

圈长在图中的分布

Distribution of Cycle Lengths in Graphs

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1 Introduction

- **Definition**

- ★ A *cycle* is a connected 2-regular graph.
- ★ A *block* of G is a maximal nonseparable subgraph of G , and an *endblock* of G is a block that contains at most one cut vertex of G .
- ★ For $A, B \in V(G)$, $e(A, B)$ is the number of edges with one end in A and the other end in B .
- ★ For $x, y \in V(G)$, an (x, y) -*path* is a path from x to y ; and an (x, y) -path is *trivial* if $x = y$, in which the path consists of a single vertex.
- ★ An edge is *contracted* if it is removed and its two ends are identified.

• Background

Erdős asked whether every graph with minimum degree at least three contains two cycles whose lengths differ by one or two. Using nonseparating induced cycles, Bondy and Vince answered the question affirmatively. Fan proved the following more general result.

THEOREM 1.1.[Fan,2002] If G is a graph with minimum degree $\delta(G) \geq 3k$ for any positive integer k , then G contains $k + 1$ cycles C_0, C_1, \dots, C_k such that $k + 1 < |E(C_0)| < |E(C_1)| < \dots < |E(C_k)|$, $|E(C_i) - E(C_{i-1})| = 2$, $1 \leq i \leq k - 1$, and $1 \leq |E(C_k) - E(C_{k-1})| \leq 2$, and furthermore, if $\delta(G) \geq 3k + 1$, then $|E(C_k) - E(C_{k-1})| = 2$.

2 Main results

THEOREM 2.1. Let G be a graph with minimum degree $\delta(G) \geq 3$. For any positive integer k (if $k \geq 2$, then $\delta(G) \geq 4$), if $d_G(u) + d_G(v) \geq 6k - 1$ for every pair of adjacent vertices $u, v \in V(G)$, then G contains $k + 1$ cycles C_0, C_1, \dots, C_k such that $k + 1 < |E(C_0)| < |E(C_1)| < \dots < |E(C_k)|$, $|E(C_i)| - |E(C_{i-1})| = 2$, $1 \leq i \leq k - 1$, and $1 \leq |E(C_k)| - |E(C_{k-1})| \leq 2$, and furthermore, if $d_G(u) + d_G(v) \geq 6k + 1$, then $|E(C_k)| - |E(C_{k-1})| = 2$.

• Definition

- ★ For an edge $uv \in E(G)$, by *replacing uv with a cycle*, we mean the operation of deleting the edge uv and adding a new cycle C such that $V(C) \cap V(G) = \{u, v\}$.
- ★ An (x, y) -*string* (of k cycles) is the graph obtained from an (x, y) -path by replacing k edges of the path with k cycles, one edge with one cycle.
- ★ In a string, if C is the cycle replacing uv , then u and v are called the *connection vertices* of C .
- ★ C is *t-defective* if the two segments of C divided by u and v differ in length by t .
- ★ A string is *t-defective* if each of its cycles is t-defective.

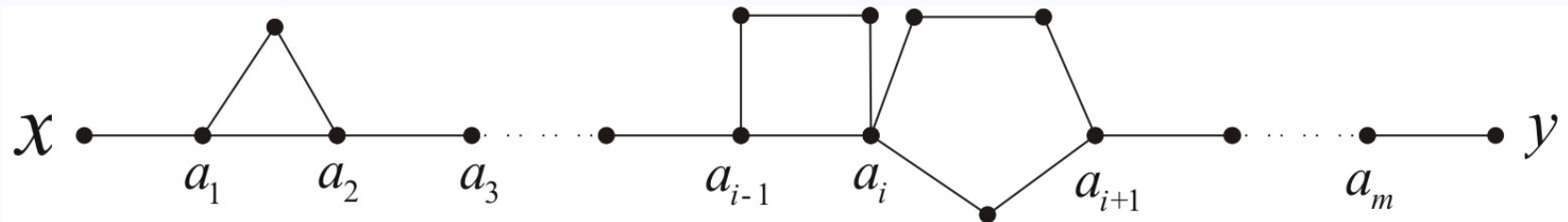


FIGURE 1

• Definition

★ An (x, y) -string S of k cycles can be represented by $S = P_0C_1P_1C_2 \cdots C_kP_k$. For each i , $1 \leq i \leq k$, let C'_i and C''_i be the two segments of C_i divided by its connection vertices such that $|E(C''_i)| \geq |E(C'_i)|$. The *length of S* is defined by

$$\ell(S) = \sum_{i=1}^k |E(C'_i)| + \sum_{i=0}^k |E(P_i)|.$$

★ Let $S = P_0C_1P_1 \cdots C_kP_k$ be an (x, y) -string of k cycles in a graph G . S is *feasible* (with respect to k and G) if all the following three statements hold.

- (1) $\sum_{i=0}^k |E(P_i)| \neq 0$.
- (2) C_i is 2-defective for every i , $1 \leq i \leq k$, with at most one exception.
- (3) If C_j is the exceptional cycle in (2), then C_j is 1-defective, and moreover, there is $uv \in E(C_j)$ such that $\{u, v\} \cap \{x, y\} = \emptyset$ and $d_G(u) + d_G(v) \in \{6k - 1, 6k\}$.

