Long heterochromatic paths in edge-colored graphs

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Definitions and notations :

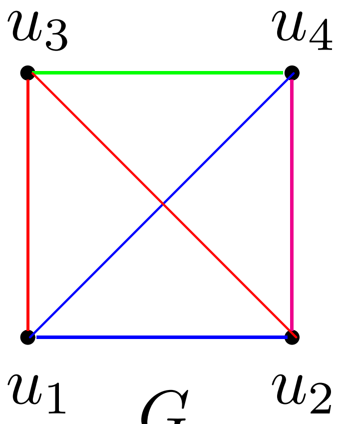
Let $G = (V, E)$ be a graph. By an edge-coloring of G we mean a surjective function $C : E \longrightarrow \{1, 2, \cdots, r\}$.

If G is assigned such a coloring, then we say that G is an edge-colored graph, or r-edge-colored graph. Denote the colored graph by (G, C).

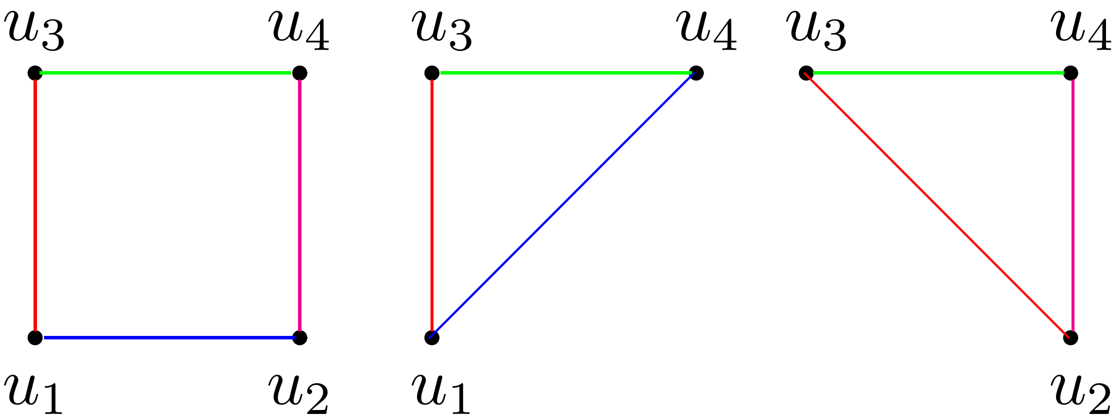
For a subgraph H of G, H is called heterochromatic if its any two edges have different colors.

This kind of subgraph is also called rainbow, multicolored, polychromatic, colorful.

For example :



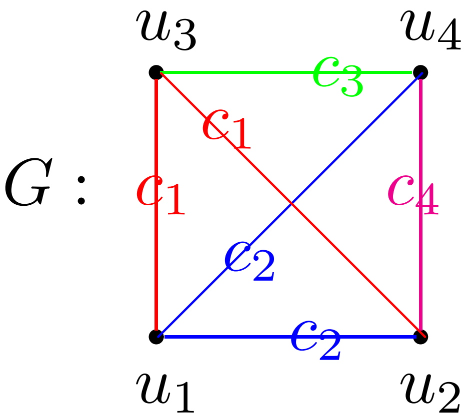
G



(heterochromatic subgraphs of G)

For a vertex v of G, the color neighborhood $CN(v)$ of v is defined as the set $\{C(e) \mid e \text{ is incident with } v\}$ and the color degree $d^c(v)$ is $d^c(v) = |CN(v)|$.

For example :



$$CN(u_1) = \{c_1, c_2\},\ CN(u_2) = \{c_1, c_2, c_4\},$$
$$CN(u_3) = \{c_2, c_3, c_4\},\ CN(u_4) = \{c_1, c_3\}.$$

$$d^c(u_1) = d^c(u_4) = 2,\ d^c(u_2) = d^c(u_3) = 3.$$

The long heterochromatic paths was first considered in edge-colored complete graphs.

Some conditions for the existence of the heterochromatic hamiltonian cycles or the heterochromatic hamiltonian paths in edge-colored complete graphs was given.

Frieze and Reed showed that if the edges of the complete graph K_n are colored so that no color appears more than $\lceil n/(A \ln n) \rceil$ times, for some sufficiently large A, then there is always a heterochromatic Hamiltonian cycle.

A. Frieze and B. Reed, Polychromatic Hamilton cycles, *Discrete Math.* **118** (1993), 69-74.

Albert, Frieze and Reed showed that if n is sufficiently large and the edges of the complete graph K_n are colored so that no color appears more than $\lceil cn \rceil$ times, where $c < 1/32$ is a constant, then there is a heterochromatic Hamiltonian cycle.

M. Albert, A. Frieze and B. Reed, Multicolored hamilton cycles, *Electronic J. Combin.* **2** (1995), ♯R10, 1-13.

Hahn and Thomassen showed that there exists a constant c such that if $n \geq ck^3$ and the edges of K_n are colored using no color more than k times, then there is a heterochromatic Hamiltonian path.

