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# Decycling Numbers of Some Box-cross Product Graphs

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# 1 Introduction

The graphs considered in this paper are finite, simple, undirected and denoted by $G = (V(G), E(G))$. Now let $G = (V(G), E(G))$ be a graph. If $S \subseteq V(G)$ and $G - S$ is acyclic, then $S$ is said to be a *decycling set* of $G$. A decycling set of the smallest size is said to be a *minimum decycling set* and denoted by $\phi$-set. The size of a minimum decycling set of $G$ is said to be the *decycling number* of $G$ and denoted by $\phi(G)$. If we denote by $I(G)$ the greatest order of an induced forest in $G$, then finding the greatest order of an induced forest of a graph is equivalent to determining the decycling number of this graph, and $I(G) + \phi(G) = |G|$.

In the following $C_n$ denotes a circuit on $n$ vertices, $P_m$ denotes a path on $m$ vertices, $K_n$ denotes a complete graph on $n$ vertices, $K_{1,n-1}$ denotes a star on $n$ vertices and $\overline{G}$ denotes the complement graph of $G$. The *independence number* $\alpha(G)$ of a graph $G$ is the maximum number of vertices in an independent set, and the *covering number* $\beta(G)$ of a graph $G$ is the minimum number of vertices in a set that meets every edge. They are connected by the elementary but useful result that $\alpha(G) + \beta(G) = |G|$ for every graph $G$. Especially, in the following we denote a vertex in a decycling set by a bigger red dot " $\bullet$ ".

# 2 Decycling Numbers of Some Box-cross Product Graphs

Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs. A *box-cross product* of two graphs $G_1$ with $G_2$ is defined to be $G_1 \boxtimes G_2 = (V, E)$ where

$$V = \{(u_i, v_j) \mid \forall\, u_i \in V_1,\ v_j \in V_2\} \quad \text{and}$$

$$E = \{((u_1, v_1), (u_2, v_2)) \mid \forall\, (u_1, v_1),\ (u_2, v_2) \in V'',\ (u_1, u_2) \in E_1 \text{ when } v_1 = v_2 \text{ or } (v_1, v_2) \in E_2 \text{ when } u_1 = u_2 \text{ or } (u_1, u_2) \in E_1 \text{ and } (v_1, v_2) \in E_2\}.$$

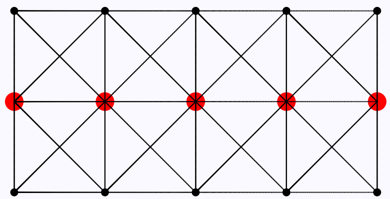

Figure 1. $P_3 \boxtimes P_5$.

**Observation 2.1 [7].** $\phi(G_1 \boxtimes G_2) = \phi(G_2 \boxtimes G_1)$.

Let the vertex sets of $G_1$ and $G_2$ be respectively $\{u_1, u_2, \cdots, u_{|G_1|}\}$ and $\{v_1, v_2, \cdots, v_{|G_2|}\}$. There are $|G_1|$ copies of $G_2$ and $|G_2|$ copies of $G_1$ in $G_1 \boxtimes G_2$. Let $x_{i,j} = (u_i, v_j)$, and let $R(i) = \{x_{i,j} \mid \forall\ v_j \in V(G_2)\}$, $i = 1, 2, \cdots, |G_1|$ and $C(j) = \{x_{i,j} \mid \forall\ u_i \in V(G_1)\}$, $j = 1, 2, \cdots, |G_2|$. $R(i)$ is called the $i$-th *row* and $C(j)$ is called the $j$-th *column* of $G_1 \boxtimes G_2$. For any $i$, the induced subgraph $(G_1 \boxtimes G_2)|_{R(i)}$ is a copy of $G_2$. For any $j$, the induced subgraph $(G_1 \boxtimes G_2)|_{C(j)}$ is a copy of $G_1$.

In order to prove the following results, we introduce some useful notations as follow: If $S$ is a vertex subset of $G_1 \boxtimes G_2$, we denote the vertices of $S$ in the $i$-th row of $G_1 \boxtimes G_2$ by $S(R(i))$ and the vertices of $S$ in the $j$-th column of $G_1 \boxtimes G_2$ by $S(C(j))$, and by extension, for $k \leq l$, denote the set $S(R(k)) \cup S(R(k+1)) \cup \cdots \cup S(R(l))$ by $S(R(k,l))$ and the set $S(C(k)) \cup S(C(k+1)) \cup \cdots \cup S(C(l))$ by $S(C(k,l))$. Further, we let $N(R(i)) = |S(R(i))|$, $N(C(j)) = |S(C(j))|$ and $N(R(k,l)) = |S(R(k,l))|$, $N(C(k,l)) = |S(C(k,l))|$.

