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Optimal Strong (κ, d)-Orientation of Complete k-Partite Graphs

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1 Introduction

- The familiar distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest (u, v)-path in G. Equivalently, the distance is the minimum size of a connected subgraph of G containing u and v.

- Using this equivalent formulation of distance, this concept was extended by Chartrand *et al.*[2] to strongly connected digraphs, in particular to strong oriented graphs.

♦ Let u, v be vertices of strong oriented graph D. The strong distance $sd_D(u,v)$(or simply $sd(u,v)$) between u and v is defined as the minimum size of a strong sub-digraph of D containing u and v.

In Figure1, $sd(w,v) = 3$, $sd(u,w) = 5$, $sd(u,x) = 6$.

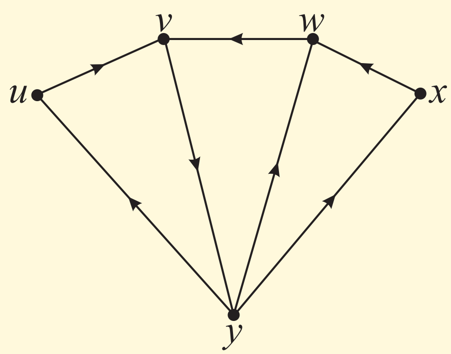

Figure1: Strong distance in a strong digraph.

♦ The strong eccentricity $se_D(v)$(or simply $se(v)$) of a vertex v in a strong oriented graph D is

$$se(v) = \max\{sd(v, x) \mid x \in V(D)\}.$$

The strong diameter $sdiam(D)$ of D is

$$sdiam(D) = \max\{se(v) \mid v \in V(D)\}.$$

♦ For a connected graph G, Lai *et al.*[5] defined the lower orientable strong diameter $sdiam(G)$ of G as

$$sdiam(G) = \min\{sdiam(D) \mid D \text{ is a strong orientation of } G\}.$$

- Chartrand *et al.* gave an upper bound on the strong diameter of a strong oriented graph D.

Theorem 1.1:(Chartrand *et al.*[2]) If D is a strong oriented graph of order $n \geq 3$, then

$$sdiam(D) \leq \lfloor \frac{5(n-1)}{3} \rfloor.$$

- Dankelmann *et al.* showed the strong diameter and strong connectivity κ of a strong oriented graph satisfy the following inequality.

Theorem 1.2:(Dankelmann *et al.*[4]) Let D be a strong oriented graph of order n and $\kappa(D) = \kappa$, then

$$sdiam(D) \leq \frac{5}{3}(1 + \frac{n-2}{\kappa}).$$

♦ Let $K(m_1, m_2, \ldots, m_k)$ be a complete k-partite graph with vertex partition of cardinalities m_1, m_2, $\ldots$, m_k, where $k \geq 2$.

♦ In [6], the present authors gave the lower orientable strong diameter of complete k-partite graphs.

Theorem 1.3:(Miao and Guo[6]) Let $2 \leq m_1 \leq m_2$. Then

$$sdiam(K_{m_1,m_2}) = \begin{cases} 4, & \text{if } m_1 \leq m_2 \leq \binom{m_1}{\lfloor \frac{m_1}{2} \rfloor}, \\ 6, & \text{if } m_2 > \binom{m_1}{\lfloor \frac{m_1}{2} \rfloor}. \end{cases}$$

Theorem 1.4:(Miao and Guo[6]) Let $k \geq 3$, $1 \leq m_1 \leq m_2 \leq \cdots \leq m_k$, where $m_k \geq 2$, and let $m = m_1 + m_2 + \cdots + m_{k-1}$. Then

$$sdiam(K(m_1, m_2, \ldots, m_k)) = \begin{cases} 4, & \text{if } \binom{m}{\lfloor \frac{m}{2} \rfloor} \geq m_k, \\ 5, & \text{if } \binom{m}{\lfloor \frac{m}{2} \rfloor} < m_k. \end{cases}$$

♦ For vertex-strong connectivity of a digraph D(denoted by $\kappa(D)$), Thomossen [7] posed the following conjecture.

Conjecture 1.5: (Thomossen [7]) Every $2k$-strong digraph has a k-strong orientation.

♦ We will give a weaker form of this conjecture.

Conjecture 1.6: Every m-connected graph has a $\lfloor m/2 \rfloor$-strong orientation.

