On (s,t)-Supereulerian Triangle Free Graphs

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Abstract

For two integers $s \geq 0, t \geq 0$, G is $(s,t) - supereulerian$, if $\forall X, Y \subset E(G)$, with $X \cap Y = \phi, |X| \leq s, |Y| \leq t$, G has a spanning eulerian subgraph H with $X \subset E(H)$ and $Y \cap E(H) = \phi$. Clearly, G is supereulerian if and only if G is (0,0)-supereulerian. In this note, we show that if G is a $(2+t)$-edge-connected triangle-free simple graph on n vertices with $\delta(G) \geq \frac{n}{10} + t$, then when $n \geq 41$, G is (2,t)-supereulerian or can be contracted to some well classified special graphs. Our result extends the result in [Journal of Graph Theory 12 (1988) 11-15].

Keywords: (s,t)-supereulerian; Collapsible graph; Reduction

1 notation

1). For a graph G, $O(G)$ denotes the set of all vertices of odd degree in G.

2).For an integer $i \geq 1$, define $D_i(G) = \{v \in V(G) | d_G(v) = i\}$.

3).For a graph G with a subgraph H, the contraction G/H is the graph obtained from G by replacing H by a vertex v_H, such that the number of edges in G/H joining any $v \in V(G) - V(H)$ to v_H in G/H equals the number of edges joining v in G to H; v_H is *nontrivial* if $E(H) \neq \phi$.

4).A graph G with $O(G) = \phi$ is an *even graph*, and a connected even graph is an *eulerian graph*.

5).A graph is *supereulerian* if it has a spanning eulerian subgraph. The collection of all supereulerian graphs will be denoted by SL.

6).Let $F(G)$ denote the minimum number of extra edges that must be added to G so that the resulting graph has 2-edge-disjoint spanning trees.

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2 Collapsible graphs , reduced graphs and some results about it

A graph G is *collapsible* if for every set $R \subset V(G)$ with $|R|$ even, there is a spanning connected subgraph H_R of G, such that $O(H_R) = R$. Thus K_1 is both supereulerian and collapsible. Denote the family of collapsible graphs by CL. Let G be a collapsible graph and let $R = \phi$. Then by definition G has a spanning connected subgraph H with $O(H) = \phi$, and so G is supereulerian. Therefore, we have $CL \subset SL$.

In [6], Catlin showed that every graph G has a unique collection of pairwise disjoint maximal collapsible subgraphs $H_1, H_2, ..., H_c$. The contraction of G obtained from G by contracting each H_i into a single vertex$(1 \leq i \leq c)$, is called the *reduction* of G. A graph is *reduced* if it is the reduction of some other graph.

Theorem 1 (Catlin Theorem 9 of [6]) Let G be a 2-edge-connected simple graph on n vertices, if $\delta(G) > \frac{n}{5} - 1$ and $n > 20$, then G is $(0,0)$-supereulerian.

Theorem 2 (Lai Theorem 5 of [3]) Let G be a 2-edge-connected triangle-free simple graph on $n > 30$ vertices, if $\delta(G) > \frac{n}{10}$, then G is $(0,0)$-supereulerian.

Theorem 3 (Catlin Theorem 8 of [6]) H is collapsible and $H \subset G$, then $G \in SL \Leftrightarrow G/H \in SL$.

Theorem 4 (Catlin, Han and Lai, Lemma 2.3 of [7]) If G is reduced with $|V(G)| \geq 3$, then $F(G) = 2|V(G)| - |E(G)| - 2$.

Theorem 5 (Catlin, Theorem 7 of [6], Catlin, Han and Lai, Theorem 1.3 of [7]) Let G be a connected reduced nontrivial graph. If $F(G) \leq 1$, then $G \in \{K_1, K_2\}$; If $F(G) \leq 2$, then $G \in \{K_1, K_2, K_{2,t}\ (t \geq 1)\}$.

Theorem 6 (Caltin, Theorem 8 and Lemma 5 of [6]) If G is reduced, then G is simple and has no K_3. Moreover, if $\kappa'(G) \geq 2$, then $\sum_{i=2}^{4} |D_i(G)| \geq 4$, and when $\sum_{i=2}^{4} |D_i(G)| = 4$, G must be eulerian.