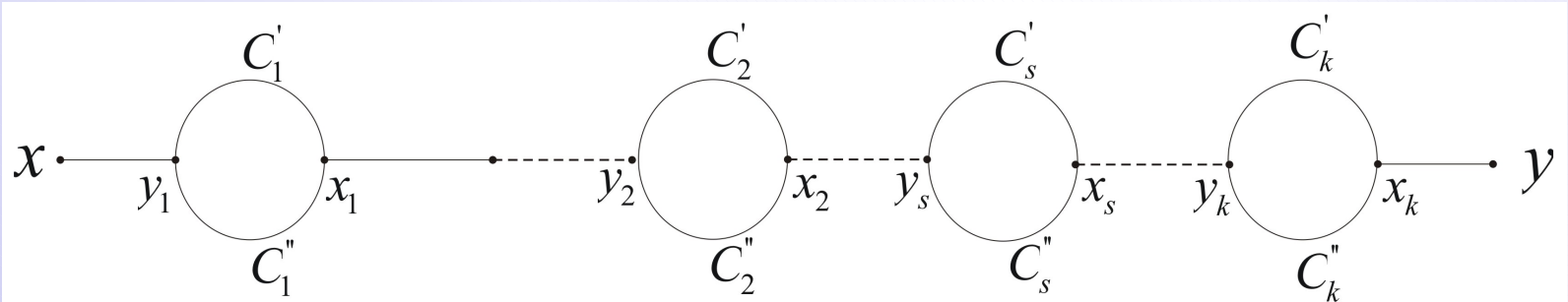


FIGURE 2 $S = P_0C_1P_1C_2 \cdots P_{s-1}C_sP_s \cdots C_kP_k$

• The theorem of cycle string

THEOREM 2.1. Let x and y be two distinct vertices in a 2-connected graph G . For any positive integer k , if every vertex other than x and y has degree at least 3 (if $k \geq 2$, then replace 3 with 4) and $d_G(u) + d_G(v) \geq 6k - 1$ for every pair of adjacent vertices $u, v \in V(G) \setminus \{x, y\}$, then G contains a feasible (x, y) -string of k cycles.

Proof. We prove this assertion by induction on $|V(G)|$. It is clear that $|V(G)| \geq 4$. The result is trivial for $|V(G)| = 4$.

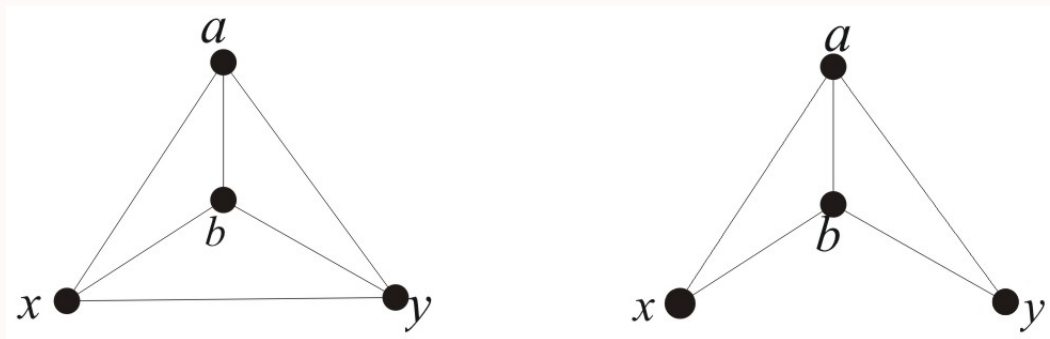


FIGURE 3 K_4 and $K_4 - xy$

Suppose therefore that $|V(G)| \geq 5$ and the result holds for all graphs G' with $|V(G')| < |V(G)|$. By symmetry, we may assume that $d(x) \leq d(y)$. The proof is divided into two parts.

Part I. G contains no vertex $a \in V(G) \setminus \{x, y\}$ such that $e(a, N(x) \setminus \{y\}) \geq 2$.

Let $X = N(x) \setminus \{y\}$, then

$$e(v, X) \leq 1 \quad \text{for every } v \in V(G) \setminus \{x, y\}. \quad (1)$$

Let G^* be the simple graph obtained from G by contracting the subgraph induced by $X \cup \{x\}$ into a single vertex x^* and then removing multiple edges. By (1), the contraction only results in multiple edges between x^* and y , and thus $d_{G^*}(v) = d_G(v)$ for every $v \in V(G^*) \setminus \{x^*, y\}$. Then we have

$$d_{G^*}(u) + d_{G^*}(v) = d_G(u) + d_G(v) \quad \text{for every pair of adjacent vertices } u, v \in V(G^*) \setminus \{x^*, y\}. \quad (2)$$

If G^* is not 2-connected, then x^* is the only cut vertex of G^* and each block of G^* is an end-block containing x^* . Let B be a block of G^* which contains y . ($B = G^*$ if G^* is 2-connected.)