**Lemma 2.2 [7].** $\phi(G_1 \boxtimes G_2) \geq \max\{|G_1| \cdot \phi(G_2),\ |G_2| \cdot \phi(G_1)\}$.

**Theorem 2.3 [7].** Suppose $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$. For each $V_1' \subseteq V_1$, $V_2' \subseteq V_2$, $\phi(G_1 \boxtimes G_2) \geq \max\{\phi(G_1|_{V_1'} \boxtimes G_2) + \phi(G_1|_{V_1 - V_1'} \boxtimes G_2),\ \phi(G_1 \boxtimes G_2|_{V_2'}) + \phi(G_1 \boxtimes G_2|_{V_2 - V_2'})\}$.

**Theorem 2.4 [7].** $\phi(G_1 \boxtimes G_2) \leq \min\{|G_1| \cdot |G_2| - \alpha(G_1) \cdot (|G_2| - \phi(G_2)),\ |G_1| \cdot |G_2| - \alpha(G_2) \cdot (|G_1| - \phi(G_1))\}$

**Lemma 2.5 [7].** $\phi(P_2 \boxtimes P_n) = 2 \cdot \left\lfloor \frac{n}{2} \right\rfloor$.

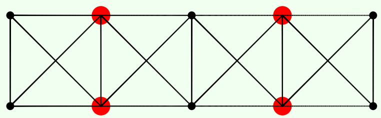

Figure 2. A $\phi$-set for $P_2 \boxtimes P_5$.

**Lemma 2.6 [7].** Every $\phi$-set of $P_2 \boxtimes P_{2k+1}$ is equal to $\{\ x_{1,2}, x_{2,2}, x_{1,4}, x_{2,4}, \cdots, x_{1,2k}, x_{2,2k}\ \}$ (see Figure 2 for the case $k = 2$).

**Theorem 2.7 [7].** $\phi(P_m \boxtimes P_n) = \min\{m \cdot \lfloor \frac{n}{2} \rfloor,\ n \cdot \lfloor \frac{m}{2} \rfloor\}$.

**Lemma 2.8 [8].** $\phi(P_2 \boxtimes C_n) = 2 \cdot \lceil \frac{n}{2} \rceil$

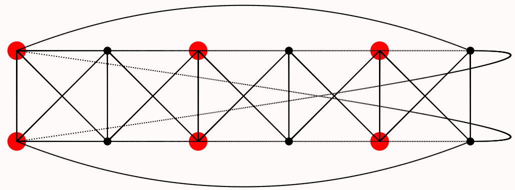

Figure 3. A $\phi$-set for $P_2 \boxtimes C_6$.

**Lemma 2.9 [8].** Every $\phi$-set of $P_2 \boxtimes C_{2k}$ is equal to $\{x_{1,1}, x_{2,1}, x_{1,3}, x_{2,3}, \cdots, x_{1,2k-1}, x_{2,2k-1}\}$ or $\{x_{1,2}, x_{2,2}, x_{1,4}, x_{2,4}, \cdots, x_{1,2k}, x_{2,2k}\}$. (see Figure 3 for the case $k = 3$).

**Theorem 2.10 [8].** $\phi(P_m \boxtimes C_n) = \min\{m \cdot \lceil \frac{n}{2} \rceil,\ n \cdot \lfloor \frac{m}{2} \rfloor + \lceil \frac{m}{2} \rceil\}$.

**Corollary 2.11 [8].** $\phi(C_m \boxtimes P_n) = \min\{m \cdot \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil,\ n \cdot \lceil \frac{m}{2} \rceil\}$.

**Theorem 2.12 [8].** When $m$ or $n$ is even, $\phi(C_m \boxtimes C_n) = \min\{m \cdot \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor, n \cdot \lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor\}$; when $m$ and $n$ are both odd and $m \leq n$, $\phi(C_m \boxtimes C_n) = n \cdot \lceil \frac{m}{2} \rceil$ or $n \cdot \lceil \frac{m}{2} \rceil + 1$ .