Definition of $\mathcal{F}$: The Peterson graph is denoted by P. Let s_1, s_2, s_3, m, l, t be natural numbers with $t \geq 2$ and $m, l \geq 1$. Let $M \cong K_{1,3}$ with center a and ends a_1, a_2, a_3. Define $K_{1,3}(s_1, s_2, s_3)$ to be the graph obtained from M by adding s_i vertices with neighbors $\{a_i, a_{i+1}\}$, where $i \equiv 1, 2, 3 \pmod 3$. Let $K_{2,t}(u, u^{'})$ be a $K_{2,t}$ with $u, u^{'}$ being the nonadjacent vertices of degree t. Let $K^{'}_{2,t}(u, u^{'}, u^{''})$ be the graph obtained from a $K_{2,t}(u, u^{'})$ by adding a new vertex $u^{''}$ that joins to $u^{'}$ only.Hence $u^{''}$ has degree 1 and u has degree t in $K^{'}_{2,t}(u, u^{'}, u^{''})$. Let $K^{''}_{2,t}(u, u^{'}, u^{''})$ be the graph obtained from a $K_{2,t}(u, u^{'})$ by adding a new vertex $u^{''}$ that joins to a vertex of degree 2 of $K_{2,t}$. Hence $u^{''}$ has degree 1 and both u and $u^{'}$ have degree t in $K^{''}_{2,t}(u, u^{'}, u^{''})$. We shall use $K^{'}_{2,t}$ and $K^{''}_{2,t}$ for a $K^{'}_{2,t}(u, u^{'}, u^{''})$ and a $K^{''}_{2,t}(u, u^{'}, u^{''})$, respectively. Let $S(m, l)$ be the graph obtained from a $K_{2,m}(u, u^{'})$ and a $K^{'}_{2,l}(w, w^{'}, w^{''})$ by identifying u with w,and $w^{''}$ with $u^{'}$. Let $J(m, l)$ denote the graph obtained from a $K_{2,m+1}$ and a $K^{'}_{2,l}(w, w^{'}, w^{''})$ by identifying w,$w^{''}$ with the two ends of an edge in $K_{2,m+1}$,respectively. Let $J^{'}(m, l)$ denote the graph obtained from a $K_{2,m+2}$ and a $K^{'}_{2,l}(w, w^{'}, w^{''})$ by identifying w,$w^{''}$ with two vertices of degree 2 in $K_{2,m+2}$, respectively. see Figure for examples of these graphs.

Let $\mathcal{F} = \{K_1, K_2, k_{2,t}, K^{'}_{2,t}, K^{''}_{2,t}, K_{1,3}(s, s^{'}, s^{''}), S(m, l), J(m, l), J^{'}(m, l), P\}$ where $t, s, s^{'}, s^{''}, m, l$ are nonnegative integers.

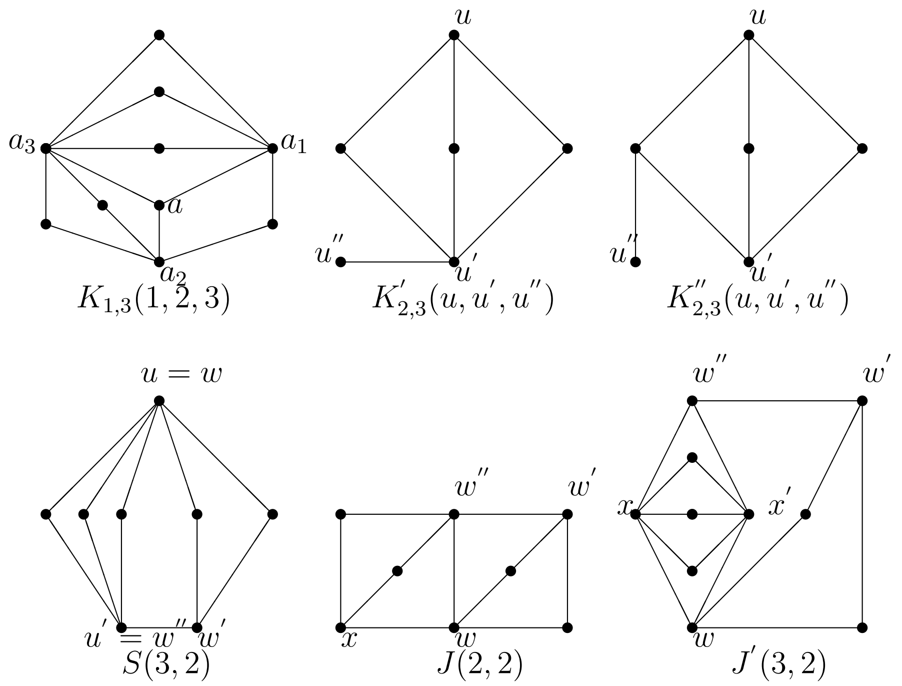

Figure: Some graphs in $\mathcal{F}$ with small parameters

Theorem 7 (Chen and Lai Theorem 2.4 of [9]) If G is a connected reduced graph with $|V(G)| \leq 11$ and $F(G) \leq 3$, then $G \in \mathcal{F}$.

