On the T-uniqueness of the Twisted Wheels

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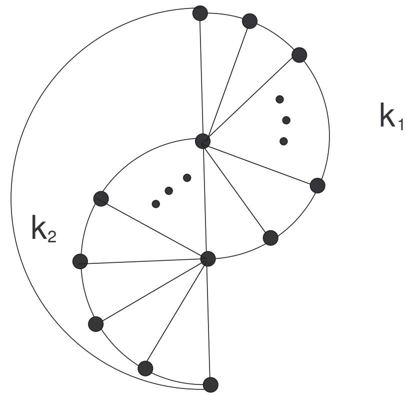
The framework of the talk

- **Definitions**
- **The history and background**
- **The outline proof of the main result**

Defination of the twisted wheel

The *twisted wheel* W_{k_1,k_2} is defined by $k_1 + k_2$ triangles constructed like:



We prove that the twisted wheel is T-unique.

Definitions

The **Tutte polynomial** of G is given by:

$$T(G;x,y) = \sum_{S \subseteq E} (x-1)^{r(E)-r(S)} (y-1)^{|S|-r(S)}.$$

For example, the Tutte polynomial of a triangle is:

$$(x-1)^2 + 3(x-1) + 3 + (y-1).$$

History and Back Ground

The Tutte polynomial is brought forward in 1954 by Tutte with the initial name dichromate polynomial.

It is an important tool either in graph theory or in related fields such as knot theory or statistical physics.

The Tutte polynomial of G contains a great deal of information about the graph G.

Rank-size generating Polynomials

Rank-size generating polynomial (Mier and Noy, 2004): $F(G;x,y)=\sum_{S\subseteq E} x^{r(S)}y^{|S|}$.

Both $F(G;x,y)$ and $T(G;x,y)$ contain exactly the same information about G.

The following is the information of a graph having been proved to be contained in its Tutte polynomial.

The information of a graph

The T-invariants:

1. G is simple or not;
2. $V(G)$ and $E(G)$;
3. $g(G)$;
4. $\lambda(G)$ and the number of the min-element bond;
5. The number of cliques of each size;
6. The number of C_3, C_4, C_5 and C_4^+.

Chromatic polynomial:

$p(G;x)$ = the number of *x-coloring* of G

A graph G is χ-*unique* if a graph H, $p(H) = p(G)$ implies that $H \cong G$.

Chromatic polynomial

Some known χ-unique graphs: complete graphs K_n, cycles C_n and bipartite graphs $K_{p,q}$ for $p, q \geq 2$ and so on.

It is proved that:

$$p(G; x) = (-1)^{r(G)} x^{k(G)} T(G; 1 - x, 0).$$

Clearly, Tutte polynomial contains more information about G.

T-equivalent and T-unique

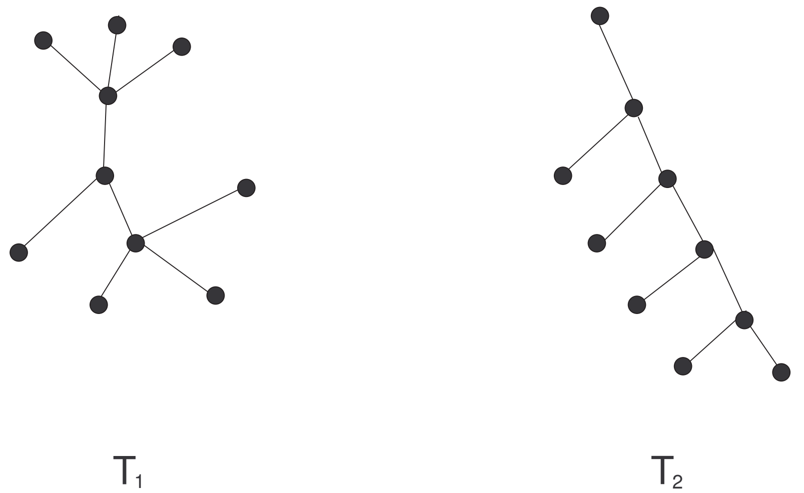
- We say that two graphs G_1 and G_2 are T-equivalent if $T(G_1; x, y) = T(G_2; x, y)$.
- We say that G is T-unique if for any graph H, $T(H) = T(G)$ implies that $H \cong G$.

Clearly, a χ-unique graph is also T-unique.

Question: If all the graphs are T-unique?

It is clearly not the case, for example, all the tree with n vertices have the same Tutte polynomial.

Counter examples



$$T(T_1; x, y) = T(T_2; x, y) = \sum_{i=0}^{10} \binom{10}{i} (x-1)^{9-i}$$

Counter examples

Bollobas and Riordan proved that: there indeed exist non-isomorphic graphs of arbitrarily high connectivity with the same Tutte polynomial.

Question: Which graphs are T-unique?

Known results on T-unique graphs

A.de Mier, M.Noy, (2004)

Wheels, squares of cycles, complte multipartite graphs, ladders, mobius ladders and hypercubes

J.S.Kuhl

Generalized Peterson graphs $P(m, 2)$

A.de Mier, M.Noy, (2005)

Line graphs

A,M$\acute{a}$rquez, A.de Mier, M.Noy, (2003)

Locally grid graphs

Triangle-graph of G:

$$G \Rightarrow G'$$

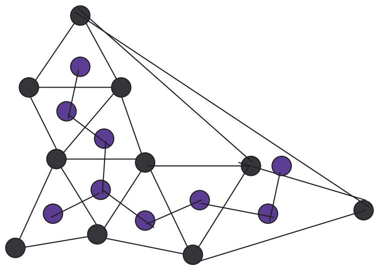
Proposition

- The number of vertices in G' is equal to the number of triangles in G;

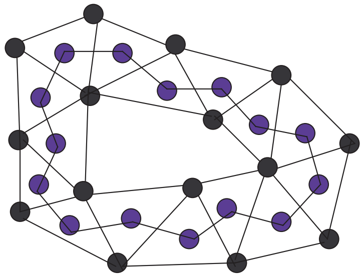
- The number of edges in G' is equal to the number of C_4^+ in G.

Definitions

- n-tree-corresponding graph $T^n \in \Gamma^n$.
- n-path-corresponding graph $P^n \in \wp^n$.
- n-cycle-corresponding graph $C^n \in \mathcal{C}^n$.



T^9



C^{15}

The Main Theorem (Duan, Wu and Yu 2006)

G is T-equivalent to the twist wheel W_{k_1,k_2}, $k_1 \geq k_2 \geq 3$. Then G is isomorphic to W_{k_1,k_2}.

The outline of the proof of the main Theorem

Let $k_1 + k_2 = n - 2$;

- G is simple and 2-connected
- $|V(G)| = n$, $|E(G)| = 2n - 2$
- the number of C_3 is $n - 2$
- the number of C_4 is $n - 2$
- the number of C_4^+ is $n - 3$
- the number of K_4 is 0
- G is 3-edge-connected, and the number of 3-element-bond is $n - 2$

The outline of the proof of the main result

G' is acyclic

$\Downarrow$

G' is a (n-2)-tree

$\Downarrow$

G' is a (n-2)-path

$\Downarrow$

G is isomorphic to $W_{k_1',k_2'}$

$\Downarrow$

G is isomorphic to W_{k_1,k_2}

Thank you

Thank You!