Case 1. $|V(B)| \geq 3$. Then B is 2-connected. By the induction hypothesis and using (2), B contains a feasible (x^*, y) -string $S^* = P_0 C_1 P_1 \cdots C_k P_k$ of k cycles, with respect to k and B . Here we distinguish two subcases: $|E(P_0)| \geq 1$ and $|E(P_0)| = 0$.

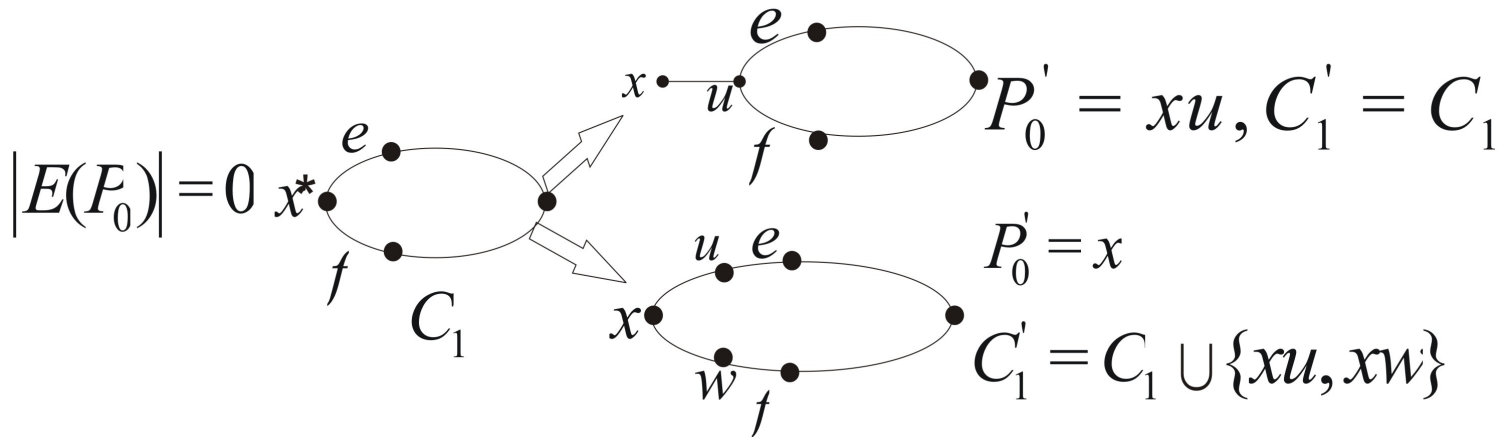
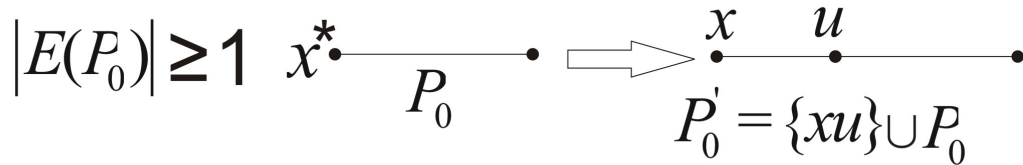
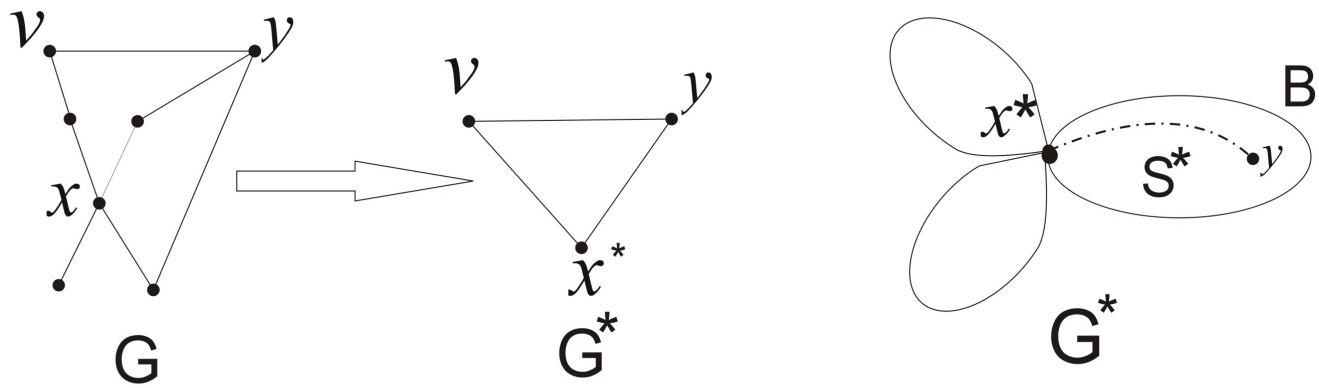


FIGURE 4 $S = P'_0 C'_1 P_1 \dots C_k P_k$

Case 2. $|V(B)| = 2$. Then $B = x^*y$ and $N(y)\setminus\{x\} \subseteq X$ in G . But $d(x) \leq d(y)$, and so $N(y)\setminus\{x\} = X$.

