

Improper coloring of plane graphs and Steinberg's conjecture

Baogang Xu

School of Math. & Computer Science

Nanjing Normal University

122 Ninghai Road, Nanjing, 210097, P. R. China

Email: baogxu@njnu.edu.cn

Research partially supported by NSFC 10371055

Colorings

Given a graph G , and a positive integer k , a k -coloring of G is a mapping $\phi : V(G) \mapsto Z_k$ such that $\phi(u) \neq \phi(v)$ whenever $uv \in E(G)$.

In other words, a k -coloring of G is a partition V_1, V_2, \dots, V_k of $V(G)$ such that

V_i is an independent set for every i , i.e.,

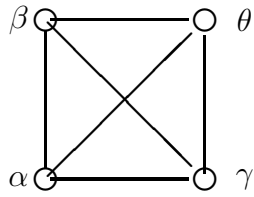
the subgraph induced by V_i has maximum degree zero.

Defective colorings

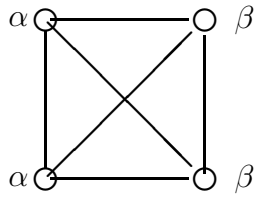
Instead of partitioning $V(G)$ into independent sets, an interesting problem is to partition $V(G)$ into subsets of which each induces a subgraph with preassigned maximum degree. This is the so called *defective coloring* problem of graphs.

Given positive integers k and d , a k -coloring with deficiency d , simply denoted a $(k, d)^*$ -coloring of G , is a mapping $\phi : V(G) \mapsto Z_k$ such that every vertex v has at most d neighbors receiving the same color as $\phi(v)$.

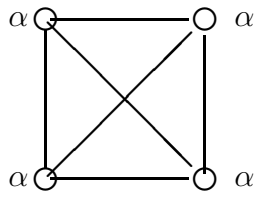
In other words, a $(k, d)^*$ -coloring of G is a partition V_1, V_2, \dots, V_k of $V(G)$ such that for every i , the subgraph induced by V_i has maximum degree at most d .



A 4-coloring of K_4



A $(2, 1)^*$ -coloring of K_4



A $(1, 3)^*$ -coloring of K_4

A $(k, 0)^*$ -coloring is just a classical k -coloring of a graph. So, defective colorings is certainly a natural generalization of the classical colorings. Defective colorability of graphs has been extensively studied. By FCT, every plane graph is $(4, 0)^*$ -colorable. But there exist non- $(3, 1)^*$ -colorable plane graphs. For further information on this topic, interested readers are referred to a survey by Woodall [List colorings of graphs, *Surveys in Combinatorics* 2001, Cambridge University Press, (2001)269-301.].

Here, we consider the $(3,1)^*$ -colorability of plane graphs. This research was motivated by Steinberg's conjecture and a conjecture of Xu and Zhang.

Steinberg's conjecture and its progress

In 1976, Steinberg conjectured that every plane graph without cycles of length 4 and 5 is 3-colorable.

In 2005, Borodin *et al* proved that every plane graph without cycles of length from 4 to 7 is 3-colorable [Borodin, Glebov, Raspaud and Salavatipour, Planar graphs without cycles of length from 4 to 7 are 3-colorable, JCTB, 93(2005)303-311].

Borodin and Raspaud's conjecture

As a variation of Steinberg's 3-coloring problem, Borodin and Raspaud considered the 3-colorability of plane graphs with neither cycles of length 5 nor triangles of shorter distance, they proved that *every plane graph with neither cycles of length 5 nor triangles of distance less than four is 3-colorable* [A sufficient condition for planar graphs to be 3-colorable, JCTB, 88(2003)17-27], and proposed a *conjecture claiming that every plane graph with neither adjacent triangles nor cycles of length 5 is 3-colorable.*

Borodin and Raspaud's conjecture is a little stronger than Steinberg's conjecture because a graph without cycles of length 4 cannot contain adjacent triangles.

Xu improved the result of [JCTB, 88(2003)17-27] by showing that *every plane graph with neither cycles of length 5 nor triangles of distance less than three is 3-colorable* [to appear in Acta Math. Sinica], and improved the result of [JCTB, 93(2005)303-311] by showing that *every plane graph with neither adjacent triangles nor cycles of length 5 and 7 is 3-colorable* [to appear in JCTB] (this result implies every plane graph without cycles of length in $\{4, 5, 7\}$ is 3-colorable).

