Connectedness of realization graphs of 4-regular graphs with minimum decycling number

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Background

[Bau and Beineke, 2002]

Problem 1. Which cubic graphs G with $|G| = 2n$ satisfy $\phi(G) = \left\lceil \frac{n+1}{2} \right\rceil$?

Problem 2. Which cubic planar graphs G with $|G| = 2n$ satisfy $\phi(G) = \left\lceil \frac{n+1}{2} \right\rceil$?

$\diamond$ Which 4-regular graphs G with $|G| = n$ satisfy $\phi(G) = \left\lceil \frac{n+1}{3} \right\rceil$?

Decycling Number

For a graph G and $S \subseteq V(G)$, if $G - S$ is acyclic, then S is said to be a *decycling set* of G. The cardinality of the smallest decycling set of G is called the *decycling number* of G and is denoted by $\phi(G)$. Let $F = G - S$. Then F is an induced forest of G and it is a *maximum induced forest* of G if $|S| = \phi(G)$.

Let $\mathcal{R}(\mathbf{d})$ be the class of all graphs having degree sequence $\mathbf{d}$.

we define

$$min(\phi, \mathbf{d}) = min\{\phi(G) : G \in \mathcal{R}(\mathbf{d})\} \text{ and } max(\phi, \mathbf{d}) = max\{\phi(G) : G \in \mathcal{R}(\mathbf{d})\}.$$

$$min\left(\phi, (4^n)\right) = \left\lceil \frac{n+1}{3} \right\rceil.$$

Switching operation

- For any graph G and independent edges ab,cd in G with ac,$bd \notin E(G)$, let

$$G^{\sigma(a,b;c,d)} = G - \{ab, cd\} + \{ac, bd\}.$$

  The operation $\sigma(a, b; c, d)$ is called a *switching operation*.

- Let $G \in \mathcal{R}(\mathbf{d})$. A switching σ is called an $\mathcal{R}(\mathbf{d})$-*safe switching* with respect to G,

  if $G^{\sigma} \in \mathcal{R}(\mathbf{d})$.

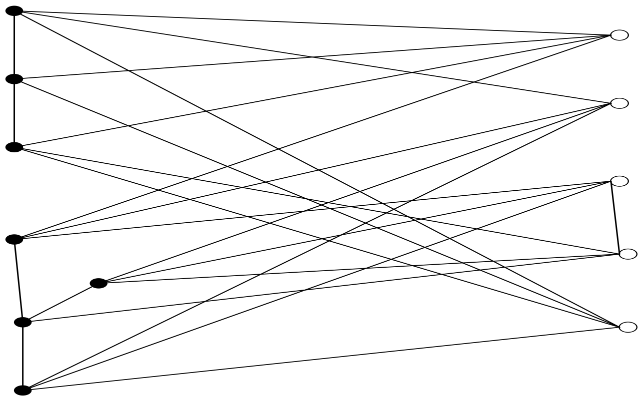
Connectedness

- For any $G_1, G_2 \in \mathcal{R}(\mathbf{d})$, G_1 and G_2 are *adjacent* if G_2 can be obtained from G_1 by a switching.
- For any $G_1, G_2 \in \mathcal{R}(\mathbf{d})$, G_1 and G_2 are *connected* if G_2 can be obtained from G_1 by a finite sequence of $\mathcal{R}(\mathbf{d})$-safe switchings.

Main result

Let $\mathcal{R}\left(4^n; \left\lceil \frac{n+1}{3} \right\rceil\right)$ be the set of all 4-regular graphs with order n whose decycling number is $\left\lceil \frac{n+1}{3} \right\rceil$.

We will prove that $\mathcal{R}\left(4^n; \left\lceil \frac{n+1}{3} \right\rceil\right)$ is connected under the switching operation.



○ : vertices in the decycling set S.
● : vertices in the maximum induced forest F.

$$e(S) + e(S,F) + e(F) = 2n$$

$$e(S,F) = 4|S| - 2e(S)$$

$$e(F) \leq |F| - 1$$

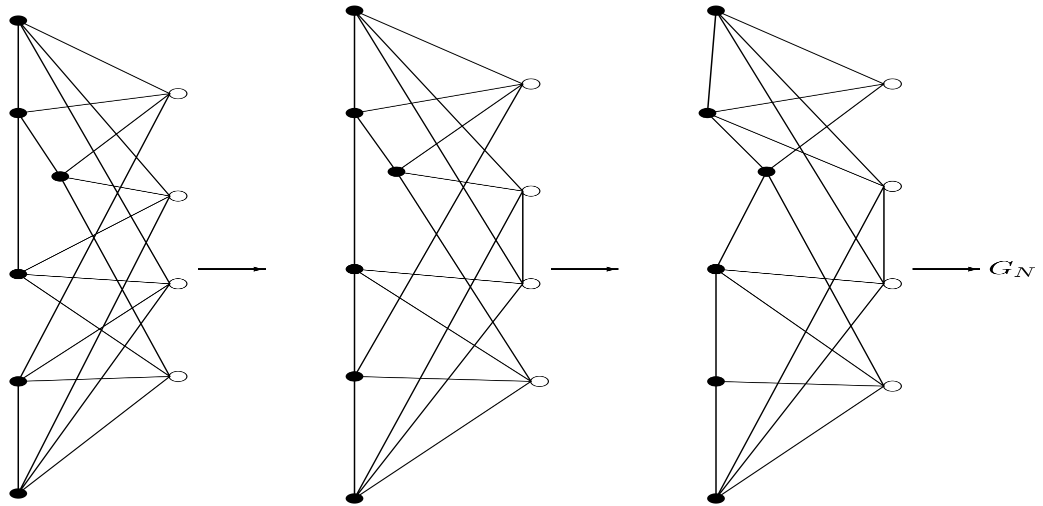
Structure

Let $G \in \mathcal{R}\left(4^n; \left\lceil \frac{n+1}{3} \right\rceil\right)$ and S be a minimum decycling set of G. Put $F = G - S$. Let $m \geq 3$ be an integer and $\omega(F)$ denote the number of components of F.

