On the Spectral Radius of Unicyclic Graphs and Bicyclic Graphs with Fixed Diameter

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Abstract

In this talk, we will determine graphs with the maximal spectral radius among all the unicyclic graphs and bi-cyclic graphs with n vertices and diameter d.

- The The investigation on the spectral radius of graphs is an important topic in the theory of graph spectra.

- Recently, the problem concerning graphs with maximal or minimal spectral radius of a given class of graphs has been studied extensively.

Some Known Results

• Brualdi and Solheid (1986) studied the spectral radius of connected graphs.

• Berman and Zhang (2001) studied the spectral radius of graphs with n vertices and k cut vertices.

• Liu, Lu and Tian (2004) studied the spectral radius of graphs with n vertices and k cut edges.

• Guo and Shao (2006) determine the first $\lfloor \frac{d}{2} \rfloor - 1$ spectral radii of trees with n vertices and diameter d.

The spectral radius of unicyclic graphs and bicyclic graphs have been studied by many authors.

- Chang and Tian (2003,2004) determined graphs of the largest and the second largest spectral radius among all the unicyclic graphs and bicyclic graphs on n vertices with perfect matching.

- Yu and Tian (2004,2005) determined graphs of the largest spectral radius among all the unicyclic graphs and bicyclic graphs on n vertices with a given size of maximum matching.

- Guo (2005) determined graphs with the largest spectral radius among all the unicyclic graphs and all bicyclic graphs with n vertices and k pendant vertices, respectively.

Preliminaries

- **Lemma 1 (Wu, Xiao, Hong).** Let G be a connected graph and $\rho(G)$ be the spectral radius of $A(G)$. Let u, v be two vertices of G. Suppose $v_1, v_2, \ldots, v_s \in N(v) \setminus N(u)$ ($1 \leq s \leq d_G(v)$) and $x = (x_1, x_2, \ldots, x_n)^t$ is the Perron vector of $A(G)$, where x_i corresponds to the vertex v_i ($1 \leq i \leq n$). Let G^* be the graph obtained from G by deleting the edges vv_i and adding the edges uv_i, $1 \leq i \leq s$. If $x_u \geq x_v$, then

$$\rho(G) < \rho(G^*).$$

• **Lemma 2 (Hoffman, Smith).** Let G be a connected graph, and let $uv \in E(G)$.

(i) If uv does not belong to an internal path of G and $G \ncong C_n$, then $\rho(G) < \rho(G_{u,v})$;

(ii) If uv belongs to an internal path of G and $G \ncong W_n$, where W_n is shown in Fig. 2, then $\rho(G) > \rho(G_{u,v})$.

Let G be a connected graph, and $uv \in E(G)$. The graph $G_{u,v}$ is obtained from G by subdividing the edge uv, that is, adding a new vertex w and edges wu, wv in $G - uv$.

• **Lemma 3 (Li, Feng).** Let G be a non-trivial connected graph, and let $u, v \in V(G)$, $d(u, v) = s$. Suppose that two paths of lengths k, m $(k \geq m \geq 1)$ are attached to G by their ends vertices at u, v, respectively, to form $G^*_{k,m}$. Then $\rho(G^*_{k,m}) > \rho(G^*_{k+1,m-1})$ for $s = 0$, $d(u) \geq 1$ or $s = 1$, $d(u), d(v) \geq 2$.

• Lemma 4 (Hong, 1986): Let G be a unicyclic graph of order n, then $\rho(G) \geq \rho(C_n)$ and equality holds if and only if $G \cong C_n$.

• Lemma 5 (Simić 1989): Let G be a bicyclic graph of order n, then $\rho(G) \geq \rho(B(\lceil \frac{n}{3} \rceil, n+1-2\lceil \frac{n}{3} \rceil, \lceil \frac{n}{3} \rceil))$ and equality holds if and only if $G \cong B(\lceil \frac{n}{3} \rceil, n+1-2\lceil \frac{n}{3} \rceil, \lceil \frac{n}{3} \rceil)$.

