Color Degree and Color Neighborhood Condition for Large Heterochromatic Matchings in Edge-Colored Bipartite Graphs

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1 Introduction

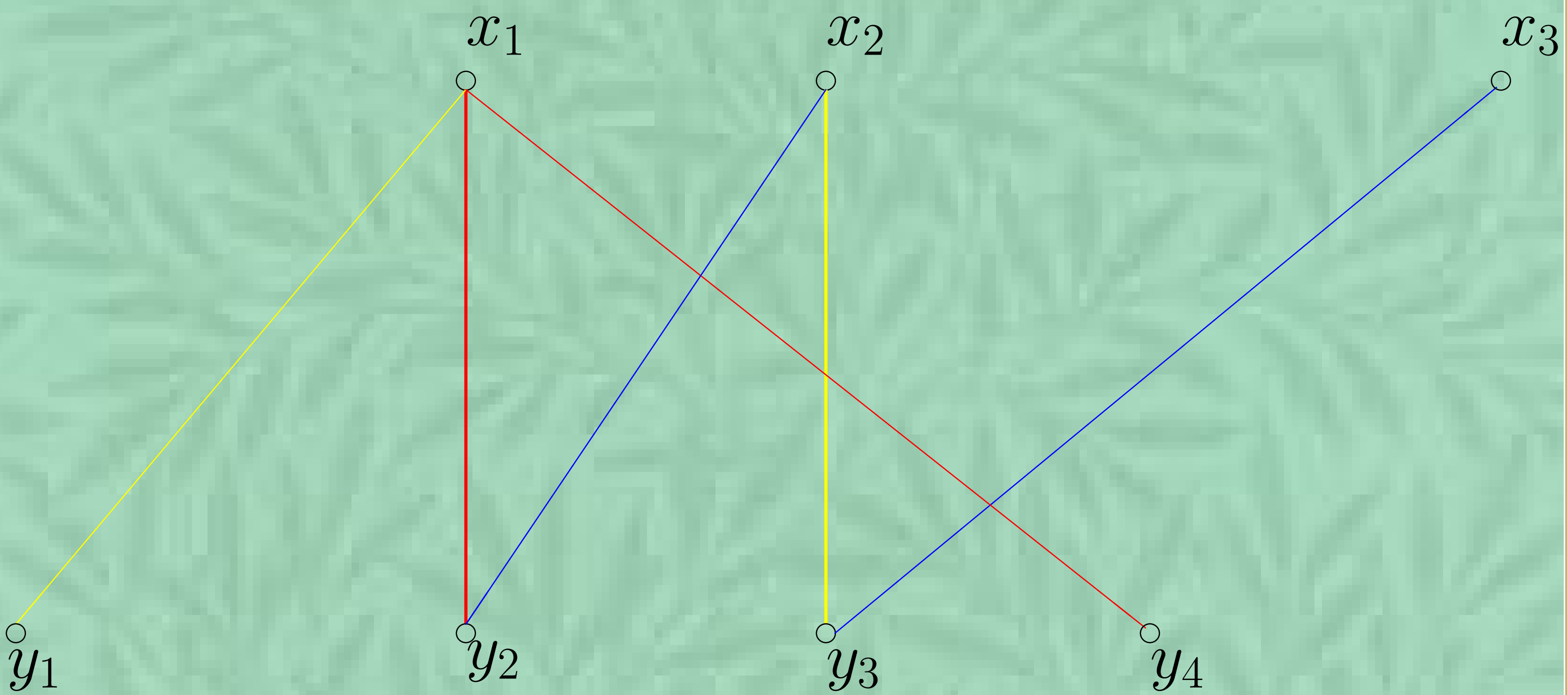
Let $G = (V, E)$ be a graph. By an *edge-coloring* of G we mean a surjective function $C : E \to \{1, 2, \cdots, r\}$, the set of nonnegative integers. If G is assigned such a coloring, then G is called an *edge-colored graph* or, simply *colored graph*. Denote the colored graph by (G, C). We call $C(e)$ the *color* of the edge $e = uv \in E$. Note that C is not necessarily a proper edge-coloring, i.e., two adjacent edges may have the same color. For a vertex v of G, the *color neighborhood* $CN(v)$ of v is defined as the set $\{C(e) : e \text{ is incident with } v\}$. Then, $CN(S) = \bigcup_{v \in S} CN(v)$ for $S \subseteq V$. For a subgraph H of G, let $C(H) = \{C(e) : e \in E(H)\}$. A subgraph H of G is called *monochromatic* if its any two edges have the same color, whereas H is called *heterochromatic*, or *rainbow*, or *colorful* if its any two edges have different colors.