Case 2a. $G^* = B(= x^*y)$. Then $N(y)\setminus\{x\} = X = V(G)\setminus\{x, y\}$. By (1), $k = 1$ and $d_G(v) = 3$ for every $v \in X$.

Case 2b. $G^* \neq B$. Then there exists a block B' of G^* other than B . Since the given degree condition, by (1) and (2), $|V(B')| \geq 4$. Split x^* (in B') into two new vertices v_1 and v_2 such that (i) each v_i is incident with at least one edge of B' , $i = 1, 2$; (ii) if two edges incident with the same $v \in X$ in G , then they are incident with the same v_1 or v_2 . Since G is 2-connected, such a splitting of x^* exists. Add a new edge joining v_1 and v_2 , and denote the resulting graph by H .

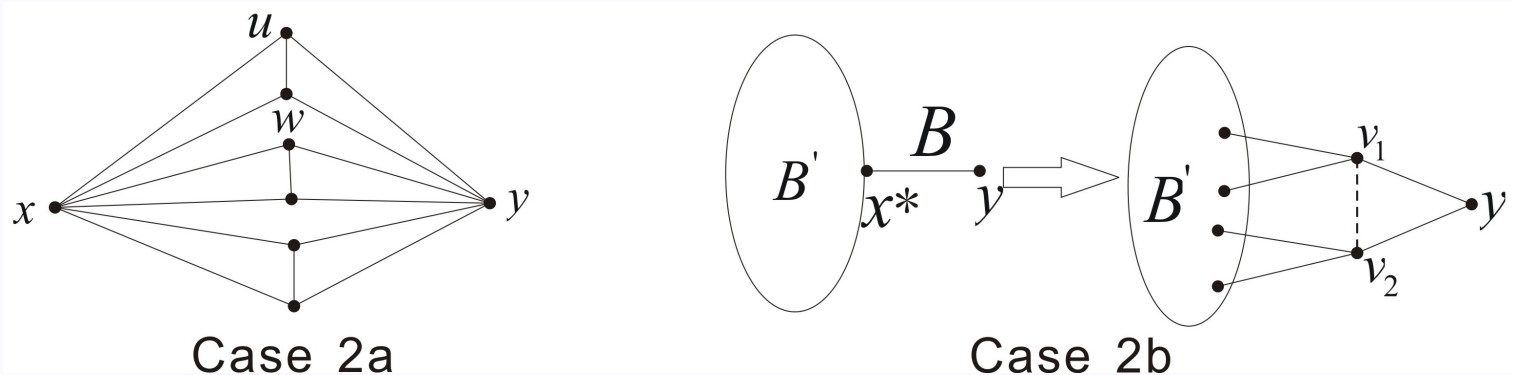


FIGURE 5

Part II. G contains a vertex $a \in V(G) \setminus \{x, y\}$ such that $e(a, N(x) \setminus \{y\}) \geq 2$.

Let $\{x_1, x_2\} \subseteq N(a) \cap (N(x) \setminus \{y\})$ and denote by Q the cycle xx_1ax_2x of length 4. Suppose that a has been chosen so that the component H in $G - V(Q)$ containing y is as large as possible.

If $e(\{x_1, x_2\}, H) = 0$, then, since G is 2-connected, we have $e(a, H) \geq 1$, and moreover, if we let $R = G - V(H)$, then R is 2-connected and $d_R(v) = d_G(v)$ for every $v \in V(R) \setminus \{x, a\}$. (see fig 6.)