**Proof.** When $m$ or $n$ is even. Applying Theorem 2.4, $\phi(C_m \boxtimes C_n) \leq \min\{m \cdot n - \lfloor \frac{m}{2} \rfloor \cdot (n-1), m \cdot n - \lfloor \frac{n}{2} \rfloor \cdot (m-1)\} = \min\{n \cdot \lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor, m \cdot \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor\} = \min\{m \cdot \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor, n \cdot \lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor\}$. So in the rest of the proof, we only have to prove that $\phi(C_m \boxtimes C_n) \geq \min\{m \cdot \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor, n \cdot \lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor\}$. Since $\phi(C_m \boxtimes C_n) = \phi(C_n \boxtimes C_m)$, without loss of generality, we can assume that $m \leq n$. So it suffices to prove the following three cases.

**Case 1.** $m = 2k,\ n = 2s+1\ (k \le s)$.

Since $m = 2k,\ n = 2s+1\ (k \le s)$, hence $\min\{m \cdot \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor,\ n \cdot \lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor\} = n \cdot \lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor$. So we only have to prove that $\phi(C_m \boxtimes C_n) \ge n \cdot \lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor$. Since $\phi(P_2 \boxtimes C_n) = 2 \cdot \lceil \frac{n}{2} \rceil = n+1$, hence $\phi(C_m \boxtimes C_n) \ge \phi(P_m \boxtimes C_n) \ge k \cdot \phi(P_2 \boxtimes C_n) = k \cdot (n+1) = \frac{m}{2} \cdot (n+1) = n \cdot \lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor$.

**Case 2.** $m = 2k+1,\ n = 2s\ (k < s)$.

Since $m = 2k+1,\ n = 2s\ (k < s)$, hence $\min\{m \cdot \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor,\ n \cdot \lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor\} = m \cdot \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor$. So it suffices to prove that $\phi(C_m \boxtimes C_n) \ge m \cdot \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor$. Since $\phi(C_m \boxtimes P_2) = 2 \cdot \lceil \frac{m}{2} \rceil = m+1$, hence $\phi(C_m \boxtimes C_n) \ge \phi(C_m \boxtimes P_n) \ge s \cdot \phi(C_m \boxtimes P_2) = s \cdot (m+1) = \frac{n}{2} \cdot (m+1) = m \cdot \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor$.

Case 3. $m = 2k,\ n = 2s\ (k \le s)$.

Since $m = 2k,\ n = 2s\ (k \le s)$, hence $\min\{m \cdot \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor,\ n \cdot \lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor\} = n \cdot \lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor$. So it suffices to prove that $\phi(C_m \boxtimes C_n) \ge n \cdot \lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor$. Let $S$ be any $\phi$-set of $C_m \boxtimes C_n$. By Lemma 2.8, we know that for any $i$ ($1 \le i \le m$ when $i = m$, $i+1 = 1$), $N(R(i, i+1)) \ge \phi(P_2 \boxtimes C_n) = n$. Now if for any $i$ ($1 \le i \le m$ when $i = m$, $i+1 = 1$), $N(R(i, i+1)) \ge \phi(P_2 \boxtimes C_n) + 1 = n+1$, then we can get that $|S| = N(R(1,2)) + N(R(3,4)) + \cdots + N(R(2k-1, 2k)) \ge k \cdot (n+1) = \frac{m}{2} \cdot (n+1) = n \cdot \lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor$. Otherwise, there exists some $i$ ($1 \le i \le m$ when $i = m$, $i+1 = 1$) so that $N(R(i, i+1)) = \phi(P_2 \boxtimes C_n) = n$. Without loss of generality, we assume that $i = 1$. Hence $N(R(1,2)) = \phi(P_2 \boxtimes C_n) = n$. By Lemma 2.9 and symmetry, $S(R(1,2)) = \{x_{1,1}, x_{2,1}, x_{1,3}, x_{2,3}, \cdots, x_{1,n-1}, x_{2,n-1}\}$. Hence $\{x_{1,1}, x_{2,1}\} \subset S(C(1,2))$. Again by Lemma 2.9, we conclude that $N(C(1,2)) \ge m+1$. Using the same method, we know that $N(C(3,4)) = \cdots = N(C(n-1, n)) \ge m+1$. Hence $|S| = N(C(1,2)) + N(C(3,4)) + \cdots + N(C(n-1,n)) \ge s \cdot (m+1) = \frac{n}{2} \cdot (m+1) = m \cdot \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor \ge n \cdot \lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor$. So this case also has been proved.

