

On Distance Constrained Labeling of Graphs

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Outline

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Channel Assignment Problem

The *Channel Assignment Problem* is to assign a channel (nonnegative integer) to each radio transmitter so that interfering transmitters are assigned channels whose separation is not in a set of disallowed separations.

William K. Hale ¹ formulated $F * D - CCAP$.

Fred S. Roberts ² proposed the distinction of “close” and “very close” .

¹William K. Hale, *Frequency assignment: theory and applications*, Proc. IEEE 68, 1980, 1497-1513.

²Fred S. Roberts, *T-Colorings of graphs: recent results and open problems*, Discrete Math. 93, 1991, 229-245.

Interference Graph

The *interference graph* $G = (V, E)$ is defined as follows:

Each vertex $v \in V$ corresponds to a transmitter location, and adjacent vertices correspond to pairs of transmitters that interfere with each other due to their proximity.


- ▶ weighted graphs
- ▶ distance constrained

$L(2, 1)$ -Labeling

Griggs and Yeh³ proposed $L(2, 1)$ -Labeling as a variation of the T -coloring problem.

An $L(2, 1)$ -Labeling of a graph G is an **integer** assignment f to the vertices of G such that

$$|f(u) - f(v)| \geq \begin{cases} 2, & d(u, v) = 1; \\ 1, & d(u, v) = 2. \end{cases}$$

³Jerrold R. Griggs and Roger K. Yeh(叶光清), *Labelling graphs with a condition at distance 2*, SIAM J. Discrete Math. 5(4), 1992, 586-595. 

λ -Number

To optimize the assignment f , we seek to minimize the *span*

$$sp(f) := \max_{v \in V} f(v) - \min_{v \in V} f(v).$$

And define

$$\lambda(G) = \min_f sp(f).$$

Results


Many results were developed: P_n, C_n, Q_n, T, W_n ; planar graphs, chordal graphs, products, etc..

Hemant Balakrishnan and Narsingh Deo ⁴ gave the Nordhaus-Gaddum Type Result:

$$2\sqrt{n} - 2 \leq \lambda(G) + \lambda(\bar{G}) \leq 3n - 3$$

$$0 \leq \lambda(G) \cdot \lambda(\bar{G}) \leq \left(\frac{3n - 3}{2}\right)^2$$

And presented a parallel algorithm: LF (Large-degree First).

⁴Hemant Balakrishnan and Narsingh Deo, Parallel Algorithm for Radiocoloring a Graph, Congr. Numer. 160, 193-204, 2003. 


Conjecture

For any graph G with maximum degree $\Delta \geq 2$, $\lambda(G) \leq \Delta^2$.

Proving Progress

- ▶ $\lambda(G) \leq \Delta^2 + 2\Delta$, 1992, Griggs and Yeh;
- ▶ $\lambda(G) \leq \Delta^2 + \Delta$, 1996, Chang and Kuo⁵;
- ▶ $\lambda(G) \leq \Delta^2 + \Delta - 1$, 2003, Král' and Skrekovski⁶.
- ▶ ...

⁵Gerard J. Chang(张镇华) and David Kuo(郭大卫), *The $L(2,1)$ -labeling problem on graphs*, SIAM J. Discrete Math. 9(2), 1996, 309-316.

⁶Daniel Král' and Riste Škrekovski, *A theorem about the channel assignment problem*, SIAM J. Discrete Math. 16(3), 2003, 426-437. 

General Proximity-interference Problem

The natural generalization of $\lambda(G)$ to deal with multiple levels of interference was **reconsidered** in Griggs and Jin⁷: Let \mathbb{N} denote the set of natural numbers, $\{0, 1, 2, \dots\}$. Given integers $k_1, \dots, k_p \in \mathbb{N}$, let $L(k_1, \dots, k_p)$ (abbreviated $L(\mathbf{k})$) denote the set of labellings $f : V \rightarrow \mathbb{N}$ such that for all $u, v \in V$,

$$|f(u) - f(v)| \geq k_i, \quad \text{if } d(u, v) = i \leq p.$$

⁷J. R. Griggs and X. T. Jin, *Real Number Graph Labelings with Distance Conditions*, IMI-RR 2004:24, USC; SIAM J. Discrete Math. 20(2), 2006, 302-327.

General Proximity-interference Problem

We seek the optimal span of any labeling f , denoted by

$$\lambda(G; k_1, \dots, k_p) := \min_{f \in L(\mathbf{k})} sp(f).$$

Griggs and Jin , by doing a greedy first-fit labeling, proved

$$\lambda(G; \mathbf{k}) \leq \sum_{i=1}^p 2k_i \Delta (\Delta - 1)^{i-1}.$$

But they mainly focused on the PL(Piecewise Linearity) properties of $\lambda(G; \mathbf{k})$.