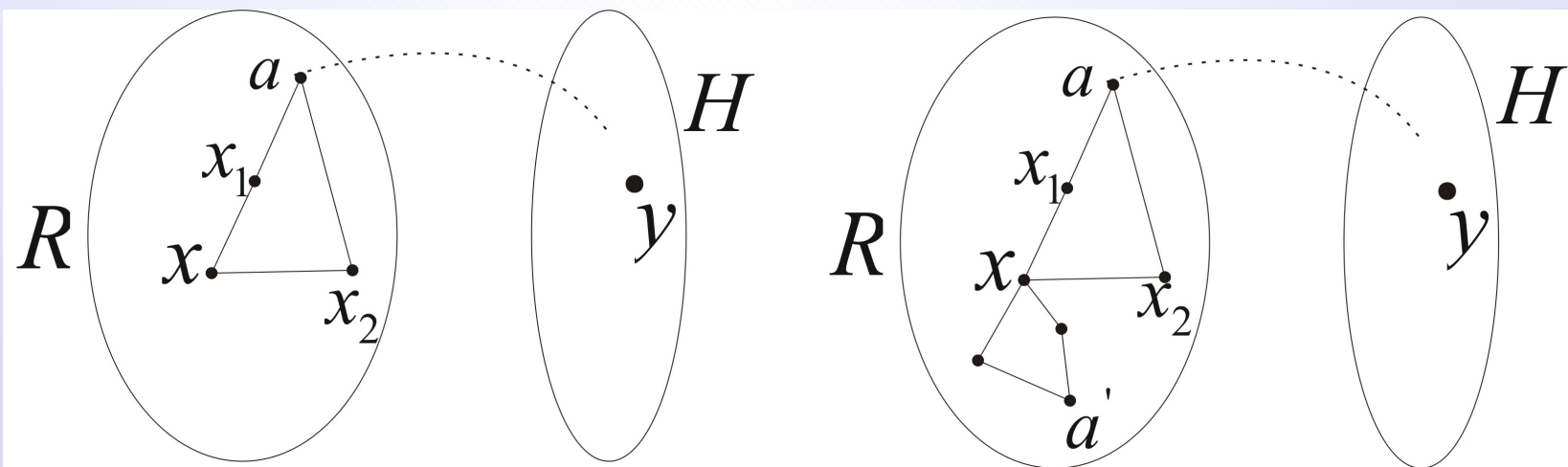


FIGURE 6 $e(\{x_1, x_2\}, H) = 0$

$e(\{x_1, x_2\}, H) \geq 1$

Suppose therefore that $e(\{x_1, x_2\}, H) \geq 1$, say $e(x_1, H) \geq 1$.

If $k = 1$, clearly, G contains a feasible (x, y) -string in G . In the rest of the proof, assume that $k \geq 2$. We claim that either we are done or

$$\text{if } e(a, H) \geq 1, \text{ then } H \text{ is the only component of } G - V(Q). \quad (3)$$

If (3) is not true, let $R = G - V(H)$. Then R is 2-connected and $|V(R)| \geq 5$. If there is $a' \in V(R) \setminus \{x, a\}$ such that $e(a', N_R(x) \setminus \{a\}) \geq 2$, then, it is easy to get a contradiction (see Fig 6.). Suppose therefore that $e(v, N_R(x) \setminus \{a\}) \leq 1$ for every $v \in V(R) \setminus \{x, a\}$. By the given degree condition, we have that $|V(R \setminus Q) \setminus N(x)| \geq 3$. Then, if R^* denotes the simple graph obtained from R by contracting the subgraph induced by $(N_R(x) \setminus \{a\}) \cup \{x\}$ into a single vertex x^* and then removing multiple edges between x^* and a , then $|V(R^*)| \geq 5$. Now, we are in situations similar to Case 1 and Case 2b of Part I, with R^* and (x^*, a) in place of G^* and (x^*, y) , respectively.

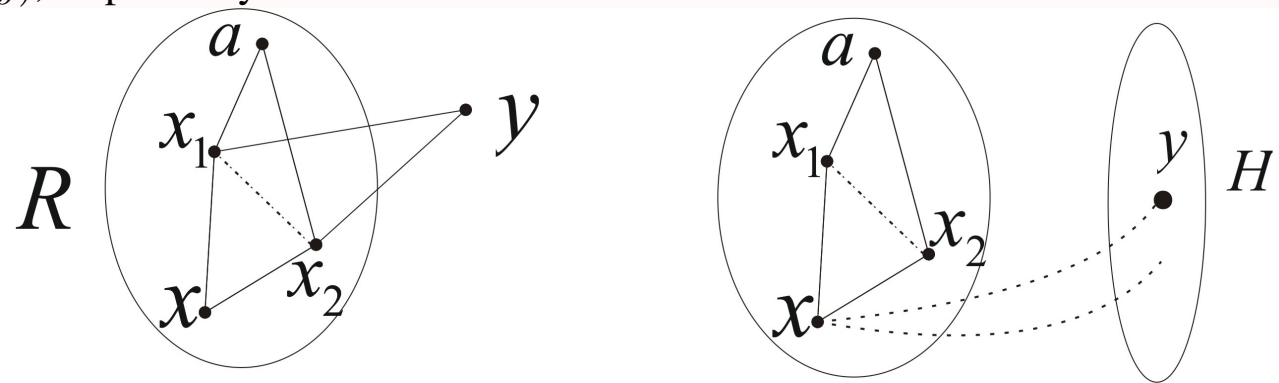


FIGURE 7 $|V(H)| = 1$

$|V(H)| \geq 2$ and $e(\{x_1, x_2\}, H - y) = 0$

If $|V(H)| = 1$, that is, $H = y$, let $R = G - y$. Let R' be the graph obtained from $G - \{x, y\}$ by adding x_1x_2 (if $x_1x_2 \notin E(G)$). (see fig 7.)

Suppose therefore that $|V(H)| \geq 2$.

If $e(\{x_1, x_2\}, H - y) = 0$, then H is not the only component of $G - V(Q)$, and by (3), we have that $e(a, H) = 0$. It follows, since G is 2-connected, that $e(x, H - y) \neq 0$. Let H' be the graph induced by $V(H) \cup \{x\}$ plus xy if $xy \notin E(G)$ (see fig 7.).