(1) If $n = 3m - 1$, then $e(S) = 0$ and $\omega(F) = 1$.

(2) If $n = 3m$, then
$e(S) = 0$ and $\omega(F) = 3$;
$e(S) = 1$ and $\omega(F) = 2$;
$e(S) = 2$ and $\omega(F) = 1$.

(3) If $n = 3m + 1$, then
$e(S) = 0$ and $\omega(F) = 2$;
$e(S) = 1$ and $\omega(F) = 1$.



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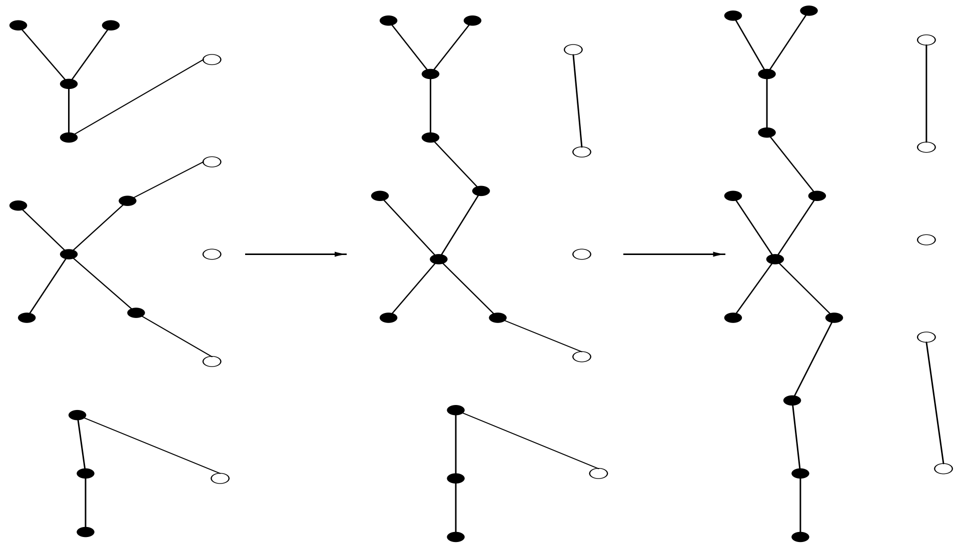

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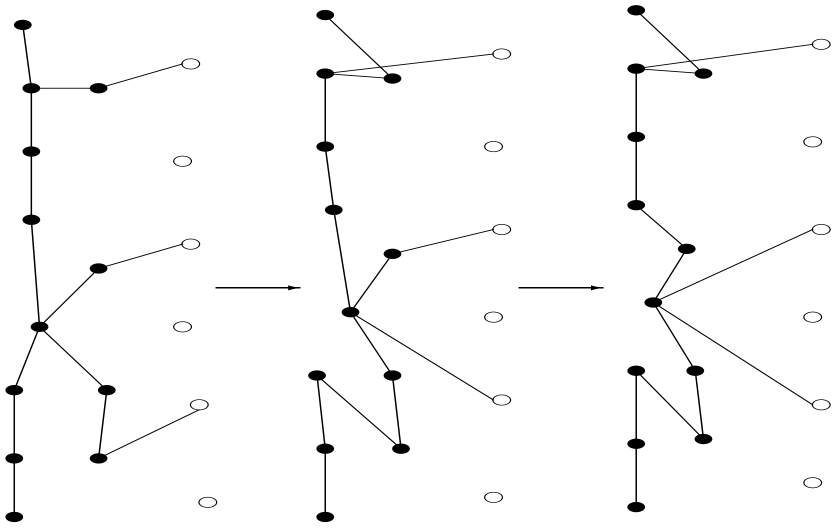

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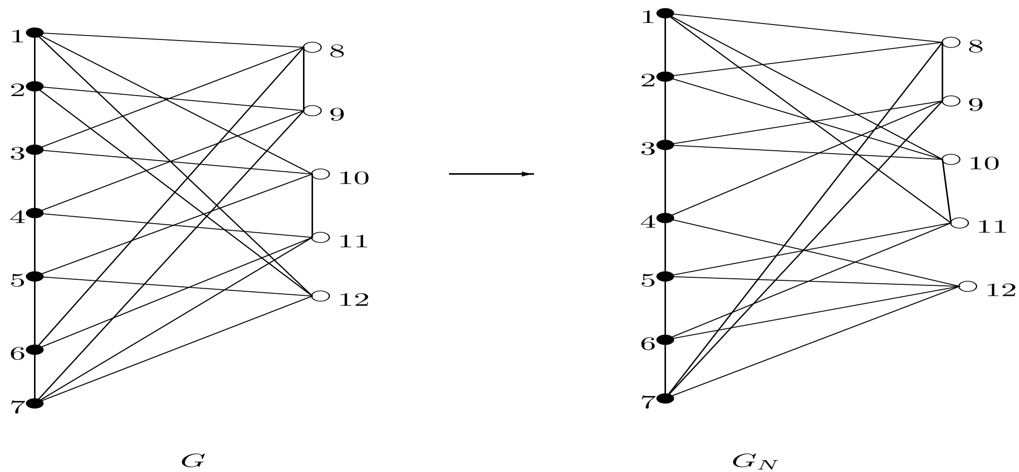

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Thanks for your attention