When $m$ and $n$ are both odd and $m \leq n$. Let $S$ be any $\phi$-set of $C_m \boxtimes C_n$. By Lemma 2.8, we know that for any $i$ ($1 \leq i \leq m$ when $i = m$, $i+1 = 1$), $N(C(i, i+1)) \geq \phi(C_m \boxtimes P_2) = 2 \cdot \lceil \frac{m}{2} \rceil$. Hence there exist some $j$ ($1 \leq j \leq n$) so that $N(C(j)) \geq \lceil \frac{m}{2} \rceil$. Without loss of generality, we assume that $j = 1$. Hence $N(C(1)) \geq \lceil \frac{m}{2} \rceil$. Then again by Lemma 2.8, $|S| = N(C(1)) + N(C(2,3)) + \cdots + N(C(n-1, n)) \geq \lceil \frac{m}{2} \rceil + \frac{n-1}{2} \cdot 2 \cdot \lceil \frac{m}{2} \rceil = n \cdot \lceil \frac{m}{2} \rceil$. In addition, we can construct a vertex set $S'$ of $C_m \boxtimes C_n$ as follow:

$S' = S'(C(1)) \cup S'(C(2)) \cup \cdots \cup S'(C(n))$ ,

$S'(C(1)) = \{x_{1,1}, x_{\lceil \frac{m}{2} \rceil,1}, x_{\lceil \frac{m}{2} \rceil+1,1}, \cdots, x_{m-1,1}, x_{m,1}\}$ ,

$S'(C(i)) = \{x_{f(1+i\cdot\lfloor \frac{m}{2} \rfloor),i}, x_{f(2+i\cdot\lfloor \frac{m}{2} \rfloor),i}, \cdots, x_{f(\lfloor \frac{m}{2} \rfloor+i\cdot\lfloor \frac{m}{2} \rfloor),i}, x_{f(\lceil \frac{m}{2} \rceil+i\cdot\lfloor \frac{m}{2} \rfloor),i}\} \quad (2 \leq i \leq m)$ ,

$$\text{where} \quad f(x) = \begin{cases} \bmod (x, m) & m \nmid x, \\ m & m \mid x. \end{cases}$$

$S'(C(i)) = \{x_{\lceil \frac{m}{2} \rceil,i}, x_{\lceil \frac{m}{2} \rceil+1,i}, \cdots, x_{m-1,i}, x_{m,i}\} \quad (m+1 \leq i \leq n$ and $\bmod (i,2) = 0)$ ,

$S'(C(i)) = \{x_{1,i}, x_{2,i}, \cdots, x_{\lfloor \frac{m}{2} \rfloor,i}, x_{\lceil \frac{m}{2} \rceil,i}\} \quad (m+1 \leq i \leq n$ and $\bmod (i,2) = 1)$.

It is easy to know that $C_m \boxtimes C_n - S'$ is a tree and $|S'| = n \cdot \lceil \frac{m}{2} \rceil + 1$. Hence $\phi(C_m \boxtimes C_n) \leq n \cdot \lceil \frac{m}{2} \rceil + 1$. So $n \cdot \lceil \frac{m}{2} \rceil \leq \phi(C_m \boxtimes C_n) \leq n \cdot \lceil \frac{m}{2} \rceil + 1$. Since $\phi(C_m \boxtimes C_n)$ is an integer, hence $\phi(C_m \boxtimes C_n) = n \cdot \lceil \frac{m}{2} \rceil$ or $n \cdot \lceil \frac{m}{2} \rceil + 1$. This completes the proof of Theorem 2.12.

**Theorem 2.13 [8].** $\phi(K_m \boxtimes P_n) = m \cdot n - 2 \cdot \left\lceil \frac{n}{2} \right\rceil$.

**Theorem 2.14 [8].** $\phi(K_m \boxtimes C_n) = m \cdot n - 2 \cdot \left\lfloor \frac{n}{2} \right\rfloor$.

**Theorem 2.15 [8].** $\phi(K_m \boxtimes K_n) = m \cdot n - 2$.

**Theorem 2.16 [8].** $\phi(\overline{K_m} \boxtimes P_n) = 0$, $\phi(\overline{K_m} \boxtimes C_n) = m$, $\phi(\overline{K_m} \boxtimes K_n) = m \cdot (n-2)$, $\phi(\overline{K_m} \boxtimes \overline{K_n}) = 0$.

**Theorem 2.17 [8].** When $m \geq 3$, $\phi(K_{1,m-1} \boxtimes P_n) = n$.

**Theorem 2.18 [8].** When $m \geq 3$, $\phi(K_{1,m-1} \boxtimes C_n) = n + m - 1$.

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