

Toughness in graphs: structural and algorithmic aspects

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Motivation

- ▶ Most of the research on toughness is based on a number of **conjectures** in the following paper:

V. Chvátal,
Tough graphs and hamiltonian circuits,
Discrete Mathematics **5** (1973) 215–228.

- ▶ We will discuss most of the conjectures and the **progress** made towards proving or refuting them.
- ▶ More details can be found in a **survey paper** by Bauer, B. and Schmeichel that appeared in **Graphs and Combinatorics**, volume **22** (2006) 1–35 .
- ▶ The survey contains more than 160 references to papers related to toughness.

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- ▶ Hence if G is not complete, $\tau(G) = \min\{|S|/\omega(G - S)\}$, where the minimum is taken over all **cut sets** of vertices in G .

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- ▶ It is not difficult to check that **a hamiltonian graph is 1-tough**.
- ▶ Is there a **level** of toughness that **guarantees** the existence of **k -factors** or **Hamilton cycles**?

Toughness and factors

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- ▶ He observed that the **validity for $k = 1$** follows from **Tutte's matching theorem**.
- ▶ All questions were answered by the following two results.

Chvátal's conjecture is true and sharp

Theorem (Enomoto, Jackson, Katerinis, Saito '85)

Let G be a k -tough graph on n vertices with $n \geq k + 1$ and kn even. Then G has a k -factor.

Theorem (Enomoto, Jackson, Katerinis, Saito '85)

Let $k \geq 1$. For every $\epsilon > 0$, there exists a $(k - \epsilon)$ -tough graph G on n vertices with $n \geq k + 1$ and kn even which has no k -factor.

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- ▶ So what about **2-tough** graphs?

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- ▶ As we have seen, **Enomoto et al.** have found $(2 - \epsilon)$ -tough graphs having no 2-factor for arbitrary $\epsilon > 0$.
- ▶ So what about **2-tough** graphs?
- ▶ Another conjecture of **Chvátal** was that every 2-tough graph in which every vertex has a **connected neighbourhood**, is hamiltonian.

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- ▶ Moreover, it would imply the truth of two other conjectures that have been open for about twenty years:
 - ▶ Every 4-connected line graph is hamiltonian (**Thomassen**).
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- ▶ However, it turns out that **not** all 2-tough graphs are hamiltonian.

Counterexamples

Theorem (Bauer, B., Veldman 2000)

For every $\epsilon > 0$, there exists a $(\frac{9}{4} - \epsilon)$ -tough nonhamiltonian graph.

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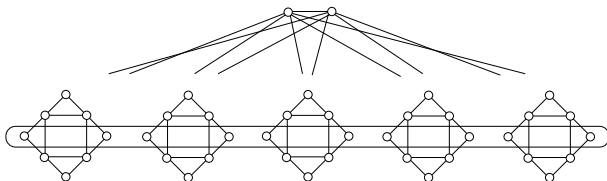


Figure: A 2-tough nonhamiltonian graph

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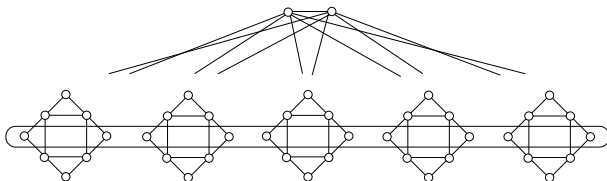


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All counterexamples are **neighbourhood-connected**, so the weaker conjecture of Chvátal is also false.

The t_0 -tough conjecture

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- ▶ This conjecture is **still (wide) open** for general graphs, but it is true within many special graph classes:

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 - ▶ There exist $(\frac{7}{4} - \epsilon)$ -tough chordal **nonhamiltonian** graphs ([Bauer](#), [B.](#), [Veldman](#)).
 - ▶ Every t -tough **planar chordal graph** with $t > 1$ is hamiltonian ([Böhme](#), [Harant](#), [Tkáč](#)).

