

**Extreme Values
for the Sum of Powers of Degrees of Graphs**

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Joint Work with Yonglin Guo

1. Definitions and the problem

By $\mathcal{G}(n, m)$ we denote the set of simple graphs with n vertices and m edges. For a simple graph G , we define

$$\sigma_2(G) = \sum_{v \in V(G)} d(v)^2.$$

Problem.

Let n and m be two integers with $n \geq 2$ and $m \geq 0$. Determine

$$\min\{\sigma_2(G) \mid G \in \mathcal{G}(n, m)\}$$

and

$$\max\{\sigma_2(G) \mid G \in \mathcal{G}(n, m)\}.$$

2. Motivation

- A basic extremal problem
- Algebraic graph theory
- Information theory and network reliability
- Chemical graph theory

3. Results for general graphs

An Trivial Fact.

Let G be a simple graph with $n \geq 2$ vertices and $m \geq 0$ edges. Then, among all graphs with n vertices and m edges, $\sigma_2(G)$ attains the smallest value if and only if G is an **almost regular graph**, i.e., $\Delta(G) - \delta(G) \leq 1$.

For given integers $n \geq 2$ and $m \geq 0$, define

$$G_1(p, q, r) = K_p \oplus (E_q \cup K_{1,r}),$$

where $n = p + q + r + 1$, $m = \binom{p}{2} + p(q + r + 1) + r$; and

$$G_2(p, q, r) = E_p \cup (K_q \oplus \overline{K_{1,r}}),$$

where $n = p + q + r + 1$, $m = \binom{q}{2} + \binom{r}{2} + q(r + 1)$.

Theorem (Ahlsweide & Katona, 1978; Boesch et al. 1990)
 Among all graphs with $n \geq 2$ vertices and $m \geq 0$ edges, at least one of $\sigma_2(G_1(p, q, r))$ and $\sigma_2(G_2(p, q, r))$ attains the maximum value.

For given nonnegative integers a, b, c, d , define

$$G_1^*(a, b, c, d) = K_a \oplus (E_b \cup (K_c \oplus E_d))$$

and

$$G_2^*(a, b, c, d) = E_a \cup (K_b \oplus (E_c \cup K_d)).$$

Theorem (Peled, Petreschi & Sterbini, JGT, 1999)
 Among all graphs with $n \geq 2$ vertices and $m \geq 0$ edges, if $\sigma_2(G)$ attains the maximum value, then G has the form $G_1^*(a, b, c, d)$ or $G_2^*(a, b, c, d)$ with $b \leq 1$ or $c \leq 1$ or $d \leq 1$.

4. Results for bipartite graphs

We define a bipartite graph $B^s(n, m)$ with n vertices and m edges. Let $2m = nt + r$, where $0 \leq r < n$. If n is even, then set

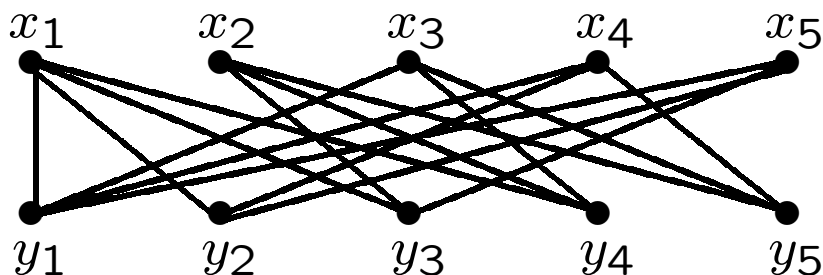
$$V(B^s(n, m)) = \{x_i, y_i \mid 1 \leq i \leq \frac{n}{2}\}$$

and

$$E(B^s(n, m)) = \{x_i y_j \mid i < j \leq i + t, 1 \leq i, j \leq \frac{n}{2}\} \cup \{x_i y_i \mid 1 \leq i \leq \frac{r}{2}\},$$

where addition is taken modulo $\frac{n}{2}$.

Example. The bipartite graph $B^s(10, 16)$.