3 Main result: $(s,t)-supereulerian$

We shall consider the problem of finding eulerian subgraphs that contain given edge subsets and exclude given edge subsets.

Definition: For two integers $s \geq 0, t \geq 0$, G is $(s,t)-supereulerian$, if $\forall X, Y \subset E(G)$, with $X \cap Y = \phi, |X| \leq s, |Y| \leq t$, G has a spanning eulerian subgraph H with $X \subset E(H)$ and $Y \cap E(H) = \phi$. Clearly, G is supereulerian if and only if G is (0,0)-supereulerian. The following have been proved.

The purpose of paper is to extend Theorem 2 to its (s,t)-supereulerian version.

Let $\mathcal{F}' = \mathcal{F} + \{\cup_{X \subset E(K_{2,t})}(K_{2,t})_X$: t is nonehative integer}-{Eulerian graph}.

Let G be g graph and let $X \subset E(G)$. The graph G_X is obtained from G by replacing each edge $e \in X$ with end u_e and v_e by a (u_e, v_e)-path P_e of length 2, where the internal vertex $w(e)$ of the path P_e is newly added.

Lemma 8 Let G be a graph and let $X \subset E(G), Y \subset E(G)$ with $X \cap Y = \phi$, then G has a spanning eulerian graph such that $X \subset E(H)$ and $Y \cap E(H) = \phi$ if and only if $(G - Y)_X \in SL$.

Lemma 8 follows from the definitions.

Theorem 9 Let $s \leq 2$ and $t \geq 0$ be two integers. Let G be a $2+t$-edge-connected triangle-free simple graph on n vertices. If $\delta(G) \geq \frac{n}{10} + t$, then when $n \geq 41$, $\forall X \subset E(G), Y \subset E(G)$ with $X \cap Y = \phi, |X| \leq s, |Y| \leq t$, exactly one of the following holds:

(i) G has a spanning eulerian graph H, such that $X \subset E(H)$ and $Y \cap E(H) = \phi$ (e.g. G is (2,t)-supereulerian).

(ii) The reduction of $(G - Y)_X$ is a member of $\mathcal{F}'$.

Note that in Theorem 9, G must be $2+t$-edge-connected because that $G - Y$ is supereulerian only if $G - Y$ is 2-edge-connected.

In the proof of Theorem 9, we need the following lemma:

Lemma 10 Let G be a triangle-free simple graph with $\delta(G) \geq 5$. G_0 is the reduction of G. For $v \in V(G_0)$ with $d_{G_0}(v) \leq 4$, let H_v be the preimage of v, then $|V(H_v)| \geq 2\delta(G)$.

Proof: Let $v \in V(G_0)$ with $d_{G_0}(v) \leq 4$, H_v the preimage of v. Let $e_1, e_2, \cdots, e_i$ be incident with v in G_0. Since $\delta(G) \geq 5$, H_v is nontrivial and there is a vertex in $V(H_v)$,denoted by $v^{'}$, not incident with $e_1, e_2, \cdots, e_i$. Then the neighbor of $v^{'}$ is completely in H_v. Similarly, by $d_G(v^{'}) \geq 5$, there must be a vertex in $V(H_v)$, denoted by $v^{''}$,$v^{'} \neq v^{''}$, not incident with $e_1, e_2, \cdots, e_i$. Since G is triangle-free, the intersection of the neighbor of $v^{'}$ and $v^{''}$ is empty. It follows that $|V(H_v)| \geq d_G(v^{'}) + d_G(v^{''}) \geq 2\delta(G)$.

the outline of Proof of Theorem 9: Let G_0 be the reduction of $(G-Y)_X$. By Lemma 8, to prove Theorem 9, we need only to prove that G_0 is either supereulerian or a member of $\mathcal{F}^{'}$. As G is $2+t$-edge-connected, so $(G-Y)_X$ and G_0 is 2-edge-connected.By Theorem 4-7,we discuss G_0 recording to the value of $F(G_0)$.

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