Optimal Strong $(\kappa, d)$-Orientation of Complete $\kappa$-Partite Graphs

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1 Introduction

♦ The familiar distance $d(u,v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $(u,v)$-path in $G$. Equivalently, the distance is the minimum size of a connected subgraph of $G$ containing $u$ and $v$.

♦ Using this equivalent formulation of distance, this concept was extended by Chartrand et al.[2] to strongly connected digraphs, in particular to strong oriented graphs.
Let $u, v$ be vertices of strong oriented graph $D$. The strong distance $sd_D(u, v)$ (or simply $sd(u, v)$) between $u$ and $v$ is defined as the minimum size of a strong sub-digraph of $D$ containing $u$ and $v$.

In Figure 1, $sd(w, v) = 3$, $sd(u, w) = 5$, $sd(u, x) = 6$.

**Figure 1**: Strong distance in a strong digraph.
The strong eccentricity $se_D(v)$ (or simply $se(v)$) of a vertex $v$ in a strong oriented graph $D$ is

$$se(v) = \max\{sd(v, x) \mid x \in V(D)\}.$$ 

The strong diameter $sdiam(D)$ of $D$ is

$$sdiam(D) = \max\{se(v) \mid v \in V(D)\}.$$ 

For a connected graph $G$, Lai et al. [5] defined the lower orientable strong diameter $sdiam(G)$ of $G$ as

$$sdiam(G) = \min\{sdiam(D) \mid D \text{ is a strong orientation of } G\}.$$
Chartrand et al. gave an upper bound on the strong diameter of a strong oriented graph $D$.

**Theorem 1.1:** (Chartrand et al.[2]) If $D$ is a strong oriented graph of order $n \geq 3$, then

$$sdiam(D) \leq \left\lfloor \frac{5(n - 1)}{3} \right\rfloor.$$  

Dankelmann et al. showed the strong diameter and strong connectivity $\kappa$ of a strong oriented graph satisfy the following inequality.

**Theorem 1.2:** (Dankelmann et al.[4]) Let $D$ be a strong oriented graph of order $n$ and $\kappa(D) = \kappa$, then

$$sdiam(D) \leq \frac{5}{3} \left(1 + \frac{n - 2}{\kappa}\right).$$
Let $K(m_1, m_2, \ldots, m_k)$ be a complete $k$-partite graph with vertex partition of cardinalities $m_1, m_2, \ldots, m_k$, where $k \geq 2$.

In [6], the present authors gave the lower orientable strong diameter of complete $k$-partite graphs.

**Theorem 1.3:** (Miao and Guo [6]) Let $2 \leq m_1 \leq m_2$. Then

$$sdiam(K_{m_1,m_2}) = \begin{cases} 4, & \text{if } m_1 \leq m_2 \leq \left(\frac{m_1}{[m_1/2]}\right), \\ 6, & \text{if } m_2 > \left(\frac{m_1}{[m_1/2]}\right). \end{cases}$$

**Theorem 1.4:** (Miao and Guo [6]) Let $k \geq 3$, $1 \leq m_1 \leq m_2 \leq \cdots \leq m_k$, where $m_k \geq 2$, and let $m = m_1 + m_2 + \cdots + m_{k-1}$. Then

$$sdiam(K(m_1, m_2, \ldots, m_k)) = \begin{cases} 4, & \text{if } \left(\frac{m}{[m/2]}\right) \geq m_k, \\ 5, & \text{if } \left(\frac{m}{[m/2]}\right) < m_k. \end{cases}$$
For vertex-strong connectivity of a digraph $D$ (denoted by $\kappa(D)$), Thomassen [7] posed the following conjecture.

**Conjecture 1.5:** (Thomassen [7]) Every $2k$-strong digraph has a $k$-strong orientation.

We will give a weaker form of this conjecture.

**Conjecture 1.6:** Every $m$-connected graph has a $\lfloor m/2 \rfloor$-strong orientation.
Let $k \geq 2$, $m_1 \leq m_2 \leq \cdots \leq m_k$ and $m = m_1 + m_2 + \cdots + m_{k-1} \geq 2$. Then $\kappa(K(m_1, m_2, \ldots, m_k)) = \delta(K(m_1, m_2, \ldots, m_k)) = m$. We have known that for any orientation $K$ of $K(m_1, m_2, \ldots, m_k)$, $\kappa(K) \leq \min\{\delta^-(K), \delta^+(K)\} \leq \lfloor m/2 \rfloor = \lfloor \kappa(K(m_1, m_2, \ldots, m_k))/2 \rfloor$.