• Let $\triangle_n^d$ be a graph of order n obtained from a triangle by attaching $n-d-2$ pendant edges and a path of length $\lfloor \frac{d}{2} \rfloor$ at one vertex of triangle, and a path of length $\lceil \frac{d}{2} \rceil - 1$ to another vertex of triangle, respectively.

• Let ∇_n^d be a graph of order n obtained from a triangle by attaching $n-d-2$ pendant edges and a path of length $\lceil \frac{d}{2} \rceil - 1$ at one vertex triangle, and a path of length $\lfloor \frac{d}{2} \rfloor$ to another vertex of triangle, respectively.

• Note that if $d = n-2$ or $d \equiv 1 (\mathrm{mod}\ 2)$, then $\triangle_n^d \cong \nabla_n^d$.

- Let $\triangle_n^{2t}$ and ∇_n^{2t} be the above two graphs shown in Fig. 1. Suppose that $2 \leq t \leq \lfloor \frac{n-3}{2} \rfloor$. Then

$$\rho(\triangle_n^{2t}) > \rho(\nabla_n^{2t}).$$

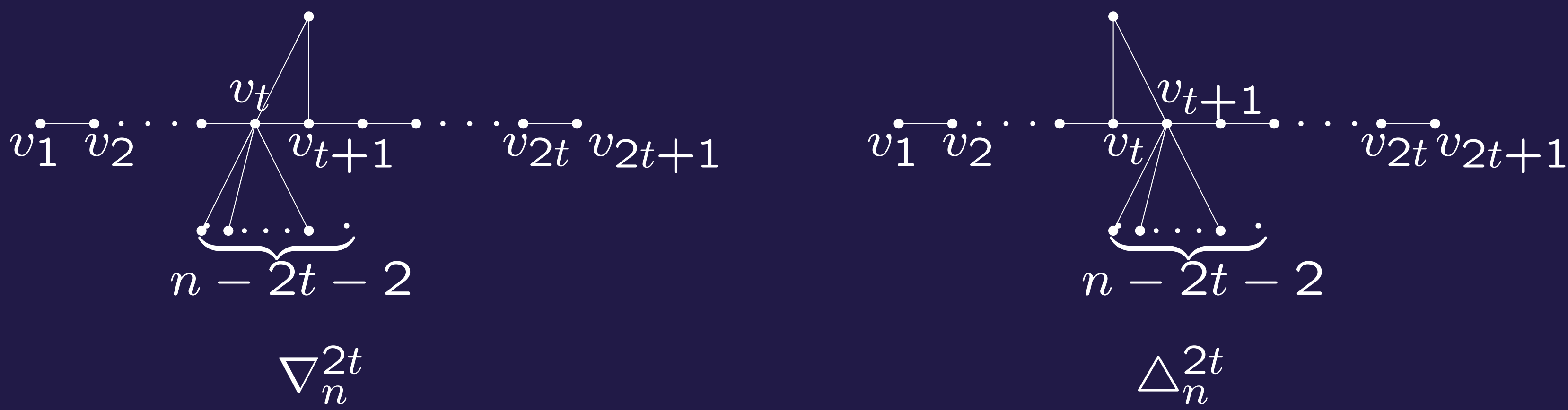

Fig. 1

Let U_0 be a unicyclic graph of order $d+2$ shown in Fig. 2. Let $U_0(p_2,\ldots,p_d,p_{d+2})$ be a graph of order n obtained from U_0 by attaching p_i pendant vertices to each $v_i \in V(U_0)\setminus\{v_1,v_{d+1}\}$, respectively, where $p_{d+2}=0$ when $k=1$ or $k=d$. Denote
$\widetilde{\mathcal{U}}_n^d = \left\{U_0(p_2,\ldots,p_d,p_{d+2}) : \sum_{i=2}^d p_i + p_{d+2} = n-d-2\right\}$,
$\overline{\mathcal{U}}_n^d = \{U_0(0,\ldots,0,p_i,0,\ldots,0) \ : \ U_0(0,\ldots,0,p_i,0,\ldots,0) \in \widetilde{\mathcal{U}}_n^d,\ p_i \geq 0\}$.