For a vertex v of G, the *color neighborhood* $CN(v)$ of v is defined as the set $\{C(e) : e$ is incident with $v\}$ and the *color degree* $d^c(v) = |CN(v)|$.

Let (B, C) be a colored bipartite graph with bipartition (X, Y). For a vertex set $S \subseteq X$ or Y, a *color neighborhood* of S is defined as a set $T \subseteq N_B(S)$ such that there are $|T|$ edges between S and T that are adjacent to distinct vertices of T and have distinct colors. A *maximum color neighborhood* $N^c(S)$ is a color neighborhood of S and $|N^c(S)|$ is maximum. Given an S and a color neighborhood T, denote by $C(S, T)$ a set of $|T|$ distinct colors on the $|T|$ edges between S and distinct vertices of T. Note that there might be more than one such T, $N^c(S)$ and set $C(S, T)$.

$$S = \{x_1, x_2\}, \qquad T = \{y_1, y_2\}$$

$$N^c(S) = \{y_1, y_2, y_4\}$$



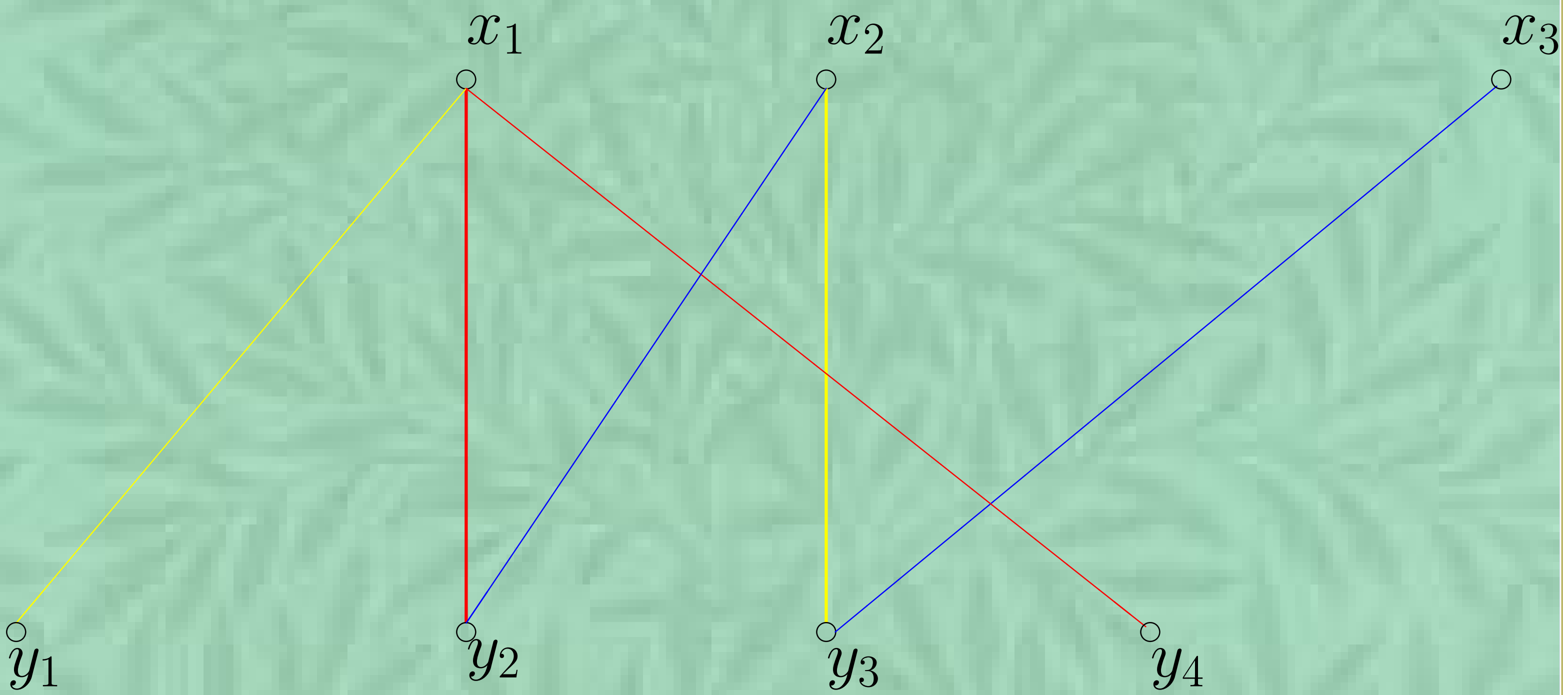
A matching M is called *heterochromatic* in a colored graph G, if its any two edges have different colors. The two ends of an edge in M are said to be *matched under* M. M *saturates* a vertex u, or u is *M-saturated*, if there is an edge in M incident with u; otherwise, u is *M-unsaturated*. If every vertex of G is M-saturated, the heterochromatic matching M is called *perfect*. M is a *maximum heterochromatic matching*, if G has no heterochromatic matching M' such that $|M'| > |M|$. We denote by $C(M) = \{C(e) : e \in E(M)\}$ and $\overline{C(M)} = C(G) \backslash C(M)$.