Xu and Zhang's conjecture

For attacking a conjecture by Xu (and independently by Lih and Wang) which claims that *every plane graph without adjacent triangles is 4-list colorable*, Xu and Zhang proved that *every toroidal graph without adjacent triangles is $(4, 1)^*$ -list colorable* [to appear in Disc. Applied Math.], and conjectured that *every plane graph without adjacent triangles is $(3, 1)^*$ -list colorable*.

This conjecture is still open. We even do not know whether it is true that every plane graph without adjacent triangles is $(3, 1)^*$ -colorable. We believe this is true, but cannot prove it yet.

Motivated by Borodin and Raspaud's conjecture and the conjecture just mentioned, we consider the $(3,1)^*$ -colorability of plane graphs, and proved that *plane graphs with neither cycles of length 5 nor adjacent triangles are $(3,1)^*$ -colorable*. **This result indicates that every plane graph G which contains neither adjacent triangles nor cycles of length 5 contains a matching M such that $G \setminus M$ is 3-colorable (this provides a partial solution to Borodin and Raspaud's conjecture [JCTB, 93(2003)17-27]).**

In fact, we prove a little stronger result. We first introduce a new concept—*super-extendability*.

Let C be a separating cycle of a plane graph G . Suppose that $G \setminus \text{int}(C)$ admits a k -coloring ϕ , $G \setminus \text{ext}(C)$ admits a k -coloring ψ , and ϕ and ψ coincide on C . Then, ϕ together with ψ gives a k -coloring of G . In this situation, we usually say that the coloring on C is extended to G . This is not true again while we consider defective colorings. Even $G \setminus \text{int}(C)$ admits a $(k, 1)^*$ -coloring ϕ' , $G \setminus \text{ext}(C)$ admits a $(k, 1)^*$ -coloring ψ' , and ϕ' and ψ' coincide on C , ϕ' and ψ' may not offer a $(k, 1)^*$ -coloring of G .

For applying the technique of extending a local coloring to global, we introduce a new property named as *super-extendability*. Let G be a graph, H be an induced subgraph of G , and ϕ be a $(k, 1)^*$ -coloring of H for some integer $k \geq 2$. We call (H, ϕ) a *super-extendable pair* of G if ϕ can be extended to a $(k, 1)^*$ -coloring ϕ' of G in such a way that for every vertex $v \in V(G) \setminus V(H)$ and every neighbor u of v in H , $\phi'(v) \neq \phi(u)$. H is called a *k-super-extendable subgraph* if (H, ϕ) is super-extendable for every $(k, 1)^*$ -coloring ϕ of H .

Theorem 1 *Let G be a plane graph with neither adjacent triangles nor cycles of length 5, C be a triangle, or a cycle of length 7. Then, C is a 3-super-extendable cycle.*

This theorem tells that every $(3, 1)^*$ -coloring of an arbitrary triangle or cycle of length 7 can be extended to whole graph.

Let G be a plane graph containing neither adjacent triangles nor cycles of length 5. If G contains no triangles, by Grötzsch's theorem, G is 3-colorable and certainly $(3, 1)^*$ -colorable. If G contains a triangle, then any $(3, 1)^*$ -coloring of the triangle can be super-extended to G .

So, every plane graph with neither adjacent triangles nor cycles of length 5 is $(3, 1)^*$ -colorable.

First we find some *reducible configurations* concerning with $(3,1)^*$ -colorability. The proof is completed then by applying *discharging method* on the plane graphs without these reducible configurations.

This result is sharp since there exist non- $(2,1)^*$ -colorable plane graphs with neither triangles nor 5-cycles. Let $l \geq 3$ be an integer, H be the graph obtained from two paths $v_0v_1 \dots v_{2l}$ and $u_0u_1 \dots u_{2l}$ by adding edges u_iv_i ($0 \leq i \leq 2l$), and let B be the graph obtained from a $(2l+1)$ -cycle $x_0x_1 \dots x_{2l}x_0$ and copies of H by identifying x_{2i} with u_0 , identifying x_{2i+1} with v_0 and v_{2l} , and identifying x_{2i+2} with u_{2l} , $i = 0, 1, 2, \dots, l$, and the summations of the subindices are taken modulo $2l + 1$.

Thank you for your attendance!