Suppose therefore that $e(\{x_1, x_2\}, H - y) \geq 1$. Without loss of generality, suppose that

$$\text{there is } w \in V(H - y) \text{ such that } x_1w \in E(G). \quad (4)$$

Let B be an endblock of H and b the unique cut vertex of H contained in B such that $y \notin V(B - b)$. Let P be a (b, y) -path in H .

If $w \in V(B - b)$ (w is defined in (4)), denote by F the graph induced by $V(B) \cup \{x_1\}$ plus x_1b if $x_1b \notin E(G)$. (See Fig. 8.)

Thus, by the arbitrariness of w , we may assume that $e(x_1, B - b) = 0$, and similarly, $e(x_2, B - b) = 0$. Therefore,

$$e(\{x_1, x_2\}, B - b) = 0, \quad (5)$$

Hence, for any $u \in V(B - b)$,

$$B \text{ contains a feasible 2-defective } (u, b)\text{-string of } k - 1 \text{ cycles.} \quad (6)$$

(See Fig. 8.)

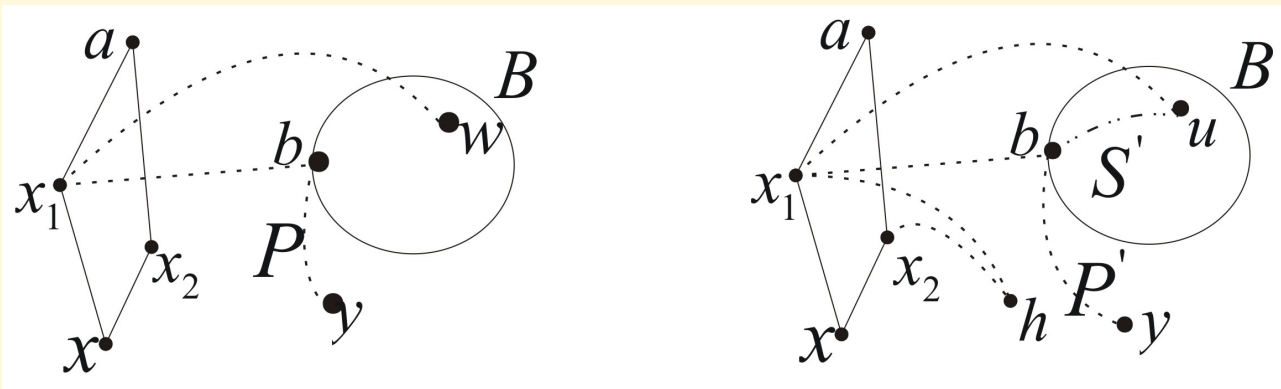


FIGURE 8 $w \in V(B - b)$

$$e(\{x_1, x_2\}, B - b) = 0$$

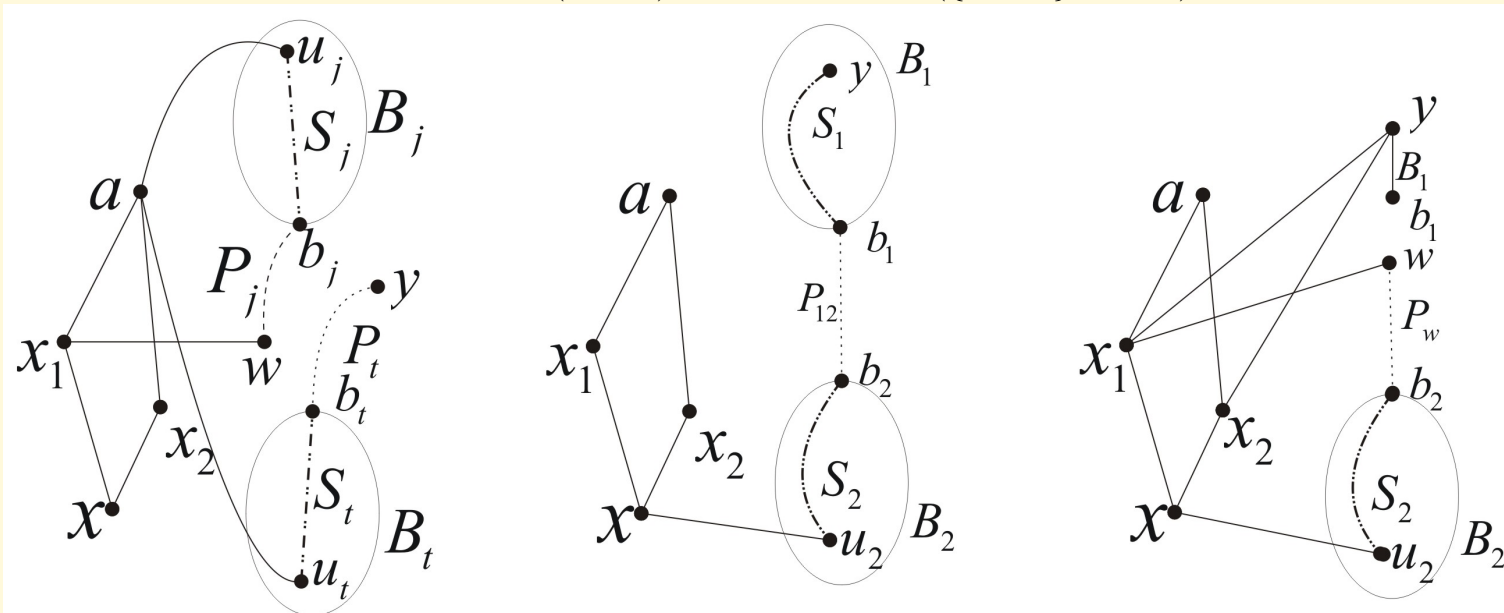


FIGURE 9 $y \notin V(B_i - b_i)$

$$|V(B_1)| \geq 3$$

$$|V(B_1)| = 2 \text{ and } e(y, \{x_1, x_2\}) = 2$$

If $N_G(x) \cap V(B - b) = \emptyset$, then, since G is 2-connected, $N_G(a) \cap V(B - b) \neq \emptyset$. Denote by F_a the graph induced by $V(B) \cup \{a\}$ plus ab if $ab \notin E(G)$. (See Fig. 8.)

Suppose therefore that