- Let $k \geq 2$, $m_1 \leq m_2 \leq \cdots \leq m_k$ and $m = m_1 + m_2 + \cdots + m_{k-1} \geq 2$. Then $\kappa(K(m_1, m_2, \ldots, m_k)) = \delta(K(m_1, m_2, \ldots, m_k)) = m$. We have known that for any orientation K of $K(m_1, m_2, \ldots, m_k)$, $\kappa(K) \leq \min\{\delta^-(K), \delta^+(K)\} \leq \lfloor m/2 \rfloor = \lfloor \kappa(K(m_1, m_2, \ldots, m_k))/2 \rfloor$.

- Note that there exists some orientation of $K(m_1, m_2, \ldots, m_k)$ which satisfies Conjecture 1.6, that is, the upper bound $\lfloor m/2 \rfloor$ can be obtained, but its strong diameter is more than the lower orientable strong diameter of $K(m_1, m_2, \ldots, m_k)$. Furthermore, there exists some orientation K of $K(m_1, m_2, \ldots, m_k)$ in which $sdiam(K)$ equals to the lower orientable strong diameter of $K(m_1, m_2, \ldots, m_k)$, but its strong connectivity is less than $\lfloor m/2 \rfloor$.

♦ The digraph D_1 shown in Figure 2 is a strong orientation of $K(2,2,2)$. By the orientation of D_1, we know that $\kappa(D_1) = 2 = \lfloor \kappa(K(2,2,2))/2 \rfloor$. But vertices u and v cannot be contained in any directed 3-cycle or 4-cycle in D_1. Furthermore, by Theorem 1.4, we have $sdiam(K(2,2,2)) = 4$. Hence, $sdiam(D_1) \geq sd(u,v) > sdiam(K(2,2,2))$.

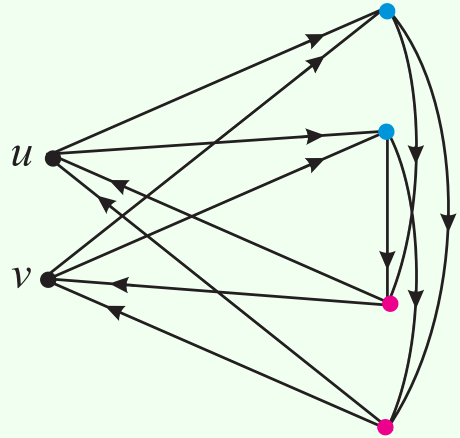

Figure 2: The strong orientation D_1 of $K(2,2,2)$ with $\kappa(D_1) = \lfloor \kappa(K(2,2,2))/2 \rfloor$,

$$sdiam(D_1) > sdiam(K(2,2,2)).$$

♦ The digraph D_2 shown in Figure 3 is a strong orientation of $K(4,4)$. By the orientation of D_2, we know that any two vertices are contained in a directed 4-cycle. And by Theorem 1.3, $4 \geq sdiam(D_2) \geq sdiam(K(4,4)) = 4$. Hence, $sdiam(D_2) = sdiam(K(4,4)) = 4$. But for any vertex $u \in V(D_2)$, $\min\{\delta^+(u), \delta^-(u)\} = 1$. Hence, $\kappa(D_2) = 1 < \lfloor \kappa(K(4,4))/2 \rfloor = 2$.

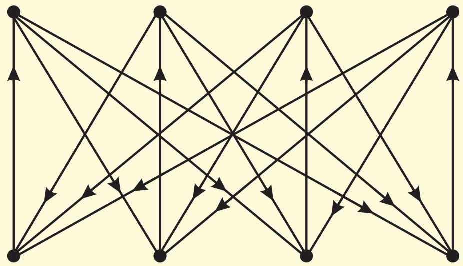

Figure 3: The strong orientation D_2 of $K(4,4)$ with $\kappa(D_2) < \lfloor \kappa(K(4,4))/2 \rfloor$, $sdiam(D_2) = sdiam(K(4,4))$.

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♦ An orientation D of a graph G is defined by optimal strong (κ, d)-orientation of G if and only if $\kappa(D) = \lfloor \kappa(G)/2 \rfloor$ and $sdiam(D) = sdiam(G)$. We have the following main result.

♦ **Theorem 1.7:** Let $k \geq 2$, $m_1 \leq m_2 \leq \cdots \leq m_k$, $m = m_1 + m_2 + \cdots + m_{k-1} \geq 2$. There exists an optimal strong (κ, d)-orientation of $K(m_1, m_2, \ldots, m_k)$.

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2 Proof of the Main Result

- A family $\mathcal{F}$ of subsets of $\{1, 2, \ldots, n\}$ is an *antichain* if no set in $\mathcal{F}$ is contained in another. The following is the well-known Sperner's Lemma.

  Sperner's Lemma: Let $\mathcal{F}$ be an antichain on $\{1, 2, \ldots, n\}$. Then $|\mathcal{F}| \le \binom{n}{\lfloor n/2 \rfloor}$. The bound is attained by taking $\mathcal{F}$ to be the family of all subsets of size $\lfloor n/2 \rfloor$.