Degree conditions

- ▶ Many known sufficient conditions for the existence of a Hamilton cycle are based on the **degrees** of the vertices of a graph.
- ▶ The earliest ones are due to **Dirac** and to **Ore**:

Theorem (Dirac '52)

Let G be a graph on $n \geq 3$ vertices with $\delta \geq \frac{n}{2}$. Then G is hamiltonian.

Theorem (Ore '60)

Let G be a graph on $n \geq 3$ vertices with $\sigma_2 \geq n$. Then G is hamiltonian.

- ▶ These conditions can be **weakened** if one imposes a **toughness condition**. We give a few examples.

Weakened degree conditions

Theorem (Jung '78)

Let G be a 1-tough graph on $n \geq 11$ vertices with $\sigma_2 \geq n - 4$. Then G is hamiltonian.

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Theorem (Bauer, B., Van den Heuvel, Veldman '95)

Let G be a t -tough graph on $n \geq 3$ vertices with $\delta > n/(t + 1) - 1$. Then G is hamiltonian.

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- ▶ This implies that [Chvátal's](#) t_0 -tough conjecture is true within the class of graphs having $\delta(G) \geq \epsilon n$, for any fixed $\epsilon > 0$.

Toughness and pancyclicity

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There exist arbitrarily tough, triangle-free graphs.

- ▶ Subsequently, Alon proved a stronger result.

Theorem (Alon '95)

For every t and g there exists a t -tough graph of girth greater than g .

Toughness and complexity

- ▶ The problem of determining the complexity of recognizing t -tough graphs was first raised by [Chvátal](#), but not in the original paper.
- ▶ The question later appeared in a paper by [Thomassen](#) '81.
- ▶ Consider the following **decision problem**, where t is any positive rational number.

t-TOUGH

INSTANCE: Graph G .

QUESTION: Is $\tau(G) \geq t$?

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- ▶ The complexity was settled about ten years later:

Theorem (Bauer, Hakimi, Schmeichel '90)

For any positive rational number t , t -TOUGH is NP-hard.

Special graph classes

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- ▶ For example, [Matthews and Sumner '84](#) have shown that for **claw-free graphs**, $\tau = \kappa/2$.
- ▶ Hence the toughness of claw-free graphs, and consequently of **line graphs**, can be determined in **polynomial** time.
- ▶ Thus, while it is NP-complete to determine whether a line graph is hamiltonian ([Bertossi '81](#)), it is polynomial to determine whether a line graph is 1-tough.

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Theorem (Kratsch, Lehel, Müller '96)

The class of 1-tough split graphs can be recognized in polynomial time.

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- ▶ On the other hand, we have the following results:

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The class of 1-tough split graphs can be recognized in polynomial time.

Theorem (Woeginger '98)

For any rational number $t \geq 0$, the class of t -tough split graphs can be recognized in polynomial time.

Other graph classes

- ▶ For many subclasses of graphs, however, it is NP-hard to recognize t -tough graphs, e.g., for graphs having **minimum degree** “almost” high enough to ensure that they are t -tough:

Theorem (Bauer, Morgana, Schmeichel '94)

Let $t \geq 1$ be a rational number. If $\delta \geq (\frac{t}{t+1})n$, then G is t -tough. On the other hand, for any fixed $\epsilon > 0$, it is NP-hard to determine whether G is t -tough for graphs G with $\delta \geq (\frac{t}{t+1} - \epsilon)n$.

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Theorem (Kratsch, Lehel, Müller '96)

1-TOUGH remains NP-hard for bipartite graphs.

- ▶ Consequently, 1-TOUGH is also NP-hard for the larger class of **triangle-free graphs**.

Regular graphs

- ▶ An important class of graphs that has received considerable attention is the class of **regular graphs**. Note that the maximum possible toughness of an r -regular noncomplete graph is $r/2$.
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- ▶ But for all odd $r \geq 5$, **Doty '91** and **Jackson and Katerinis '94** independently constructed an infinite family of r -regular, $r/2$ -tough graphs on n vertices with $n \not\equiv 0 \pmod{r}$.

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- ▶ The complexity of recognizing r -regular, t -tough graphs remains completely **open** when $2t < r < 3t$, and the complexity when $r = 2t + 1$ seems especially intriguing.

