{r}(n_i - 1)$, it is easy to see that $D(G) \geq M(G)$. Let C_n^k be the direct product of k copies of C_n.

Theorem 1 *Let G be a finite abelian group, and let p be a prime. Then $D(G) = M(G)$ provided that G is one of the following forms.*

*1. G is a p-group . (*Olson, 1969*)*

*2. $r \leq 2$. (*Olson, 1969*)*

3. $G = C_{2p^n} \oplus C_{2p^m} \oplus C_{2p^s}$. (Emde Boas and Kruyswijk, 1969*)*

4. $G = H \oplus C_{p^n m}$ with H is a p-group and $p^n \geq M(H)$. (Emde Boas and Kruyswijk, 1969*)*

5. $G = C_2 \oplus C_{2na} \oplus C_{2nb}$ with $n = 2^t 3^u 5^v 7^w, t, u, v, w \geq 0$ and either $a = 1, b$ arbitrary or $a =$

$p^r, b = p^s$. (Emde Boas and Kruyswijk, 1969)

6. $G = C_2^3 \oplus C_{2m}$ *with* m *odd .* (Baayen, 1969)

7. $G = C_3^2 \oplus C_{6m}$ *with* $3|m$. (Emde Boas and Kruyswijk, 1969)

8. $G = C_{3 \cdot 2^n} \oplus C_{3 \cdot 2^m} \oplus C_{3 \cdot 2^s}$. (Emde Boas and Kruyswijk, 1969)

9. $G = C_3 \oplus C_{6na} \oplus C_{6nb}$ *with* n, a, b *like in 5 .* (Emde Boas and Kruyswijk, 1969)

10. $G = C_2 \oplus C_{2la} \oplus C_{2lbn}$ *with* $l = 2^t 3^u 5^v 7^w$, $t, u, v, w \geq 0$, $a = p^r, b = p^s$ *for some prime* p *and* $b \geq 4la^2 - 2a$. (Chapman, Freeze, Gao and Smith, 2002).

11. $G = C_{2p^a} \oplus C_{2p^b} \oplus C_{2p^c n}$ *with* p *a prime and* $p^c \geq 4p^{a+2b} - 2p^b - p^a + 1$. (Chapman, Freeze, Gao and Smith, 2002).

12. $G = C_{3 \cdot 2^a} \oplus C_{3 \cdot 2^b} \oplus C_{3 \cdot 2^c}$ *with* $2^c \geq 9 \cdot 2^{a+2b} - 2^{b+1} - 2^a + 1$. (Chapman, Freeze, Gao and Smith, 2002).

*13. $G = C_3 \oplus C_{6la} \oplus C_{6lbn}$ with l, a, b like in 16 and $b \geq 18la^2 - 2a$. (*Chapman, Freeze, Gao and Smith, 2002*).*

Conjecture 1. $D(C_n^r) = M(C_n^r) = r(n-1)+1$. *(Emde Boas and Kruyswijk, 1969)*

Conjecture 2. *Let* $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ *with* $1 < n_1 | \cdots | n_r$. *Suppose that* $n_r | n_1^m$ *for some positive integer* m. *Then,* $D(G) = M(G)$.

Conjecture 3. *If* $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ *with* $1 < n_1 | \cdots | n_r$ *then* $D(G) \leq n_1 + \cdots + n_r$.

Theorem 2 *Let S be a zero-sum free sequence of elements in C_n with $|S| \geq \frac{n+3}{2}$. Then, there exists an element in C_n which occurs at least $\frac{n+6}{6} + \frac{1}{12}$ times in S. (Gao and Geroldinger, Combinatorica, 1998)*

Open Problem 4 Let $f(n, k)$ be the smallest integer t such that there is a zero-sum free sequence S in C_n such that $|S| = k$ and no element occurs more than t times in S. To determine $f(n, k)$.

Open Problems 5. Let $g(n, k)$ be the greatest integer t such that there is a zero-sum free sequence S in C_n such that $|S| = t$ and S contains exactly k distinct elements. To determine $g(n, k)$.

Let G be a finite abelian group of order n. Let $ZS(G)$ be the smallest integer r such that every sequence of r elements in G contains a zero-sum sequence of length n.

Theorem 3 *(Erdős, Ginzburg, Ziv, 1961)* $ZS(G) \leq 2|G| - 1$.

Theorem 4 *(Gao, 1996)* $ZS(G) = |G| + D(G) - 1$.

The following example shows that $ZS(G) \geq |G| + D(G) - 1$

$$S = (\underbrace{0, \cdots, 0}_{n-1}, a_1, \cdots, a_{D(G)-1})$$

where $n = |G|$, $a_i \in G$, and $a_1, \cdots, a_{D(G)-1}$ contains no nonempty zero-sum subsequence.

Let G be a finite abelian group. Let $\exp(G)$ be the exponent of G, i.e., the maximal order of an element in G.

Let G be a finite abelian group of order n, and let $m = \exp(G)$. Let $s(G)$ be the smallest integer r such that every sequence of r elements in G contains a zero-sum sequence of length m.
Let $\rho(G)$ to be the smallest integer ρ such that every sequence S of elements in G with $|S| \geq \rho$ contains a nonempty zero subsequence of length not exceeding m.