G. Hahn and C. Thomassen, Path and cycle sub-Ramsey numbers and an edge-coloring conjecture, *Discrete Math.* **62** (1986), 29-33.

Broersma, Li, Woeginger and Zhang showed that for an edge-colored graph G,

(1) if $d^c(v) \geq k$ for every vertex v of G, then for every vertex z of G there exists a heterochromatic z-path of length $\lceil \frac{k+1}{2} \rceil$,

(2) if $|CN(u) \cup CN(v)| \geq s > 1$ for every pair of vertices u and v of G, then G contains a heterochromatic path of length $\lceil \frac{s}{3} \rceil + 1$.

H.J. Broersma, X. Li, G. Woeginger and S. Zhang, Paths and cycles in colored graphs, *Australalasian J. Combin.* **31** (2005), 297-309.

Our Main Results :

Theorem 1 *Let G be an edge-colored graph and $k \geq 3$ an integer. Suppose that $d^c(v) \geq k$ for every vertex v of G. Then :*

(1) G has a heterochromatic path of length at least $k-1$ if $3 \leq k \leq 7$.

(2) G has a heterochromatic path of length at least $\lceil \frac{3k}{5} \rceil + 1$ if $k \geq 8$.

Proof. (1) The case when $3 \leq k \leq 7$ is easy, we show that G has a heterochromatic path of length at least $k-1$ by considering every cases for each k.

(2) For $k \geq 8$, we use induction on k. Then G has a heterochromatic path of length at least $\lceil \frac{3(k-1)}{5} \rceil + 1$ which is equal to $\lceil \frac{3k}{5} \rceil$ if $k \equiv 1, 2, 4$ $(mod\ 5)$ and $\lceil \frac{3k}{5} \rceil + 1$ otherwise. So we shall only consider the case when $k \equiv 1, 2, 4$ $(mod\ 5)$, now we will proceed by contradictions, we suppose that the longest heterochromatic path in G is of length $l = \lceil \frac{3k}{5} \rceil$.

Let $P = u_1u_2u_3 \ldots u_{l-1}u_lu_{l+1}v_1v_2 \ldots v_s$ be a path in G such that :

(a) u_1Pu_{l+1} is a longest heterochromatic path in G;

(b) $C(u_{l+1}v_1) = C(u_{k_0}u_{k_0+1})$ and $1 \leq k_0 \leq l$ is as small as possible, subject to (a);

(c) v_1Pv_s is a heterochromatic path in G with $C(u_1Pu_{l+1}) \cap C(v_1Pv_s) = \varnothing$ and v_1Pv_s is as long as possible, subject to (a) and (b).

Then we can get a contradiction. So G has a heterochromatic path of length $\lceil \frac{3k}{5} \rceil + 1$. ∎

Theorem 2 *If $d^c(v) \geq k \geq 7$ for any $v \in V(G)$, then G has a heterochromatic path of length at least $\lceil \frac{2k}{3} \rceil + 1$.*

Proof. We use induction on k. Then G has a heterochromatic path of length at least $\lceil \frac{2(k-1)}{3} \rceil + 1$ which is equal to $\lceil \frac{2k}{3} \rceil$ if $k \equiv 1, 2\ (mod\ 3)$ and $\lceil \frac{2k}{3} \rceil + 1$ otherwise. So we shall only consider the case when $k \equiv 1, 2\ (mod\ 3)$, now we will proceed by contradictions, we suppose that the longest heterochromatic path in G is of length $l = \lceil \frac{2k}{3} \rceil$.

We first show that G has a heterochromatic path $P = u_1 u_2 \ldots u_l u_{l+1}$ of length $l = \lceil \frac{2k}{3} \rceil$ and there exists a $v_1 \in V(G) - V(P)$ such that $C(u_{l+1} v_1) = C(u_1 u_2)$.

Then we find a heterochromatic path of length $l + 1$ in all the possible cases. So we get a contradiction. G has a heterochromatic path of length at least $\lceil \frac{2k}{3} \rceil + 1$. ∎

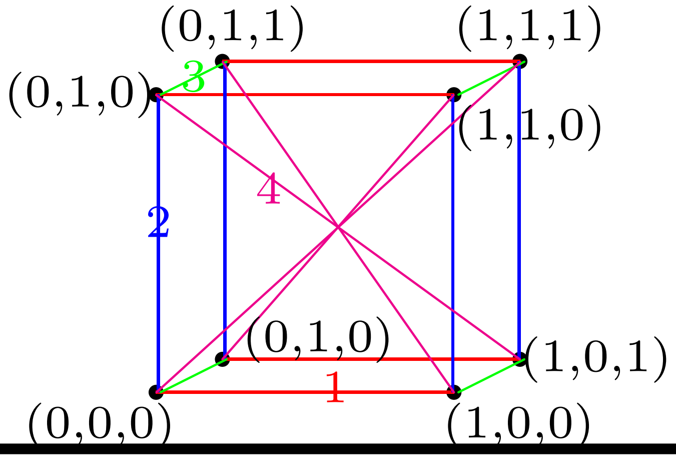
Actually, we can show that for $1 \leq k \leq 5$ any graph G with $d^c(v) \geq k$ for every vertex v of G has a heterochromatic path of length at least k, with only one exceptional graph K_4 for $k = 3$, one exceptional graph for $k = 4$ and three exceptional graphs for $k = 5$, for which (all the exceptional graphs) G has a heterochromatic path of length at least $k - 1$. If $k = 8$, by **Theorem 2** G also has a heterochromatic path of length at least $k - 1$. So, we propose the following conjecture :

