

Laplacian Characteristic Polynomial of Trees

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Based on

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A Connection between Ordinary and Laplacian
Spectra of Bipartite Graphs

Introduction

Let G be a graph on n vertices. Let $A(G)$ be the adjacency matrix of G and $D(G)$ be the diagonal matrix of vertex degrees. The Laplacian matrix of G is $L(G) = D(G) - A(G)$.

The (ordinary) characteristic polynomial of G is defined as $\phi(G, x) = \det(xI_n - A(G))$.

The Laplacian characteristic polynomial of G is defined as $\psi(G, x) = \det(xI_n - L(G))$. In what follows we write it as

$$\psi(G, x) = \sum_{k=0}^n (-1)^k c_k(G) x^{n-k}.$$

Then

$$c_0(G) = 1, \quad c_1(G) = 2m, \quad c_{n-1}(G) = nt, \quad c_n(G) = 0,$$

where m is the number of edges, t is the number of spanning trees of G .

Recall that the Wiener number $W(G)$ of a connected graph G is defined as

$$W(G) = \sum_{u,v \in V(G)} d_G(u, v)$$

where $d_G(u, v)$ denotes the distance between u and v in G .

If G is a tree on n vertices, then

- $c_{n-2}(G) = W(G)$.
- $W(S_n) \leq W(G) \leq W(P_n)$, where S_n and P_n denote respectively the star and the path on n vertices.

Conjecture 1. (Gutman & Pavlovic, 2003) The inequalities

$$c_k(S_n) \leq c_k(T) \leq c_k(P_n)$$

holds for all trees T on n vertices with $n \geq 1$ and for all values of k with $0 \leq k \leq n - 1$.

Ordinary and Laplacian Characteristic Polynomials of Bipartite Graphs

The subdivision graph $S(G)$ of a graph G is obtained by inserting a new vertex on each edge of G .

Let G be a graph with vertex set $\{v_1, v_2, \dots, v_n\}$ and edge set $\{e_1, e_2, \dots, e_m\}$.

The vertex-edge incidence matrix of G is the $n \times m$ matrix $B = (b_{ij})$ where $b_{ij} = 1$ if v_i is incident with e_j and $b_{ij} = 0$ otherwise. Then

- $BB^T = D(G) + A(G)$,
- $A(S(G)) = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$.

Suppose that G is bipartite. Then $A(G)$ and $D(G)$ can be written as the forms:

$$A(G) = \begin{pmatrix} 0 & C \\ C^T & 0 \end{pmatrix}, \quad D(G) = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}.$$

Obviously

$$\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} D_1 & C \\ C^T & D_2 \end{pmatrix} \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} D_1 & -C \\ -C^T & D_2 \end{pmatrix}$$

i.e.,

- $D(G) + A(G)$ and $D(G) - A(G) = L(G)$ are similar (if G is bipartite).

Now it follows that

$$\begin{aligned}
\phi(S(G), x) &= \det(xI_{n+m} - A(S(G))) \\
&= \det \begin{pmatrix} xI_n & -B \\ -B^T & xI_m \end{pmatrix} \\
&= x^{m-n} \det(x^2I_n - BB^T) \\
&= x^{m-n} \det[x^2I_n - (D(G) + A(G))] \\
&\stackrel{\text{bipartite}}{=} x^{m-n} \det(x^2I_n - L(G)) = \psi(G, x^2).
\end{aligned}$$

It is well-known that if G is a bipartite graph, then $\phi(G)$ can be written as

$$\phi(G) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k b_k(G) x^{n-2k},$$

where $b_k(G) \geq 0$ for all $k = 0, 1, \dots, \lfloor n/2 \rfloor$. In particular, if G is acyclic, then

$$\phi(G) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k m(G, k) x^{n-2k},$$

where $m(G, k)$ is the number of k -matchings of G .

Theorem 2. If G is a bipartite graph (on n vertices and m edges), then

- $\phi(S(G), x) = \psi(G, x^2)$,
- $c_k(G) = b_k(S(G))$ for $0 \leq k \leq \min\{\lfloor (n+m)/2 \rfloor, n\}$ and $c_k(G) = 0$ otherwise,
- if G is acyclic, then $c_k(G) = m(S(G), k)$ for all $0 \leq k \leq n$.