- We will give the proof of Theorem 1.7 in terms of the relation between m and m_k, where $m = m_1 + \cdots + m_{k-1}$.

3 Examples

♦ $m \le m_k \le \binom{m}{\lfloor m/2 \rfloor}$.

- Figure 4 gives the optimal strong (κ, d)-Orientation of $K(5,5)$.

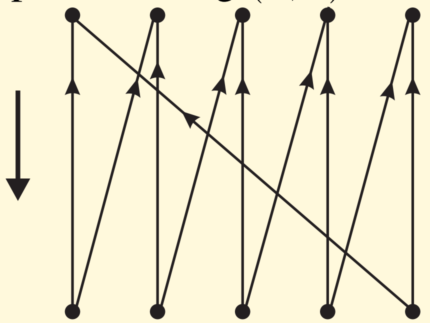

Figure 4: The optimal strong (κ, d)-Orientation of $K(5,5)$.

- Figure 5 gives the optimal strong (κ, d)-Orientation of $K(2,2,6)$.

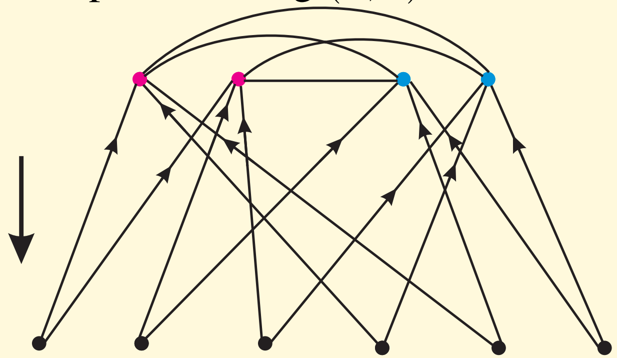

Figure 5: The optimal strong (κ, d)-Orientation of $K(2,2,6)$.

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♦ $m_k > \binom{m}{\lfloor m/2 \rfloor}$.

- Figure 6 gives the optimal strong (κ, d)-Orientation of $K(3, 4)$.

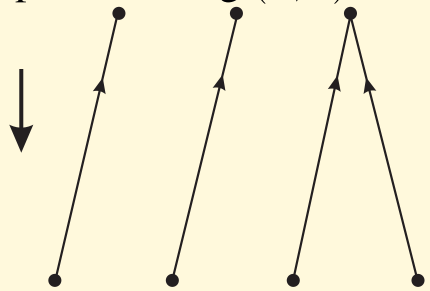

Figure 6: The optimal strong (κ, d)-Orientation of $K(3, 4)$.

- Figure 7 gives the optimal strong (κ, d)-Orientation of $K(2, 2, 7)$.

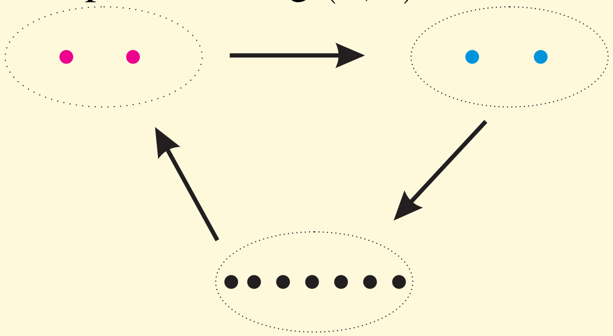

Figure 7: The optimal strong (κ, d)-Orientation of $K(2, 2, 7)$.

♦ $m_{k-1} \le m_k < m$.

- Figure 8 gives the optimal strong (κ, d)-Orientation of $K(2,4,5)$.

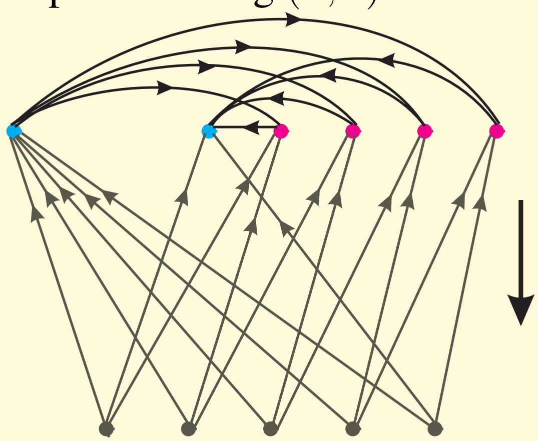

Figure 8: The optimal strong (κ, d)-Orientation of $K(2,4,5)$.

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