Note that there exists some orientation of $K(m_1, m_2, \ldots, m_k)$ which satisfies Conjecture 1.6, that is, the upper bound $\lfloor m/2 \rfloor$ can be obtained, but its strong diameter is more than the lower orientable strong diameter of $K(m_1, m_2, \ldots, m_k)$. Furthermore, there exists some orientation $K$ of $K(m_1, m_2, \ldots, m_k)$ in which $s\text{diam}(K)$ equals to the lower orientable strong diameter of $K(m_1, m_2, \ldots, m_k)$, but its strong connectivity is less than $\lfloor m/2 \rfloor$. 
The digraph $D_1$ shown in Figure 2 is a strong orientation of $K(2, 2, 2)$. By the orientation of $D_1$, we know that $\kappa(D_1) = 2 = \lceil \kappa(K(2, 2, 2))/2 \rceil$. But vertices $u$ and $v$ cannot be contained in any directed 3-cycle or 4-cycle in $D_1$. Furthermore, by Theorem 1.4, we have $sdiam(K(2, 2, 2)) = 4$. Hence, $sdiam(D_1) \geq sd(u, v) > sdiam(K(2, 2, 2))$.

Figure 2: The strong orientation $D_1$ of $K(2, 2, 2)$ with $\kappa(D_1) = \lceil \kappa(K(2, 2, 2))/2 \rceil$, $sdiam(D_1) > sdiam(K(2, 2, 2))$. 
The digraph $D_2$ shown in Figure 3 is a strong orientation of $K(4, 4)$. By the orientation of $D_2$, we know that any two vertices are contained in a directed 4-cycle. And by Theorem 1.3, $4 \geq sdiam(D_2) \geq sdiam(K(4, 4)) = 4$. Hence, $sdiam(D_2) = sdiam(K(4, 4)) = 4$. But for any vertex $u \in V(D_2)$, $\min\{\delta^+(u), \delta^-(u)\} = 1$. Hence, $\kappa(D_2) = 1 < \lceil \kappa(K(4, 4))/2 \rceil = 2$.

Figure 3: The strong orientation $D_2$ of $K(4, 4)$ with $\kappa(D_2) < \lceil \kappa(K(4, 4))/2 \rceil$,

$$sdiam(D_2) = sdiam(K(4, 4)).$$
An orientation $D$ of a graph $G$ is defined by optimal strong $(\kappa, d)$-orientation of $G$ if and only if $\kappa(D) = \lceil \kappa(G)/2 \rceil$ and $sdiam(D) = sdiam(G)$. We have the following main result.

**Theorem 1.7:** Let $k \geq 2$, $m_1 \leq m_2 \leq \cdots \leq m_k$, $m = m_1 + m_2 + \cdots + m_{k-1} \geq 2$. There exists an optimal strong $(\kappa, d)$-orientation of $K(m_1, m_2, \ldots, m_k)$. 
2 Proof of the Main Result

A family $\mathcal{F}$ of subsets of $\{1, 2, \ldots, n\}$ is an antichain if no set in $\mathcal{F}$ is contained in another. The following is the well-known Sperner’s Lemma.

Sperner’s Lemma: Let $\mathcal{F}$ be an antichain on $\{1, 2, \ldots, n\}$. Then $|\mathcal{F}| \leq \left(\binom{n}{\lfloor n/2 \rfloor}\right)$. The bound is attained by taking $\mathcal{F}$ to be the family of all subsets of size $\lfloor n/2 \rfloor$.

We will give the proof of Theorem 1.7 in terms of the relation between $m$ and $m_k$, where $m = m_1 + \cdots + m_{k-1}$.
3 Examples

\[ m \leq m_k \leq \left( \frac{m}{\lfloor m/2 \rfloor} \right). \]

- Figure 4 gives the optimal strong \((\kappa, d)\)-Orientation of \(K(5, 5)\).

![Figure 4: The optimal strong \((\kappa, d)\)-Orientation of \(K(5, 5)\).]

- Figure 5 gives the optimal strong \((\kappa, d)\)-Orientation of \(K(2, 2, 6)\).

![Figure 5: The optimal strong \((\kappa, d)\)-Orientation of \(K(2, 2, 6)\).]
$m_k > \left( \frac{m}{\lfloor m/2 \rfloor} \right)$.

- Figure 6 gives the optimal strong $(\kappa, d)$-Orientation of $K(3, 4)$.

![Figure 6: The optimal strong $(\kappa, d)$-Orientation of $K(3, 4)$.

- Figure 7 gives the optimal strong $(\kappa, d)$-Orientation of $K(2, 2, 7)$.

![Figure 7: The optimal strong $(\kappa, d)$-Orientation of $K(2, 2, 7)$.
\[ m_{k-1} \leq m_k < m. \]

- Figure 8 gives the optimal strong \((\kappa, d)\)-Orientation of \(K(2, 4, 5)\).

\[ \text{Figure 8: The optimal strong } (\kappa, d)\text{-Orientation of } K(2, 4, 5). \]
References


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