On the Determination Problem for P_4-Transformation of Graphs

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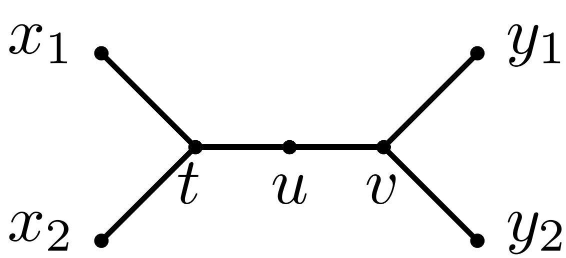
Definition of path graph

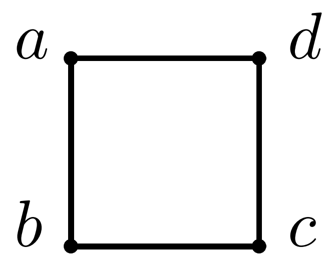
Denote by $\Pi_k(G)$ the set of all paths of G on k vertices ($k \geq 1$).

The *path graph* $P_k(G)$ of a graph G has vertex set $\Pi_k(G)$ and edges joining pairs of vertices that represent two paths P_k, the union of which forms either a path P_{k+1} or a cycle C_k in G.

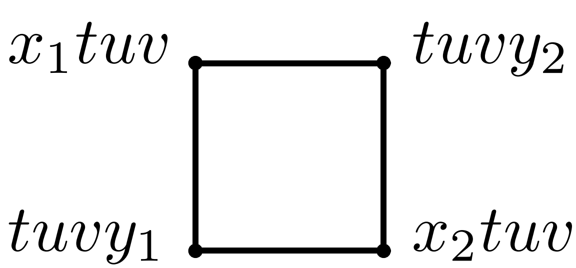
Example :



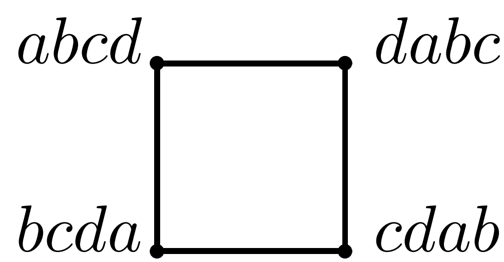
G



$G^{'}$



$P_4(G)$



$P_4(G^{'})$

For a graph transformation, there are two general problems.

Characterization Problem : Characterize those graphs that are the P_k-graph of some graph.

Determination Problem : Determine which graphs have a given graph as their P_k-graphs.

Definitions and Notations

A *vertex-isomorphism* from G to $G^{'}$ is a bijection $f : V(G) \to V(G^{'})$ such that two vertices are adjacent in G if and only if their images are adjacent in $G^{'}$. We let $\Gamma(G, G^{'})$ denote the set of all vertex-isomorphisms of G to $G^{'}$.

An *edge-isomorphism* from G to $G^{'}$ is a bijection $f : E(G) \to E(G^{'})$ such that two edges are adjacent in G if and only if their images are adjacent in $G^{'}$. We let $\Gamma_e(G, G^{'})$ denote the set of all edge-isomorphisms of G to $G^{'}$.

Definitions and Notations

We shorten $\Gamma(P_4(G), P_4(G^{'}))$ to $\Gamma_4(G, G^{'})$ and call the members P_4-isomorphisms from G to $G^{'}$.

For $f \in \Gamma_e(G, G^{'})$, define a mapping f^* by $f^*(tuvw) = f(tu)f(uv)f(vw)$ for a P_4-path $tuvw$ in G, and call f^* the mapping induced by f.

Determination Problem for $k = 2$, i.e., line graphs

Theorem 1 (H. Whitney, Amer. J. Math., 1932) *If G and G' are connected and have isomorphic line graphs, then G and G' are isomorphic unless one is $K_{1,3}$ and the other is K_3.*

Determination Problem for $k = 3$

(H.J. Broersma and C. Hoede, J. Graph Theory, 1989)
Broersma and Hoede showed that the answer to the determination problem for $k = 3$ was not as simple as for $k = 2$, and they found two pairs and two classes of nonisomorphic connected graphs with isomorphic connected P_3-graphs.

