Long heterochromatic paths in edge-colored graphs

Chen He and Li Xueliang

Center for Combinatorics, LPMC
Nankai University
Definitions and notations:

Let $G = (V, E)$ be a graph. By an edge-coloring of $G$ we mean a surjective function $C : E \rightarrow \{1, 2, \cdots, r\}$.

If $G$ is assigned such a coloring, then we say that $G$ is an edge-colored graph, or $r$-edge-colored graph. Denote the colored graph by $(G, C)$.
For a subgraph $H$ of $G$, $H$ is called heterochromatic if its any two edges have different colors. This kind of subgraph is also called rainbow, multicolored, polychromatic, colorful.

For example:
For a vertex $v$ of $G$, the color neighborhood $CN(v)$ of $v$ is defined as the set $\{C(e) \mid e \text{ is incident with } v\}$ and the color degree $d^c(v)$ is $d^c(v) = |CN(v)|$.

For example:

$CN(u_1) = \{c_1, c_2\}, \quad CN(u_2) = \{c_1, c_2, c_4\},$

$CN(u_3) = \{c_2, c_3, c_4\}, \quad CN(u_4) = \{c_1, c_3\}.$

$d^c(u_1) = d^c(u_4) = 2, \quad d^c(u_2) = d^c(u_3) = 3.$
The long heterochromatic paths was first considered in edge-colored complete graphs.

Some conditions for the existence of the heterochromatic hamiltonian cycles or the heterochromatic hamiltonian paths in edge-colored complete graphs was given.
Frieze and Reed showed that if the edges of the complete graph $K_n$ are colored so that no color appears more than $\left\lceil \frac{n}{A \ln n} \right\rceil$ times, for some sufficiently large $A$, then there is always a heterochromatic Hamiltonian cycle.

Albert, Frieze and Reed showed that if \( n \) is sufficiently large and the edges of the complete graph \( K_n \) are colored so that no color appears more than \( \lceil cn \rceil \) times, where \( c < 1/32 \) is a constant, then there is a heterochromatic Hamiltonian cycle.

Hahn and Thomassen showed that there exists a constant $c$ such that if $n \geq ck^3$ and the edges of $K_n$ are colored using no color more than $k$ times, then there is a heterochromatic Hamiltonian path.

Broersma, Li, Woeginger and Zhang showed that for an edge-colored graph $G$,

(1) if $d^c(v) \geq k$ for every vertex $v$ of $G$, then for every vertex $z$ of $G$ there exists a heterochromatic $z$-path of length $\lceil \frac{k+1}{2} \rceil$,

(2) if $|CN(u) \cup CN(v)| \geq s > 1$ for every pair of vertices $u$ and $v$ of $G$, then $G$ contains a heterochromatic path of length $\lceil \frac{s}{3} \rceil + 1$.

Our Main Results:

**Theorem 1** Let $G$ be an edge-colored graph and $k \geq 3$ an integer. Suppose that $d^c(v) \geq k$ for every vertex $v$ of $G$. Then:

1. $G$ has a heterochromatic path of length at least $k - 1$ if $3 \leq k \leq 7$.
2. $G$ has a heterochromatic path of length at least $\lceil \frac{3k}{5} \rceil + 1$ if $k \geq 8$. 
Proof. (1) The case when $3 \leq k \leq 7$ is easy, we show that $G$ has a heterochromatic path of length at least $k - 1$ by considering every cases for each $k$. 

(2) For $k \geq 8$, we use induction on $k$. Then $G$ has a heterochromatic path of length at least $\lceil \frac{3(k-1)}{5} \rceil + 1$ which is equal to $\lceil \frac{3k}{5} \rceil$ if $k \equiv 1, 2, 4 \ (mod \ 5)$ and $\lceil \frac{3k}{5} \rceil + 1$ otherwise. So we shall only consider the case when $k \equiv 1, 2, 4 \ (mod \ 5)$, now we will proceed by contradictions, we suppose that the longest heterochromatic path in $G$ is of length $l = \lceil \frac{3k}{5} \rceil$. 
Let \( P = u_1 u_2 u_3 \ldots u_{l-1} u_l u_{l+1} v_1 v_2 \ldots v_s \) be a path in \( G \) such that:

(a) \( u_1 Pu_{l+1} \) is a longest heterochromatic path in \( G \);
(b) \( C(u_{l+1} v_1) = C(u_{k_0} u_{k_0+1}) \) and \( 1 \leq k_0 \leq l \) is as small as possible, subject to (a);
(c) \( v_1 Pv_s \) is a heterochromatic path in \( G \) with \( C(u_1 Pu_{l+1}) \cap C(v_1 Pv_s) = \emptyset \) and \( v_1 Pv_s \) is as long as possible, subject to (a) and (b).

Then we can get a contradiction. So \( G \) has a heterochromatic path of length \( \lceil \frac{3k}{5} \rceil + 1 \). 

\[ \square \]
Theorem 2  If $d^c(v) \geq k \geq 7$ for any $v \in V(G)$, then $G$ has a heterochromatic path of length at least $\lceil \frac{2k}{3} \rceil + 1$.