Conclusions and open problems

- ▶ Since **Chvátal** introduced toughness in 1973, much research has been done that relates toughness conditions to the existence of cycle structures.
- ▶ In our **survey paper**, we have gathered many of the important results in this area.
- ▶ Historically, the motivation for this research was based on a number of **conjectures** in **Chvátal's** paper.
- ▶ The most challenging of these conjectures is **still open**:

Is there a finite constant t_0 such that every t_0 -tough graph is hamiltonian? If so, what is the smallest such t_0 ?

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- ▶ However **Matthews and Sumner** '84 have conjectured that 4-connected (2-tough) claw-free graphs are hamiltonian.

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- ▶ The **gaps** in our knowledge for claw-free and chordal graphs imply a number of challenging **open problems**.
- ▶ The same is true for the class of triangle-free graphs.

Triangle-free graphs

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- ▶ It even appears that a class of triangle-free graphs with no 2-factor constructed by [Bauer et al. '96](#) has toughness approaching 2 from below.
- ▶ These examples suggest the **intriguing** possibility that every 2-tough triangle-free graph is hamiltonian.
- ▶ On the other hand, it remains **completely open** whether the t_0 -tough conjecture holds for the class of triangle-free graphs.

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- ▶ We know that if G is a 2-tough graphs on n vertices with $\delta \geq n/3$, then G is hamiltonian.
- ▶ What if $5 \leq \delta < n/3$?
- ▶ The early research on toughness and cycle structure concentrated on sufficient **degree conditions** which, combined with a certain level of **toughness**, would yield the existence of **long cycles**.
- ▶ The survey paper contains a wealth of results in this direction.
- ▶ One of the **major open problems** in this area is the conjecture that every 1-tough graph on n vertices with $\sigma_3 \geq n \geq 3$ has a cycle of length at least $\min\{n, (3n + 1)/4 + \sigma_3/6\}$.

Other open problems

- ▶ Another interesting problem is to find the best possible **minimum degree condition** to ensure that a 1-tough triangle-free graph is hamiltonian.
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Other open problems

- ▶ Another interesting problem is to find the best possible **minimum degree condition** to ensure that a 1-tough triangle-free graph is hamiltonian.
- ▶ We know the answer lies somewhere between $(n + 2)/4$ and $(n + 1)/3$.
- ▶ If we do not impose a degree condition, toughness conditions can still guarantee **cycles of length proportional to a function of the number of vertices** of the graph.
- ▶ Two of the most challenging open problems in this area are whether there exist positive constants A and B , depending only on t , such that every 2-connected, respectively 3-connected, t -tough graph on n vertices has a cycle of length at least $A \log n$, respectively n^B .
- ▶ Both problems have affirmative solutions for **planar graphs**.

Toughness and factors

- ▶ Another area of research has involved finding toughness conditions for the existence of certain **factors** in graphs.
- ▶ One of the challenging open problems in this area is to determine whether every $3/2$ -tough **maximal planar graph** has a 2-factor.
- ▶ If so, are they all hamiltonian?
- ▶ We also do not know whether every $3/2$ -tough planar graph has a 2-factor.

Computational complexity

- ▶ Research on toughness has also focused on computational **complexity issues**.
- ▶ In particular, we now know that recognizing t -tough graphs is **NP-hard in general**, whereas it is polynomial within the class of claw-free graphs and within the class of split graphs.
- ▶ For many other interesting classes, this complexity question is still **open**, e.g., for **(maximal) planar** graphs and for **chordal** graphs.

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- ▶ For many other interesting classes, this complexity question is still **open**, e.g., for **(maximal) planar** graphs and for **chordal** graphs.
- ▶ Within the class of **r -regular** graphs with $r \geq 3t$, recognizing t -tough graphs has been shown to be NP-hard.
- ▶ The problem is trivial if $r < 2t$, but its complexity is **open** for values of r with **$2t \leq r < 3t$** .
- ▶ It was conjectured by **Goddard and Swart '91** to be polynomial for $r = 2t$, and seems especially interesting when $r = 2t + 1$.

Main conclusion

There are still **many tough problems** to be solved!!