Determination Problem for $k = 3$

- (X. Li, J. Graph Theory, 1996) Li proved that the P_3-transformation is one-to-one on all graphs with $\delta \geq 4$.
- (X. Li, Ars Combinatoria, 1998) Li obtained the same result for all graphs with $\delta \geq 3$.
- (R.E.L. Aldred, M.N. Ellingham, R.L. Hemminger and P. Jipsen, J. Graph Theory, 1997) The authors completely solved the determination problem for $k = 3$.

Determination Problem for $k = 3$

Theorem 2 (Aldred, Ellingham, Hemminger and Jipsen) *Let τ be a P_3-isomorphism from G to H such that at least one of G or H is connected. Then τ is one of the following :*

- *T-related to a P_3-isomorphism of generalized $K_{3,3}$ type ;*
- *of special Whitney type ;*
- *D-related to a P_3-isomorphism of Whitney type $3, 4, 5$ or 6 ;*
- *D-related to a P_3-isomorphism of bipartite type ; or*
- *$TBSD$-related to an induced P_3-isomorphism.*

Determination Problem for $k \geq 4$

- (X. Li and B. Zhao, Discrete Math., 2004) Li and Zhao proved that for $k \geq 4$ the P_k-transformation is one-to-one on all graphs with minimum degree $\delta \geq k$.

- (X. Li and B. Zhao, Australian J. Combin., 1997) Li and Zhao also showed that the P_4-transformation is one-to-one on all graphs with $\delta = 3$ and satisfying one of two other conditions.

Determination Problem for $k = 4$

Denote by $\mathcal{G}_d$ the class of all connected graphs with minimum degree at least d.

Theorem 3 (Li and Zhao) *Let $G, G^{'} \in \mathcal{G}_3$. Assume G and $G^{'}$ satisfy one of the following two conditions :*

(1) if u is a vertex of some triangle in G, then $d(u) \geq 4$,
(2) G and $G^{'}$ do not contain any C_4 as a subgraph.

Then $f \in \Gamma_4(G, G^{'})$ if and only if f is induced by an isomorphism of G to $G^{'}$, i.e., $P_4(G)$ is isomorphic to $P_4(G^{'})$ if and only if G is isomorphic to $G^{'}$.

Our main results

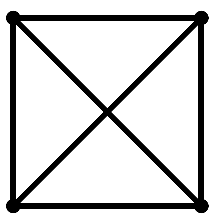
Theorem 4 *Let $G, G^{'} \in \mathcal{G}_3 \backslash \mathcal{H}$. Then $f \in \Gamma_4(G, G^{'})$ if and only if f is induced by an isomorphism from G to $G^{'}$, i.e., $P_4(G)$ is isomorphic to $P_4(G^{'})$ if and only if G is isomorphic to $G^{'}$.*

Corollary 5 *Let $G \in \mathcal{H}$ and $G^{'} \in \mathcal{G}_3$. If f is a P_4-isomorphism from G to $G^{'}$, then G is isomorphic to $G^{'}$.*

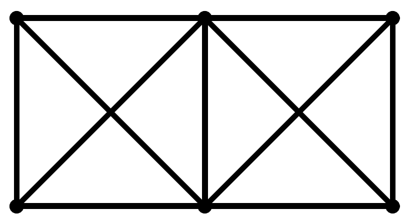
Corollary 6 *Let $G, G^{'} \in \mathcal{G}_3$, then the P_4-transformation is one to one.*

Our main results

Denote by $\mathcal{H}$ the set of all graphs obtained from n copies of K_4 by identifying one corresponding edge of each copy of K_4, where $n \geq 1$.



$n = 1$



$n = 2$

Proof of Theorem 4

If $P_4 = tuvw$, then the edge uv is called *middle edge* of the P_4 and $tuvw = wvut$.