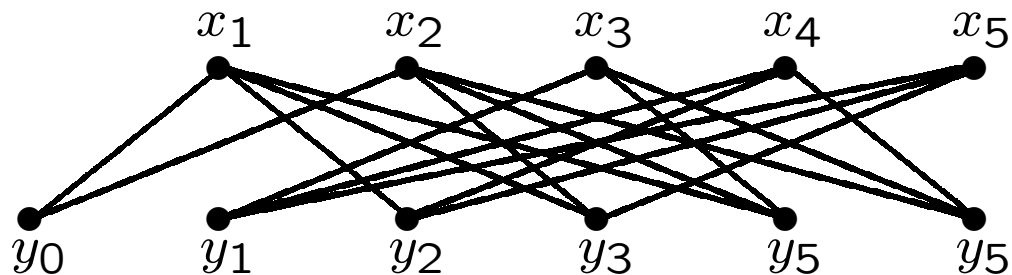


If n is odd and $nt \leq 2m < nt + t$, then define $B^s(n, m)$ as

$$B^s(n-1, m-(t-1)) \cup \{x_i y_0 \mid \frac{n+r-t+1}{2} + 1 \leq i \leq \frac{n+r+t-1}{2}\}.$$

where addition is taken modulo $\frac{n-1}{2}$.

Example. The bipartite graph $B^s(11, 17)$.

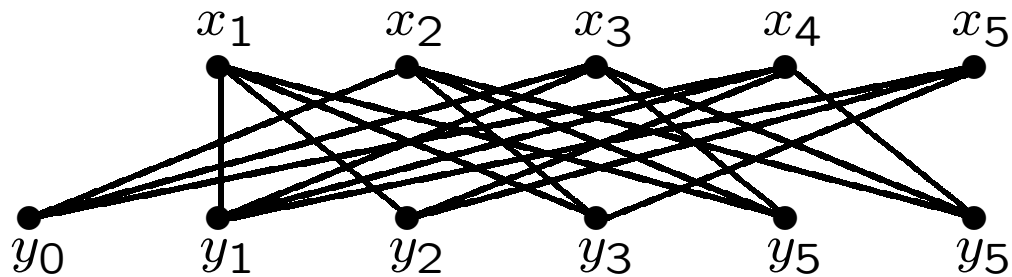


If n is odd and $nt + t \leq 2m \leq nt + n - t - 1$ or $nt + n - t + 1 \leq 2m < nt + n$, then define $B^s(n, m)$ as

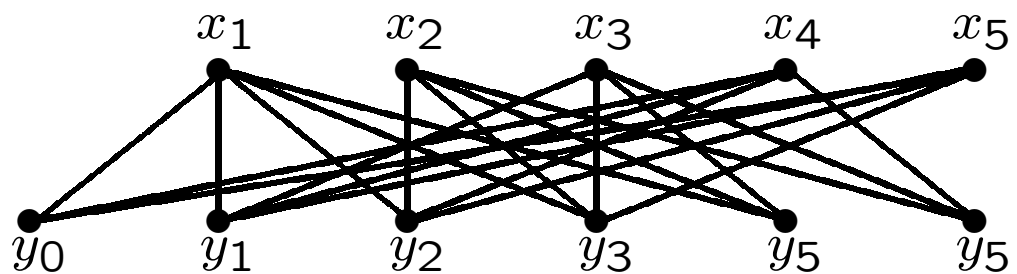
$$B^s(n, m) = B^s(n-1, m-t) \cup \{x_i y_0 \mid \frac{r-t}{2} + 1 \leq i \leq \frac{r+t}{2}\},$$

where addition is taken modulo $\frac{n-1}{2}$.

Example. The bipartite graph $B^s(11, 19)$.



Example. The bipartite graph $B^s(11, 21)$.



Theorem

Among all bipartite graphs with n vertices and m edges, $\sigma_2(B^s(n, m))$ attains the minimum value.