W_n U_0

Fig. 2

- Lemma 6. For any $G \in \widetilde{\mathcal{U}}_n^d$, $3 \leq d \leq n-2$, we have

$$\rho(G) \leq \rho(\triangle_n^d)$$

and equality holds if and only if $G \cong \triangle_n^d$.

Let Q_1 and Q_2 (shown in Fig. 3) be unicyclic graphs of order $d+2$ and order $d+3$, respectively. Let $Q_1(p_2, \ldots, p_d, p_{d+2})$ and $Q_2(p_2, \ldots, p_d, p_{d+2}, p_{d+3})$ be two graphs of order n obtained from Q_1, Q_2 by attaching p_i pendant vertices to each $v_i \in V(Q_j) \setminus \{v_1, v_{d+1}\}$, $j = 1, 2$, where $p_{d+2} = 0$ when $k = 1$ or $k = d$, and $p_{d+3} = 0$ when $k = 2$ or $k = d$, respectively. Denote

$$\mathcal{B}^1_{n,d} = \left\{Q_1(p_2, \ldots, p_d, p_{d+2}) : \textstyle\sum_{i=2}^{d} p_i + p_{d+2} = n - d - 2\right\},$$

$$\mathcal{B}^2_{n,d} = \left\{Q_2(p_2, \ldots, p_d, p_{d+2}, p_{d+3}) : \textstyle\sum_i p_i = n - d - 3\right\},$$

$$\widetilde{\mathcal{B}}^1_{n,d} = \left\{Q_1(0, \ldots, 0, p_i, 0, \ldots, 0) \in \mathcal{B}^1_{n,d} \ : \ p_i \geq 0\right\},$$

$$\widetilde{\mathcal{B}}^2_{n,d} = \left\{Q_2(0, \ldots, 0, p_i, 0, \ldots, 0) \in \mathcal{B}^2_{n,d} \ : \ p_i \geq 0, \right\}.$$

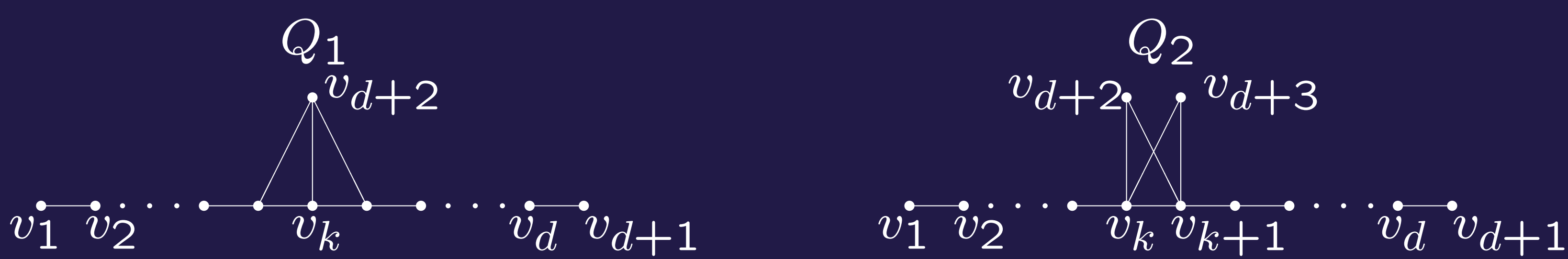

Fig. 3

Let

$$B_n^d = Q_1\left(0,\dots,0,p_{\lfloor\frac{d}{2}\rfloor+1},0,\dots,0\right) \in \widetilde{\mathcal{B}}_{n,d}^1,$$