Proof.  We use induction on $k$. Then $G$ has a heterochromatic path of length at least $\lceil \frac{2(k-1)}{3} \rceil + 1$ which is equal to $\lceil \frac{2k}{3} \rceil$ if $k \equiv 1, 2 \ (mod \ 3)$ and $\lceil \frac{2k}{3} \rceil + 1$ otherwise. So we shall only consider the case when $k \equiv 1, 2 \ (mod \ 3)$, now we will proceed by contradictions, we suppose that the longest heterochromatic path in $G$ is of length $l = \lceil \frac{2k}{3} \rceil$. 
We first show that $G$ has a heterochromatic path $P = u_1 u_2 \ldots u_l u_{l+1}$ of length $l = \lceil \frac{2k}{3} \rceil$ and there exists a $v_1 \in V(G) - V(P)$ such that $C(u_{l+1} v_1) = C(u_1 u_2)$.

Then we find a heterochromatic path of length $l + 1$ in all the possible cases. So we get a contradiction. $G$ has a heterochromatic path of length at least $\lceil \frac{2k}{3} \rceil + 1$. ■
Actually, we can show that for $1 \leq k \leq 5$ any graph $G$ with $d^c(v) \geq k$ for every vertex $v$ of $G$ has a heterochromatic path of length at least $k$, with only one exceptional graph $K_4$ for $k = 3$, one exceptional graph for $k = 4$ and three exceptional graphs for $k = 5$, for which (all the exceptional graphs) $G$ has a heterochromatic path of length at least $k - 1$. If $k = 8$, by Theorem 2 $G$ also has a heterochromatic path of length at least $k - 1$. So, we propose the following conjecture:

**Conjecture 3** If $d^c(v) \geq k \geq 3$ for any $v \in V(G)$, then $G$ has a heterochromatic path of length at least $k - 1$. 
If this conjecture is true, it would be best possible.

**Example 1:** Let $G_k$ be an edge-colored graph whose vertices are the ordered $(k - 1)$-tuples of 0’s and 1’s; two vertices are joined by an edge if and only if they differ in exactly one coordinate or they differ in all coordinates. An edge is in color $j$ ($1 \leq j \leq k - 1$) if and only if its two ends differ in exactly the $j$-th coordinate, or in color $k$ if and only if its two ends differ in all the coordinates.
It is not difficult to check that $G_k$ is an edge-colored graph such that $d^c(v) \geq k$ for all the vertices $v$, and any longest heterochromatic path of $G_k$ is of length $k - 1$. 
Example 2: Let $G'_k$ be a proper $k$-edge-colored $K_{k+1}$ when $k$ is odd. (Since $K_n$ is $(n-1)$-edge-colorable when $n$ is even, such $G'_k$ exists when $k$ is odd.)

Then, it is obvious that any longest heterochromatic path in $G'_k$ is of length $k - 1$ when $k$ is odd.
Theorem 4 Let $G$ be an edge-colored graph and $s$ a positive integer. Suppose that $|CN(u) \cup CN(v)| \geq s \geq 4$ for every pair of vertices $u$ and $v$ of $G$. Then $G$ has a heterochromatic path of length at least $\left\lfloor \frac{2s+4}{5} \right\rfloor$.

Proof. By contradiction. Suppose $P = u_1u_2\ldots u_lu_{l+1}$ is a longest heterochromatic path of length $l < \left\lfloor \frac{2s+4}{5} \right\rfloor$. Use the condition that $|CN(u_1) \cup CN(u_{l+1})| \geq s \geq 4$, we get a contradiction. So $G$ has a heterochromatic path of length at least $\left\lfloor \frac{2s+4}{5} \right\rfloor$. □
Theorem 5 Suppose $G$ is an edge-colored graph, $|CN(u) \cup CN(v)| \geq s \geq 1$ for any two vertices $u, v$ in $G$, then there exists a heterochromatic path of length $\lceil \frac{s+1}{2} \rceil$ in $G$.

Proof. (1) For $1 \leq s \leq 7$, the results are obvious.
(2) For $s \geq 8$, we use induction on $s$. Then $G$ has a heterochromatic path of length $\lceil \frac{s}{2} \rceil$ which is equal to $\lceil \frac{s+1}{2} \rceil - 1$ if $s$ is even and $\lceil \frac{s+1}{2} \rceil$ otherwise. So we shall only consider the case when $s$ is even. Now we will proceed by contradictions. Suppose $P = u_1u_2 \ldots u_lu_{l+1}$ is a longest heterochromatic path of length $l = \lceil \frac{s+1}{2} \rceil - 1$. 
Then we can get that $N(u_0) \subseteq V(P)$ and $N(u_l) \subseteq V(P)$, which implies that $|CN(u_0) \cup CN(u_l)| = 2l - 1 = s - 2$, a contradiction. So we can conclude that there exists a heterochromatic path of length $\lceil \frac{s+1}{2} \rceil$ in $G$. □

Note that the bound we gave in Theorem 5 is best possible.
Example: Let $s$ be a positive integer. If $s$ is even, let $G_s$ be the graph obtained from the complete graph $K_{\frac{s+4}{2}}$ by deleting an edge; if $s$ is odd, let $G_s$ be the complete graph $K_{\frac{s+3}{2}}$. Then, color the edges of $G_s$ by different colors for any two different edges.

So, for any $s \geq 1$ we have that $|CN(u) \cup CN(v)| \geq s$ for any pair of vertices $u$ and $v$ in $G$, and any longest heterochromatic path in $G$ is of length $\left\lceil \frac{s+1}{2} \right\rceil$. 
Thank you!