On the Trees with Maximum Nullity

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1 Introduction

- The number of zero eigenvalues in the spectrum of the graph G is called its nullity and is denoted by $\eta(G)$.

- Let D be a positive integer. Denote by $\mathcal{T}(n, D)$ the set of all n-vertex trees in which all vertex degrees are less than or equal to D.

For $D = 1$ and $n \geq 3$, $\mathcal{T}(n, D) = \emptyset$. For $D = 2$ and $n \geq 3$, each set $\mathcal{T}(n, D)$ consists of a single element (the n-vertex path P_n, for which $\eta(P_n) \leq 1$). Therefore, in what follow, we assume that $D \geq 3$.

Lemma 1.1. If T is an n-vertex tree and m is the size of its maximal matchings, then its nullity is equal to $\eta(T) = n - 2m$.

Lemma 1.2. The maximum number of vertices of a tree $T \in \mathcal{T}(D)$, such that $m(T) = k$, is equal to $kD + 1$.

Lemma 1.3. If $T \in \mathcal{T}(n, D)$, then $m(T)$ is at least $\lceil (n-1)/D \rceil$.

Lemma 1.4. For all $n \geq 1$ and $D \geq 3$, if $T \in \mathcal{T}(n, D)$, then $\eta(T) \leq n - 2\lceil (n-1)/D \rceil$. For all $n \geq 1$ and $D \geq 3$, there exsit trees $T \in \mathcal{T}(n, D)$, such that $\eta(T) = n - 2\lceil (n-1)/D \rceil$.

- An edge belonging to a matching of a graph G is said to cover its two end-vertices. A vertex is said to be perfectly covered (PC) if it is covered in all maximal matchings of G.

2 A conjecture about trees with greatest nullity

Let $\mathcal{T}(n, D, \max)$ be the set of trees from $\mathcal{T}(n, D)$ with maximum nullity (equal to $n - 2\lceil (n-1)/D \rceil$). S. Fiorini, I. Gutman and I. Sciriha made the following constructing way of the trees in $\mathcal{T}(n, D, \max)$.

- A subset of $\mathcal{T}(n, D, \max)$, denoted by $\mathcal{T}_1(n, D, \max)$, is constructed as follows: For $n = 1, 2, \cdots, D$, the unique element of $\mathcal{T}_1(n, D, \max)$ is the n-vertex star. For $n = kD + i, k \geq 1, i = 1, 2 \cdots D$, any tree$\in \mathcal{T}_1(n, D, \max)$ is obtained from tree $T' \in \mathcal{T}_1(n-D, D, \max)$ and a copy of a D-vertex star, by joining one vertex of T' of degree less than D to the center of S_D.

- Another subset of $\mathcal{T}(n, D, \max)$, denoted by $\mathcal{T}_2(n, D, \max)$, is constructed as follows: Any trees $T' \in \mathcal{T}_2(n, D, \max)$ is obtained from some tree $T \in \mathcal{T}_1(n, D, \max)$ by moving (one-by-one) some pendant vertices of T to some other PC-vertices of T following by
  (i) the vertex degrees do not exceed D, and that
  (ii) in each step the vertex to which a pendant vertex is added is PC.

