Arithmetic Progressions, Zero-sums and Subset Sums

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Arithmetic Progressions

Theorem 1 *The primes contain arbitrarily long arithmetic progressions. (B. Green and T. Tao, Annals of Mathematics)*

Conjecture 2 *If* $\Sigma_{n=1}^{\infty} \frac{1}{a_n}$ *is divergent, then the sequence* $\{a_n\}_{n=1}^{\infty}$ *contains arbitrarily long arithmetic progressions. (Erdős)*

Definition 1. Define $r_3(N)$ to be the cardinality of the largest $A \subseteq \{1, 2, \cdots, N\}$ containing no 3-term arithmetic progression. (Erdős, Turan, 1936)

$[N] = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}, A = \{1, 2, 5, 7, 10\}$. $r_3(10) = 5$.

Theorem 3 $r_3(N) \ll N(\log\log N/\log N)^{\frac{1}{2}}$. *(Bourgain, Israel J Math, 1989)*

Theorem 4 $r_3(N) \geq Ne^{-c\sqrt{\log N}}$ *for some absolute constant c. (Behrend, 1946)*

Conjecture 5 *$r_3(N) \leq Ne^{-c\sqrt{\log N}}$ for some absolute constant c.*

We even do not know if $r_3(N) \leq cN/\log N$.

Theorem 6 *Every subset of primes of positive upper density contains a 3-terms progression. (Been Green, Annals of Mathematics, 2005)*

Density Let P be the set of primes, and $[n] = \{1, 2, \cdots, n\}$. Let A be a subset of P. Set $P(n) = |P \cap [n]| = \pi(n)$, and set $A(n) = |A \cap [n]|$. Define

$$d(A) = \overline{sup}\frac{A(n)}{P(n)}.$$

Let A be a set of positive integers. Define

$$S_A = \{ \sum_{x \in B} | B \subset A, |B| < \infty \}.$$

Theorem 7 *There is a positive constant c such that the following holds. For any positive integer , if A is a subset of* $[n]$ *with at least* $cn^{\frac{1}{2}}$ *elements, then* S_A *contains an arithmetic progression of length* n *.*

(E. Szemerédi, V.H. Vu,
Annals of Mathematics, 2006)

An infinite set A is complete if S_A contains every sufficiently large positive integer.

Theorem 8 *There is a constant c such that the following holds. Any increasing sequence A of positive integers satisfying*
(a) $A(n) \geq cn^{\frac{1}{2}}$
(b) S_A *contains an element of every infinite arithmetic progression,*
is complete.
(E. Szemerédi, V.H. Vu, Annals of Mathematics, 2006, conjectured by Erdos 1962)