Conjecture 3 *If* $d^c(v) \geq k \geq 3$ *for any* $v \in V(G)$*, then* G *has a heterochromatic path of length at least* $k - 1$*.*

If this conjecture is true, it would be best possible.

Example 1 : Let G_k be an edge-colored graph whose vertices are the ordered $(k-1)$-tuples of 0's and 1's ; two vertices are joined by an edge if and only if they differ in exactly one coordinate or they differ in all coordinates. An edge is in color j $(1 \leq j \leq k-1)$ if and only if its two ends differ in exactly the j-th coordinate, or in color k if and only if its two ends differ in all the coordinates.

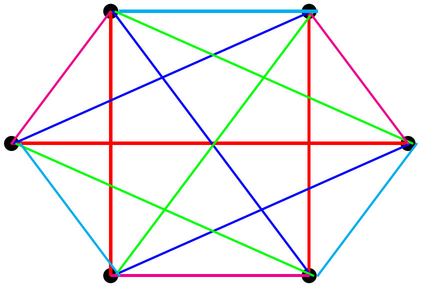
G_4 :



It is not difficult to check that G_k is an edge-colored graph such that $d^c(v) \geq k$ for all the vertices v, and any longest heterochromatic path of G_k is of length $k-1$.

Example 2 : Let G'_k be a proper k-edge-colored K_{k+1} when k is odd. (Since K_n is $(n-1)$-edge-colorable when n is even, such G'_k exists when k is odd.)

G'_5 :



Then, it is obvious that any longest heterochromatic path in G'_k is of length $k-1$ when k is odd.

Theorem 4 *Let* G *be an edge-colored graph and* s *a positive integer. Suppose that* $|CN(u) \cup CN(v)| \geq s \geq 4$ *for every pair of vertices* u *and* v *of* G*. Then* G *has a heterochromatic path of length at least* $\lfloor \frac{2s+4}{5} \rfloor$*.*

Proof. By contradiction. Suppose $P = u_1 u_2 \ldots u_l u_{l+1}$ is a longest heterochromatic path of length $l < \lfloor \frac{2s+4}{5} \rfloor$. Use the condition that $|CN(u_1) \cup CN(u_{l+1})| \geq s \geq 4$, we get a contradiction. So G has a heterochromatic path of length at least $\lfloor \frac{2s+4}{5} \rfloor$. ∎

Theorem 5 *Suppose G is an edge-colored graph, $|CN(u) \cup CN(v)| \geq s \geq 1$ for any two vertices u, v in G, then there exists a heterochromatic path of length $\lceil \frac{s+1}{2} \rceil$ in G.*

Proof. (1) For $1 \leq s \leq 7$, the results are obvious.
(2) For $s \geq 8$, we use induction on s. Then G has a heterochromatic path of length $\lceil \frac{s}{2} \rceil$ which is equal to $\lceil \frac{s+1}{2} \rceil - 1$ if s is even and $\lceil \frac{s+1}{2} \rceil$ otherwise. So we shall only consider the case when s is even. Now we will proceed by contradictions. Suppose $P = u_1 u_2 \ldots u_l u_{l+1}$ is a longest heterochromatic path of length $l = \lceil \frac{s+1}{2} \rceil - 1$.

Then we can get that $N(u_0) \subseteq V(P)$ and $N(u_l) \subseteq V(P)$, which implies that $|CN(u_0) \cup CN(u_l)| = 2l - 1 = s - 2$, a contradiction. So we can conclude that there exists a heterochromatic path of length $\lceil \frac{s+1}{2} \rceil$ in G. ■

Note that the bound we gave in **Theorem 5** is best possible.

Example : Let s be a positive integer. If s is even, let G_s be the graph obtained from the complete graph $K_{\frac{s+4}{2}}$ by deleting an edge ; if s is odd, let G_s be the complete graph $K_{\frac{s+3}{2}}$. Then, color the edges of G_s by different colors for any two different edges.

So, for any $s \geq 1$ we have that $|CN(u) \cup CN(v)| \geq s$ for any pair of vertices u and v in G, and any longest heterochromatic path in G is of length $\lceil \frac{s+1}{2} \rceil$.

Thank you!