2 Preliminaries

In order to present our main results, we need the following lemma and some definitions.

Lemma 2.1 *Let M be a maximum heterochromatic matching of a colored bipartite graph (B, C) with bipartition (X, Y). If $xy \in E(B)$ and $C(xy) \in \overline{C(M)}$, then either x or y is M-saturated.*

Let M be a maximum heterochromatic matching of a colored bipartite graph (B, C) with bipartition (X, Y). We will define an *edge-exchange transform* on M as follows: Starting from M, we select an edge $x_j y_i$ with $C(x_j y_i) \in \overline{C(M)}$ as an *input edge*, where exactly one of y_i and x_j is M-unsaturated, say y_i. Then take the edge $x_j y_l \in M$ as an *output edge*. The other edges in $M \backslash \{x_j y_l\}$ are not changed. Then we get a new maximum heterochromatic matching $M \backslash \{\text{output edge}\} \cup \{\text{input edge}\}$, i.e., $M \backslash \{x_j y_l\} \cup \{x_j y_i\}$. In this way we finish one step of the transform on a maximum heterochromatic matching M.

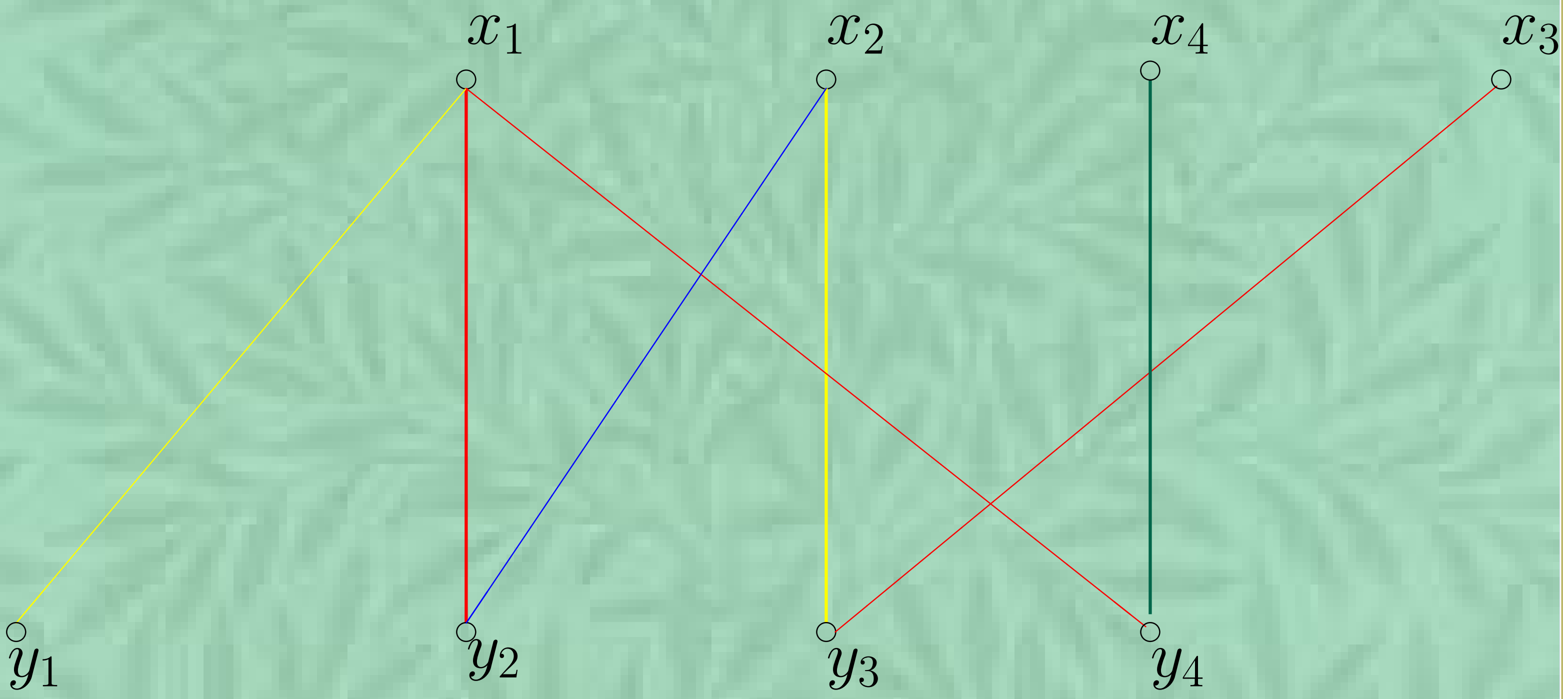

x_1
x_2
x_3
y_1
y_2
y_3
y_4

The following facts are easily seen.

(1) From Lemma 2.1, there is no edge x_jy_i of B with $C(x_jy_i) \in \overline{C(M)}$, such that x_j and y_i are both M-unsaturated.

(2) If M is a perfect heterochromatic matching of (B, C), then we cannot do the transform on any maximum (perfect) heterochromatic matching.

(3) If $CN(v) \subseteq C(M)$ for all $v \in V(B)$ and v is M-unsaturated, then we cannot do the transform on any maximum heterochromatic matching.


x_1
x_2
x_4
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y_1
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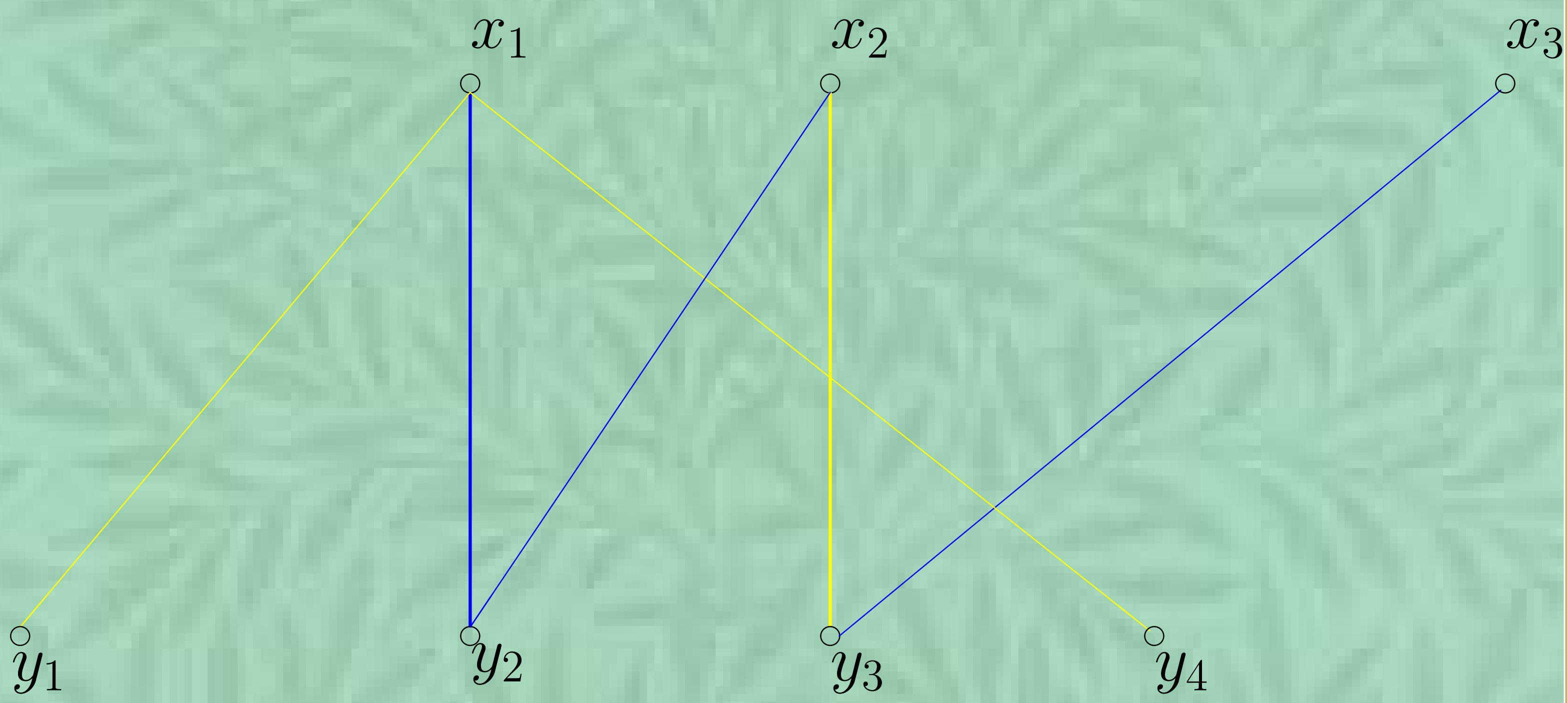
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Lemma 2.2 *(Hall [2]) Let* B *be a bipartite graph with bipartition* (X, Y)*. Then* B *contains a matching that saturates every vertex in* X *if and only if* $|N(S)| \geqslant |S|$ *for all* $S \subseteq X$*.*