$$N_G(x) \cap V(B - b) \neq \emptyset, \quad (7)$$

and similarly,
$$N_G(a) \cap V(B - b) \neq \emptyset. \quad (8)$$

Let $h \in V(H)$ which is not a cut vertex of H . If $e(h, Q) = 4$, (See Fig. 8.)

Suppose thus that
$$e(h, Q) \leq 3. \quad (9)$$

It is clear that H is not 2-connected. Let B_1, B_2, \dots, B_m be all endblocks of H and b_i the unique cut vertex of H contained in B_i , $1 \leq i \leq m$, where $m \geq 2$.

If $y \notin V(B_i - b_i)$ for each B_i , $1 \leq i \leq m$, let $D = H - (\bigcup_{i=1}^m V(B_i - b_i))$. Then $y \in V(D)$. (See Fig. 9.)

Suppose therefore that $y \in V(B_i - b_i)$ for some i , and by relabeling if necessary, we suppose that $y \in V(B_1 - b_1)$. If $|V(B_1)| \geq 3$ (See Fig. 9). Suppose thus that $|V(B_1)| = 2$. If $e(y, \{x_1, x_2\}) = 2$ (See Fig. 9.). Suppose thus that $e(y, \{x_1, x_2\}) \leq 1$. Since $|N(y) \setminus \{x\}| \leq 3$, $|N(x) \setminus \{y\}| \geq 2 + (m - 1) = m + 1$ and $d(x) \leq d(y)$, we have that $m = 2$. So H is a chain of blocks which can be represented by $B_1 A_1 A_2 \cdots A_t B_2$, where each A_i is a block containing exactly two cut vertices of H , $1 \leq i \leq t$.

If there is A_j with $|V(A_j)| \geq 3$ (See Fig. 10.). Suppose therefore that $|V(A_j)| = 2$ for each i , $1 \leq i \leq t$. (See Fig. 11.).