Coefficients of Laplacian Characteristic Polynomial of Trees

It is already known that Conjecture 1 is true for some special cases: obvious for $k = 0, 1, n - 1$, and valid for $k = 2, 3, n - 3, n - 2$, etc. .

For even n , let F_n be the tree obtained from $S(S_{n/2+1})$ by deleting a pendent vertex.

Lemma 3. Let T be a tree on n vertices with a perfect matching, $1 \leq k \leq n$. Then

$$m(T, k) \geq m(F_n, k).$$

Theorem 4. Let T be a tree on n vertices and let k be an integer $1 \leq k \leq n - 1$. Then

$$c_k(G) \leq c_k(T) \leq c_k(P_n). \quad (1)$$

Proof. Note that inequalities (1) are equivalent to

$$m(S(S_n), k) \leq m(S(T), k) \leq m(S(P_n), k).$$

Proof of inequality (2):

$$m(S(S_n), k) \leq m(S(T), k). \quad (2)$$

We proceed by induction on n . For $n \leq 4$, it is easy to check that (2) holds. Assume that

$$m(S(S_{n-1}), k) \leq m(S(T'), k)$$

holds for any tree T' on $n - 1$ vertices, and for all k .

Let v be a pendent vertex of and $S(S_n)$. Then

$$m(S(S_n), k) = m(S(S_n) - v, k) + m(S(S_{n-1}), k - 1).$$

Let v be a pendent vertex of and $S(T)$, being adjacent to u .
Then

$$m(S(T), k) = m(S(T) - v, k) + m(S(T) - v - u, k - 1).$$

Note that $S(T) - v$ is a tree with a perfect matching. By Lemma 3

$$m(S(S_n) - v, k) \leq m(S(T) - v, k).$$

On the other hand, $S(T) - v - u$ is just the subdivision of a tree T' on $n - 1$ vertices. Then by the induction assumption

$$m(S(S_{n-1}), k - 1) \leq m(S(T'), k - 1).$$

Now it is easy to see that (2) holds.

Proof of inequality (3):

$$m(S(T), k) \leq m(S(P_n), k). \quad (3)$$

It is known that among all trees on $2n - 1$ vertices, the path P_{2n-1} has the maximum number of matchings, i.e.,

$$m(T'', k) \leq m(P_{2n-1}, k)$$

holds for all tree T'' on $2n - 1$ vertices and all k . Since $S(P_n) = P_{2n-1}$, (3) follows easily. \square

Lemma 3'. (Zhang & Li) Let T be a tree on n vertices with a perfect matching, $T \neq F_n$, $2 \leq k \leq n/2$. Then

$$m(T, k) \geq m(F_n, k)$$

and this inequality is strict for $k = 2$.

Proof. Obviously, $n \geq 6$. Denote by \hat{T} be forest obtained by deleting all edges of the perfect matching of T . Since $T \neq F_n$, we have $\hat{T} \neq S_{n/2} = \hat{F}_n$. It is obvious that $m(\hat{T}, i) \geq m(\hat{F}_n, i)$ for $0 \leq i \leq k$, and $m(\hat{T}, 2) > 0 = m(\hat{F}_n, 2)$. Since any edge in \hat{T} is adjacent to at most two edges in the perfect matching, it is easy to see that

$$\begin{aligned} m(T, k) &\geq \sum_{i=0}^k m(\hat{T}, i) \binom{n/2-2i}{k-i} \\ &\geq \sum_{i=0}^k m(\hat{F}_n, i) \binom{n/2-2i}{k-i} = m(F_n, k). \end{aligned}$$

If $n \geq 8$, then $m(T, 2) > m(F_n, 2)$ can be seen from the inequalities above; if $n = 6$, then $T = P_6$ and so $m(T, 2) > m(F_6, 2)$.
□

Using Lemma 3', one can easily prove

Theorem 4'. Let T be a tree on n vertices and let k be an integer, $2 \leq k \leq n - 2$. Then

$$\begin{aligned} c_k(T) &> c_k(S_n) \text{ if } T \neq S_n \\ c_k(T) &< c_k(P_n) \text{ if } T \neq P_n. \end{aligned}$$

Thank you.