Let n and m be two integers with $n \geq 2$ and $0 \leq m \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$. Suppose that $(k+1)(n-k-1) < m \leq k(n-k)$ and $m = qk + r$, where $0 \leq r < k$. Then a bipartite graph $B^l(n, m)$ with n vertices and m edges is defined by

$$V(B^l(n, m)) = \{x_i | 1 \leq i \leq k\} \cup \{y_j | 1 \leq j \leq n - k\}$$

and

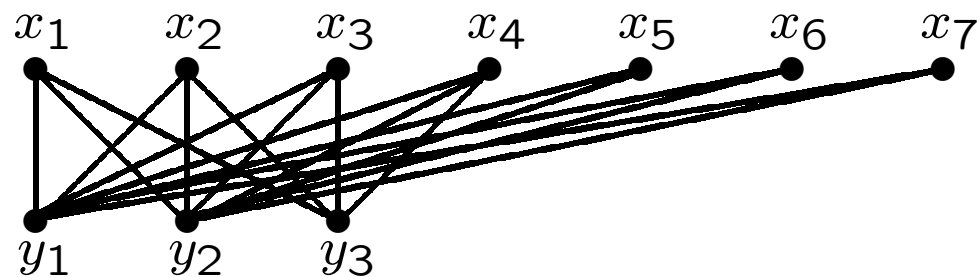
$$E(B^l(n, m)) = \{x_i y_j | 1 \leq i \leq k, 1 \leq j \leq q\}$$

if $r = 0$,

$$E(B^l(n, m)) = \{x_i y_j | 1 \leq i \leq k, 1 \leq j \leq q\} \cup \{x_i y_{q+1} | 1 \leq i \leq r\}.$$

if $r > 0$.

Example. The bipartite graph $B^l(10, 18)$.



Theorem

Among all bipartite graphs with n vertices and m edges, $\sigma_2(B^l(n, m))$ attains the maximum value.

5. Further discussions

For a graph G , define

$$\sigma_\alpha(G) = \sum_{v \in V(G)} d(v)^\alpha.$$

Problem.

Let n and m be two integers with $n \geq 2$ and $m \geq 0$, and α a real number other than 0 and 1. Determine

$$\min\{\sigma_\alpha(G) \mid G \in \mathcal{G}(n, m)\}$$

and

$$\max\{\sigma_\alpha(G) \mid G \in \mathcal{G}(n, m)\}.$$

Theorem. (Ismailescu & Stefanica, JGT, 2002)

Let G be a simple graph with $n \geq 2$ vertices and $m \geq 0$ edges, and α a real number with $0 < \alpha \leq \frac{1}{2}$. Suppose $\sigma_\alpha(G)$ attains the minimum value. If $m = \binom{k}{2}$ for some integer k , then $G \cong K_k \cup E_{n-k}$; Otherwise, let k be the unique integer such that $\binom{k-1}{2} < m < \binom{k}{2}$. Then, $G \cong E_{n-k} \cup (K_{m-\binom{k-1}{2}} \oplus K_{1, \binom{k}{2}-m})$.

Theorem. (Ismailescu & Stefanica, JGT, 2002)

Let n and m be two integers with $m \leq n - 2$, and α an integer greater than 1. Then, among all graphs with n vertices and m edges, $\sigma_\alpha(K_{1,m} \cup E_{n-m-1})$ attains the minimum value. Moreover, it is unique with this property, except the case when $\alpha = 2$ and $m = 3$.

Li and Zhao (MATCH, 2004) characterized the **trees** with the first three smallest and largest σ_α when α is an integer or a fraction $\frac{1}{k}$ for a nonzero integer k .

Zhang and Zhang (MATCH, 2006) characterized the **unicyclic graphs** with the first three smallest and largest σ_α for all α .

Zhang, Wang and Cheng (MATCH, 2006) characterized the **bicyclic graphs** with the first three smallest and largest σ_α for all α .

Our results on the minimum value of σ_2 for bipartite graphs can be generalized to the case $\alpha > 1$ and $\alpha < 0$ for σ_α .

Thank You!