Lemma 2.3 *(Tutte [2]) A graph* G *has a perfect matching if and only if*

$$o(G - S) \leqslant |S|$$

for all $S \subseteq V(G)$*, where* $o(G - S)$ *denotes the number of odd components in the remaining graph* $G - S$*.*

3 Main results

In this section, we give our main results and its proof.

Theorem 3.1 *Let* (B, C) *be a colored bipartite graph with bipartition* (X, Y)*. If* $d^c(v) \geq k$ *for every vertex* v *of* B*, then it contains a heterochromatic matching of cardinality at least* $k - 1$.

Idea of the proof: It is easy to see that for $k = 1$ and $k = 2$, any colored edge of B is a heterochromatic matching with cardinality 1, and hence the conclusion holds.

For $k \geq 3$, let M be a maximum heterochromatic matching of B and $|M| = t \leq k - 2$. By successively doing the transform on maximum heterochromatic matchings of B from M and restrained in the subgraph $H = B[V(M) \cup \{x_{t+1}, y_{t+1}\}]$, where $x_{t+1}y_{t+1} \in E(B) \backslash M$, we will prove there is an edge $xy \in E(B) \backslash E(H)$ such that every color $c \in C(B)$ can not be assigned to the edge xy, otherwise, we can get a new heterochromatic matching of B with $t + 1$ edges, a contradiction to the maximality of M. That is, there is no color in $C(B)$ which can be assigned to the edge xy, a contradiction to that C is a coloring of B.

From Theorem 3.1, the following result follows immediately.

Corollary 3.2 *Let* $(K_{n,n}, C)$ *be a properly colored complete bipartite graph, then it has a heterochromatic matching with cardinality at least* $n-1$.

Now we give the result under the color neighborhood condition.

Theorem 3.3 *Let (B, C) be a colored bipartite graph with bipartition (X, Y) and $|N^c(S)| \geq |S|$ for all $S \subseteq X$, then B has a heterochromatic matching of cardinality at least $\lceil \frac{|X|}{2} \rceil$.*

Idea of the proof: Let M be a maximum heterochromatic matching of B such that its cardinality t is at most $t < \lceil \frac{|X|}{2} \rceil$. Then by analyzing the colors of the edges, we can get the some inequations that are some contradictions. Thus we completed the theorem.

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Thank You !