We let $S(uv)$ denote the set of all P_4-paths with a common middle edge uv. Any subset of $S(uv)$ is called a *double star* at the edge uv.

A mapping $f : \Pi_4(G) \to \Pi_4(G^{'})$ is called *double star-preserving at the edge uv* if the set $f(S(uv))$ is a double star in $G^{'}$, and if the set $f(S(uv))$ is a double star in $G^{'}$ for every edge uv of G, then f is called *double star-preserving*.

Proof of Theorem 4

We let $E_1(G) = \{uv \in E(G) \mid uv$ is a common edge of two triangles with $d(u) = d(v) = 3\}$.

Denote by $E_2(G) = E(G) \backslash E_1(G)$.

Lemma 1 *Let $G, G' \in \mathcal{G}_3$ and let $f : \Pi_4(G) \rightarrow \Pi_4(G')$ be a bijective mapping. Then f is induced by an edge-isomorphism from G to G' if and only if f and f^{-1} are double star-preserving P_4-isomorphisms.*

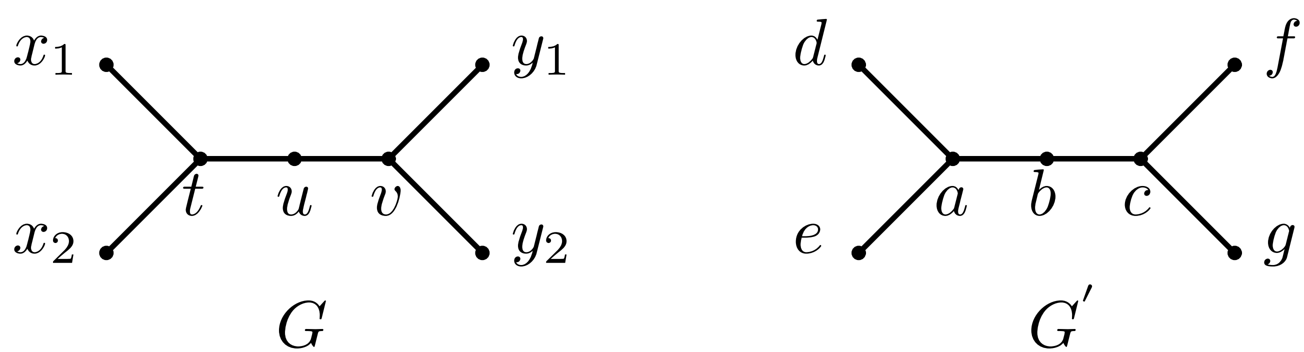
Proof of Theorem 4

Lemma 2 *Let $G, G^{'} \in \mathcal{G}_3 \backslash \mathcal{H}$ and let f be a P_4-isomorphism from G to $G^{'}$. Then f is double star-preserving at the edge uv, where $uv \in E_2(G)$.*

Lemma 3 *Let $G, G^{'} \in \mathcal{G}_3 \backslash \mathcal{H}$ and let f be a P_4-isomorphism from G to $G^{'}$. Then f is double star-preserving at the edge uv, where $uv \in E_1(G)$.*

Proof of Theorem 4

Lemma 4 *Let $G, G^{'} \in \mathcal{G}_3$ and $f \in \Gamma_4(G, G^{'})$. If x_1tuv, x_2tuv, $tuvy_1$ and $tuvy_2$ are four P_4-paths of G, then $f(x_1tuv)$ and $f(x_2tuv)$ have a common middle edge, and $f(tuvy_1)$ and $f(tuvy_2)$ have a common middle edge, respectively.*



Lemma 5 *Let $G, G' \in \mathcal{G}_3$ and $f \in \Gamma_4(G, G')$. If there is a $C_4 = xtuvx$ in G, then $f(xtuv)$, $f(tuvx)$, $f(uvxt)$ and $f(vxtu)$ form a C_4 in G'.*

Thanks !