- example

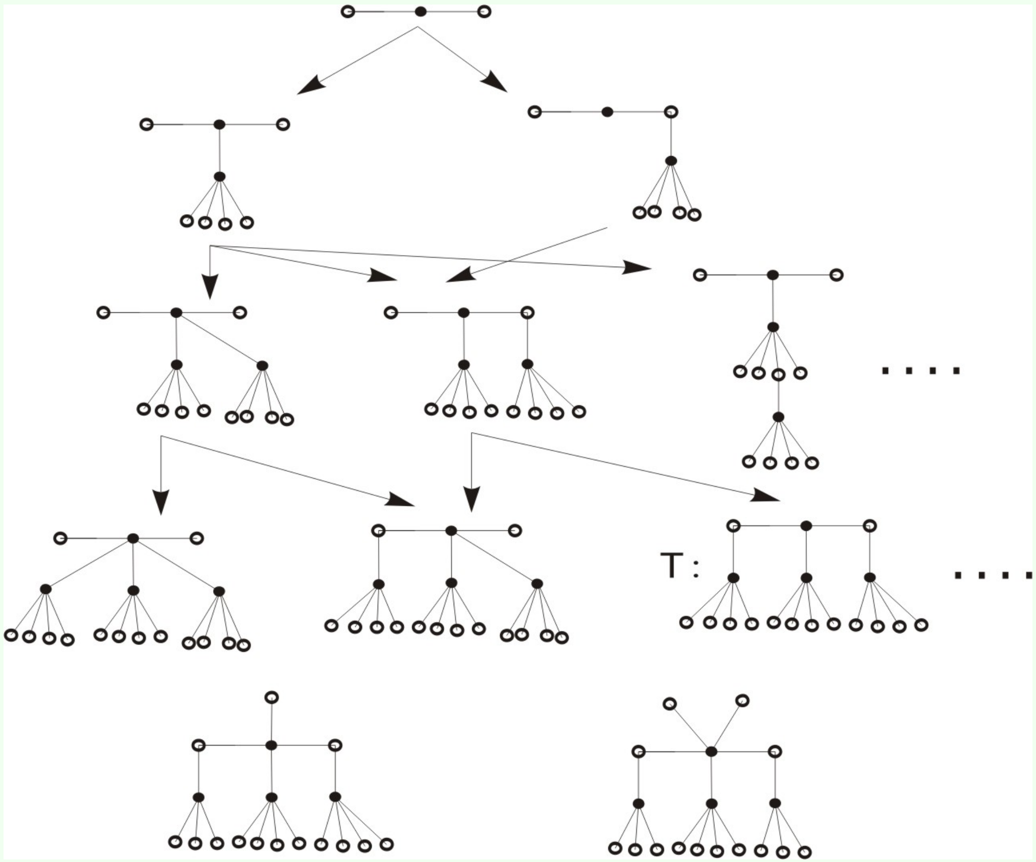

Fig. 1. some trees in $\mathcal{T}_1(18, 5, \max)$ and some trees in $\mathcal{T}_2(18, 5, \max)$ obtained from T, where $k = 3, i = 3$ and $m(G) = 4$

- **Conjecture.**
$$\mathcal{T}(n, D, \max) = \mathcal{T}_1(n, D, \max) \bigcup \mathcal{T}_2(n, D, \max)$$

However, the graph G in Fig. 2 shows that above conjecture is not true. We can easily know $G \in \mathcal{T}(18, 5, \max)$. But G cann't be constructed in the way above.

- **counterexample**

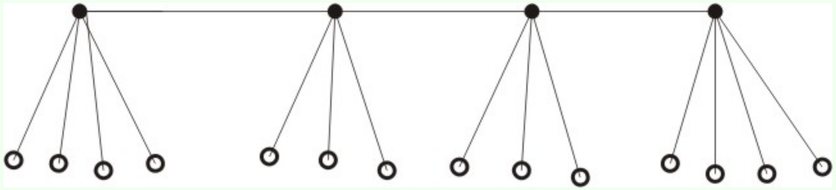

Fig. 2. Tree in $\mathcal{T}(18, 5, \max)$

3 Main result

We first modify the constructing method in section 2 as the following:

For $n = 1, 2, \cdots, D$, the unique element of $\mathcal{T}_1^*(n, D, \max)$ is the n-vertex star. For $n = kD + i, k \geq 1, i = 1, 2, \cdots, D$, any tree in $\mathcal{T}_1^*(n, D, \max)$ is obtained from tree $T' \in \mathcal{T}_1^*(n-D, D, \max) \bigcup \mathcal{T}_2^*(n-D, D, \max)$ and a copy of a D-vertex star, by joining one vertex of T' with degree less than D to the center of S_D, where $\mathcal{T}_2^*(n-D, D, \max)$ is obtained by moving (one-by-one) some pendant vertices of $T \in \mathcal{T}_1^*(n-D, D, \max)$ to some other PC-vertices, taking care above rule (i) and (ii).

- **Conclusion**

$$\mathcal{T}(n, D, \max) = \mathcal{T}_1^*(n, D, \max) \bigcup \mathcal{T}_2^*(n, D, \max)$$

Let $p(T)$ denote the number of pendant edges of T. A vertex-induced subgraph H of a graph G is called a *pendant star* of order k if

1. $H = S_k$ $(k \geq 2)$ and
2. Pendant vertices of H are also pendant vertices in G.

- **Lemma 3.2.** If T is a tree in $\mathcal{T}(n, D, \max)$ with t pendant star, $D \geq 3$, then $p(T) \geq t + D - 2$, except $S_i, i = 1, 2, \cdots, D-1$, T_1 in Fig.4 and T_2 in Fig.5.

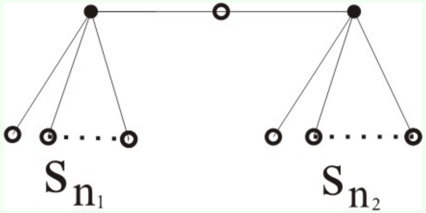

Fig. 4. T_1, $n_1 + n_2 = D + 1$

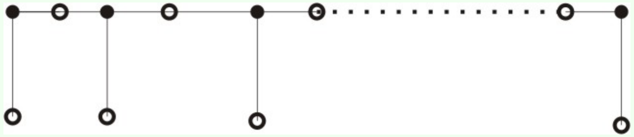

Fig. 5. T_2 $D = 3$, $k \geq 2$

Proof of the conclusion. Let $n = kD + i,\ i = 2,\ 3, \cdots, D + 1$ where $k \geq 0$. Then $m(T) = k + 1$. Moreover $n = 1$, the result is trivial.

We proceed by induction on k.

$k = 0$, then $m = 1$, $T \cong S_n$, the result is clear.

$k = 1$,

- If $T \cong T_1$, the result follows immediately by moving all pendant edges of S_{n_1}(or S_{n_2}) to the center of the other.
- If T is not isomorphic to T_1, by lemma 3.2, T has at least D pendant edges.

Assume the result holds for any positive integer less than k. We prove that the result is still true for k.

- If $T \cong T_2$, S_3 will appear in the resulting graph as a pendant star.
- If T is not isomorphic to T_2, by Lemma 3.2, $p(T) \geq t + D - 2$. Then move some pendant edges in pendant stars $S_i, i = 2, \cdots, t$ to v_1(one-by-one), where $d(v_1) = |S_1|$, taking care that every other pendant star still has at least one pendant edge. Thus, S_D appears in the resulting graph as a pendant star.

  Deleting S_D, the resulting tree has $(k-1)D + i$ vertices and $m(T_b) = k$. Thus $T_b \in \mathcal{T}(n - D, D, \max)$.

Thank you!

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