Theorem 5 *Let G be a finite abelian group of exponent m. Then,*

1. $s(C_m^2) = 4m - 3$ *. (*Reiher, 2004*)*

*2. There exists a absolute constant c such that, $s(G) \leq (cr \log_2^r)^d m$, where r denote the rank of G. (*Alon and Dubiner, 1995)

3. $s(G) \leq |G| + m - 1$*. (*Gao and Yang, 1997*)*

4. $s(C_3^3) = 19$*. (*Kemnitz, 1983*)*

5. $s(C_3^4) = 41$*. (*Kemnitz, 1983*)*

6. $s(C_3^5) = 89$*. (*Edel, Ferret, Landjev and Stonme, 2002*)*

7. $223 \leq s(C_3^6) \leq 227$*. (*Bierbrauer and Edel, 2002*)*

8. $s(C_{3^a}^3) = 3^{a+2} - 8$*. (*Gao and Thangadurai, 2004*)*

9. $s(G) \geq \rho(G) + m - 1$*.*

Conjecture 6. $s(G) = \rho(G) + m - 1$ holds for every finite abelian group G, where $m = \exp(G)$.

Conjecture 6 has been verified for
$G = C_2^r, C_3^r, C_4^r, C_5^3, C_6^3$

Conjecture 7. $s(C_n^3) = 9n - 8$ for every odd positive integer n.

It suffices to prove Conjecture 7 is true for all primes. Quite recently, it has been proved that

Theorem 6 $s(C_5^3) = 37 = 9 \times 5 - 8$. *(Gao, Geroldinger, Hou, Schmid and Thangadurai, 2006)*

Conjecture 7 is equivalent to that

Conjecture $7'$. **Every set of** $9p - 8$ **vectors (repetition allowed) in** F_p^3 **contains a** p**-subset with sum zero.**

Conjecture 8. Let n be a positive integer and let S be a zero-sum free sequence of $2n-2$ elements in $C_n \oplus C_n$. Then, S contains some element at least $n-2$ times. (Property B) (Gao and Geroldinger, 1999)

Theorem 7 *If Conjecture 8 is true for* $n = k$ *then it is true for* $n = 2k$. *(*Gao and Geroldinger, 2003*)*

Conjecture 8 has been verified for $n \leq 11$.

Conjecture 9. Let n be a positive integer and let S be a sequence of $3n-3$ elements in $C_n \oplus C_n$. Suppose that S contains no short zero-sum subsequence. Then, S consists of three distinct elements each appearing $n-1$ times. (Property C) (*Emde Boas and Kruyswijk, 1969*)

Conjecture 10. Let n be a positive integer and let S be a sequence of $4n-4$ elements in $C_n \oplus C_n$. Suppose that S contains zero-sum subsequence of length n. Then, S

consists of four distinct elements each appearing $n-1$ times. (Property D)

Theorem 8 *(1) Property B implies Property C. (Gao and Geroldinger 1999)*
(2) Both Property C and Property D are multiple. (Gao 2000)

Property C and Property D has been verified for $p = 2, 3, 5, 7, 11$.

Subset Sums

Theorem 9 *(Cauchy-Davenport) Let p be a prime, and let A, B be two nonempty subsets of C_p. Then,*

$$|A + B| \geq \min\{p, |A| + |B| - 1\},$$

where $A + B = \{a + b | a \in A, b \in B\}$. (Cauchy, 1813; Davenport, 1935)

Theorem 10 *Let $A_1, \cdots, A_k$ be finite nonempty subsets of an abelian group G, and let $H = \{h \in G | h + A_1 + \cdots + A_k = A_1 + \cdots + A_k$. Then,*

$$|A_1 + \cdots + A_k| \geq |A_1 + H| + \cdots + |A_k + H| - (k-1)|H|.$$

(Kneser, 1953)

Theorem 11 *Let* A, B *be two finite subsets of an abelian group. If* $A \cap (-B) = \{0\}$ *then*

$$|A + B| \geq |A| + |B| - 1.$$

(P. Scherk, 1956)

Conjecture 12 *Let* A, B *be two finite subsets of an abelian group. If* $A \cap (-B) = \{0\}$ *then*

$$|A \dot{+} B| \geq |A| + |B| - 3,$$

where $|A \dot{+} B| = \{a + b | a \in A, b \in B, a \neq b\}$

Theorem 13 *Let p be a prime, and let A be a nonempty subsets of C_p with $1 \leq |A| \leq p - 1$. Then, for every positive integer $\ell \in \{1, \cdots, |A|\}$ we have*

$$|\ell^* A| \geq \min\{p, k(p-k)+1\},$$

where $\ell^* A = \{a_1 + \cdots + a_\ell | a_i \in A, a_1, \cdots, a_\ell$ are pairwise distinct $\}$. (Silva and Hamidoune, 1994)

Conjecture 14 *Let A be a finite subset of positive integers. Then,* $\max\{|A+A|, |A\cdot A|\} \geq c|A|^{2-\delta}$ *for every positive δ. (Erdos, Szemeredi, 1983)*

Theorem 15 $\max\{|A+A|, |A\cdot A|\} \geq \frac{cn^{\frac{14}{11}}}{(\log n)^{\frac{3}{11}}}$, *where $n = |A|$. (Solymosi, 2006).*

$$A \cdot A = \{a \cdot b | a \in A, b \in B\}.$$

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