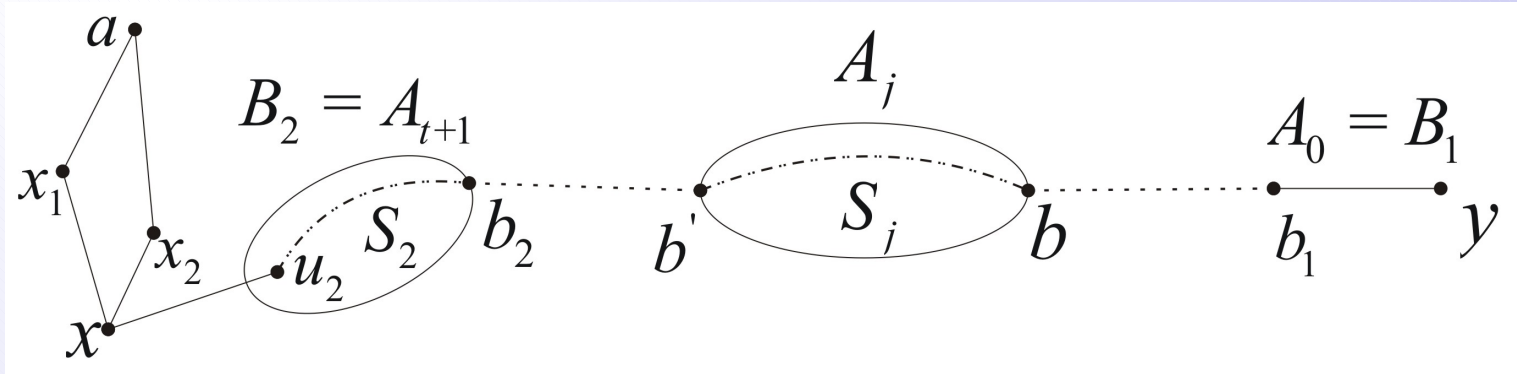


FIGURE 10 $|V(A_j)| \geq 3$

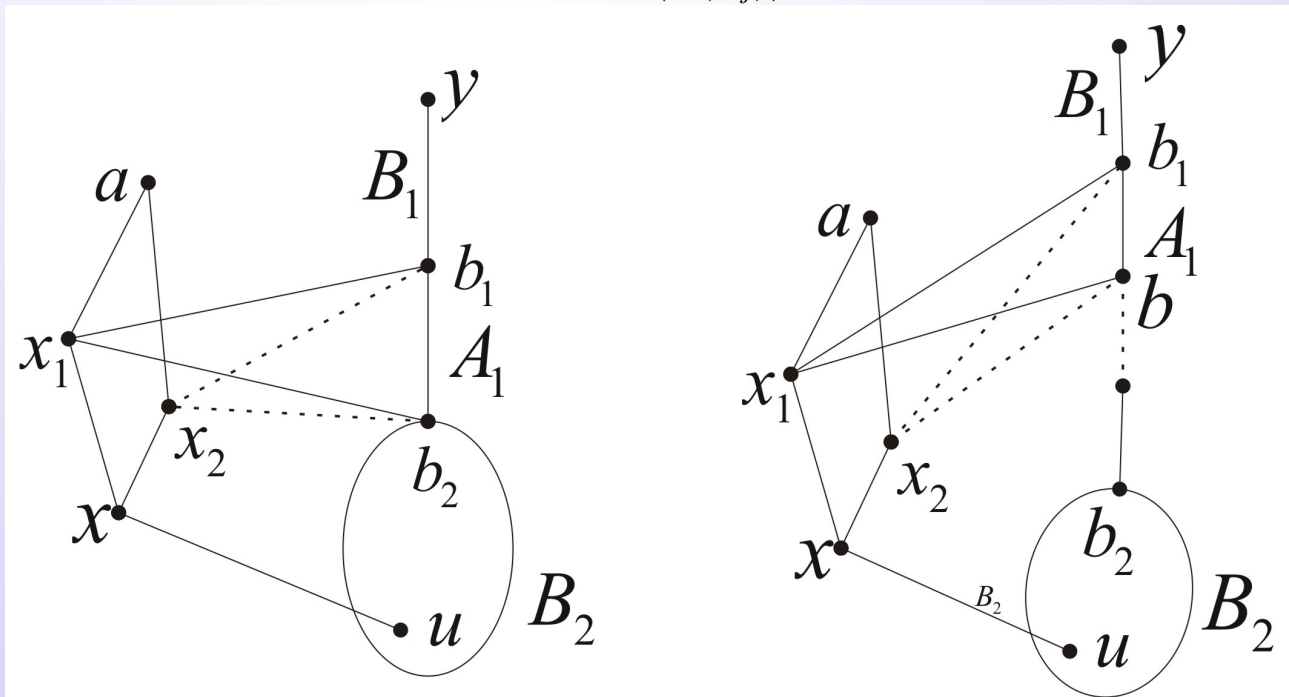


FIGURE 11 $|V(A_j)| = 2$

$t = 1$

$t \geq 2$

THEOREM 2.2. Let x and y be two distinct vertices in a 2-connected graph G . For any positive integer k , if every vertex other than x and y has degree at least 3 (if $k \geq 2$, then replace 3 with 4) and $d_G(u) + d_G(v) \geq 6k - 1$ for every pair of adjacent vertices $u, v \in V(G) \setminus \{x, y\}$, and in addition, if there is no edge $uv \in E(G - \{x, y\})$ such that $d_G(u) + d_G(v) \in \{6k - 1, 6k\}$, then G contains a feasible 2-defective (x, y) -string of k cycles.

COROLLARY 2.3. Let x and y be two distinct vertices in a 2-connected graph G . For any positive integer k , if $d_G(v) \geq 3$ (if $k \geq 2$, then $d_G(v) \geq 4$) for every $v \in V(G) \setminus \{x, y\}$, and $d_G(u) + d_G(v) \geq 6k - 1$ for every pair of adjacent vertices $u, v \in V(G) \setminus \{x, y\}$, then G contains $k + 1$ (x, y) -paths R_0, R_1, \dots, R_k such that $k < |E(R_0)| < |E(R_1)| < \dots < |E(R_k)|$, $|E(R_i)| - |E(R_{i-1})| = 2$, $1 \leq i \leq k - 1$, and $1 \leq |E(R_k)| - |E(R_{k-1})| \leq 2$.

COROLLARY 2.4. Let x and y be two distinct vertices in a 2-connected graph G . For any positive integer k , if $d_G(v) \geq 3$ (if $k \geq 2$, then $d_G(v) \geq 4$) for every $v \in V(G) \setminus \{x, y\}$, and $d_G(u) + d_G(v) \geq 6k + 1$ for every pair of adjacent vertices $u, v \in V(G) \setminus \{x, y\}$, then G contains $k + 1$ (x, y) -paths of consecutive even lengths or consecutive odd lengths $m, m + 2, m + 4, \dots, m + 2k$ for some integer $m \geq k + 1$.

谢谢大家!
Thank You!