First-Fit

```
for s from 1 to n
  c=0
  while v_s not colored do
    if v_s could be colored with color c then
      f(v_s)=c
    end if
    c=c+1
  done
next s
```

“One-Pass”

```
c=0
while not all vertices are colored do
  for s from 1 to n
    if v_s not colored
      and v_s could be colored with color c then
        f(v_s)=color
    end if
  next s
  c=c+1
done
```

Vertex Orderings

A suitable linear order is critical⁸. Here, we choose a linear order that minimizes the number of predecessors to which any vertex is (multi-step) adjacent.

A simple case is an ordering v_1, v_2, \dots, v_n such that for each vertex v_i , $1 \leq i \leq n - 1$, there is another vertex v_j ($i < j$) adjacent with v_i .

Another example is the ordering which achieves the *coloring number* $col(G)$.

⁸H. A. Kierstead and Daqing Yang, *Orderings on Graphs and Game Coloring Number*, Order 20, 2003, 255-264.

General Upper Bound

Theorem

Let G be any graph with maximum degree Δ ,

$\mathbf{k} = (k_1, \dots, k_p) \geq 0$. Then

$$\lambda(G; \mathbf{k}) \leq \sum_{i=1}^p k_i \Delta (\Delta - 1)^{i-1}.$$

Proof Sketch

Applying the “One-Pass” Algorithm.

For any vertex $u \in V$, we have

$$|N_i(u)| \leq \Delta(\Delta - 1)^{i-1}.$$

At most $k_i |N_i(u)|$ integers could be prevented labeling vertex u . Altogether, we have

$$f(u) \leq \sum_{i=1}^p k_i |N_i(u)| \leq \sum_{i=1}^p k_i \Delta(\Delta - 1)^{i-1}.$$

Motivation

The Moore graphs with diameter $D \geq 3$ and degree $\delta \geq 3$ has **defect** at least 2 to Moore bound $M_{\Delta,D}$ ⁹.

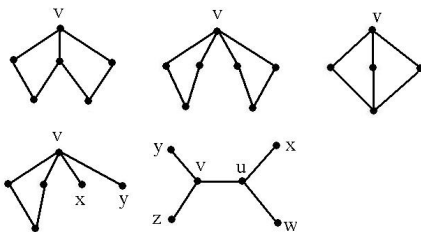
Whether we could decrease the forbidden list to improve the upper bound, by a careful-made vertices ordering?

As examples, (1) G has a vertex v with degree less than Δ ;
(2) G has a triangle.

⁹EJC, 2005, #DS14

Key Lemma

Every triangle-free regular graph G with degree $\Delta \geq 3$ has one of the following subgraphs:



$G \setminus \{x, y\}$, $G \setminus \{x, y, z\}$ are connected.

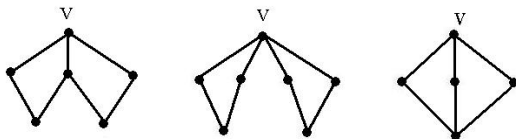
Proof Point. Reduction of 2-connected graphs.

New Vertex Orderings

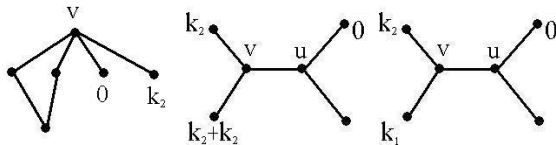
We can reorder the vertices, according to this lemma, in the following ways:

1. post-ordering of the vertices $V(G)$ produced by a depth-first search algorithm started in v ;
2. x, y + post-ordering of the vertices $V(G \setminus \{x, y\})$ produced by a depth-first search algorithm started in v .
3. x, y, z + post-ordering of the vertices $V(G \setminus \{x, y, z\})$ produced by a depth-first search algorithm started in v .

Decreasing δ



Decrease $\delta \geq 2k_2$;



$$\delta \geq 2(k_1 - k_2) \quad \delta \geq 2(k_1 - k_2) \quad \delta \geq k_1$$

Better Upper Bound with Special Sequences

Theorem

For any graph G with maximum degree $\Delta \geq 3$ and

$k_1 > k_i \geq 1, 2 \leq i \leq p$, we have

$$\lambda(G; \mathbf{k}) \leq \sum_{i=1}^p k_i \Delta (\Delta - 1)^{i-1} - 2.$$

A Little Closer to Conjecture

Back to $L(2, 1)$ -Labeling.

Since Griggs and Yeh proved $\lambda(C_n) = 4$, we get one step closer to the conjecture

$$\lambda(G) \leq \Delta^2 + \Delta - 2.$$

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Questions ?

Thanks!