$$\hat{B}_n^d = Q_1\left(0,\dots,0,p_{\lfloor\frac{d}{2}\rfloor+2},0,\dots,0\right) \in \widetilde{\mathcal{B}}_{n,d}^1,$$

$$\dot{B}_n^d = Q_1\left(0,\dots,0,p_{\lfloor\frac{d}{2}\rfloor},0,\dots,0\right) \in \widetilde{\mathcal{B}}_{n,d}^1,$$

$$\bar{B}_n^d = Q_2\left(0,\dots,0,p_{\lceil\frac{d}{2}\rceil},0,\dots,0\right) \in \widetilde{\mathcal{B}}_{n,d}^2,$$

$$\ddot{B}_n^d = Q_2\left(0,\dots,0,p_{\lceil\frac{d}{2}\rceil+1},0,\dots,0\right) \in \widetilde{\mathcal{B}}_{n,d}^2.$$

Then

(i) if $3 \le t \le \lfloor\frac{n-4}{2}\rfloor$, then $\rho(\hat{B}_n^{2t-1}) > \rho(\dot{B}_n^{2t-1})$;

(ii) if $3 \le t \le \lfloor\frac{n-3}{2}\rfloor$, then $\rho(\bar{B}_n^{2t}) > \rho(\ddot{B}_n^{2t})$;

(iii) $\rho(B_n^d) > \rho(\hat{B}_n^d)$ for $d \ge 3$;

(iv) $\rho(B_n^d) > \rho(\bar{B}_n^d)$ for $d \ge 3$.

Results

Theorem A. Let G be a graph in $\mathcal{U}_n^d$, $d \geq 1$. Then

$$\rho(G) \leq \rho(\triangle_n^d)$$

and equality holds if and only if $G = \triangle_n^d$.

Theorem B. Let $G \in \mathcal{U}_n^d \backslash \{\triangle_n^d\}$. Suppose that $d \equiv 0 (\mathrm{mod}\ 2)$ and $4 \leq d \leq n-3$. Then

$$\rho(G) \leq \rho(\nabla_n^d)$$

and equality holds if and only if $G = \nabla_n^d$.

Theorem C. Let G be a graph in $\mathcal{B}_n^d$, $d \geq 2$. Then

$$\rho(G) \leq \rho(B_n^d)$$

and equality holds if and only if $G = B_n^d$.

Proof of Theorems A and B.

Let $G \in \mathcal{U}_n^d$. If $d = 1$, then $G \cong C_3$. If $d = 2$, then $G \cong C_4$, $G \cong C_5$ or $G \cong \triangle_n^2$. Thus the result holds for $d = 1, 2$ by Lemma 4. Therefore, in the following, we can assume that $3 \le d \le n - 2$.

Choose $G \in \mathcal{U}_n^d$ such that the spectral radius of G is as large as possible. Then, by Lemma 4, we can assume that $G \neq C_n$.

Let $P_d = v_1 v_2 \cdots v_{d+1}$ be the induced path of length d and let C_q be the only cycle in G. Since $G \neq C_n$, we have $\min\{d(v_1), d(v_{d+1})\} = 1$, say $d(v_1) = 1$. We first show some claims.

Claim 1. $V(C_q) \cap V(P_d) \neq \emptyset$.

Claim 2. $d(v) = 1$ for $v \in V(G) \setminus (V(C_q) \cup V(P_d))$.

Claim 3. $k \neq l$.

Claim 4. If $l = k + 1$, then $s - d = 2$; and if $l \geq k + 2$, then $s - d = l - k$.

Claim 5. $l = k + 1$.

By Claims 4 and 5, we have $s = d + 2$. Then $G \in \widetilde{\mathcal{U}}_n^d$ by Claims 1 and 2. By Lemma 6, Theorem A follows immediately.

Thank you!