Some results on orthogonal factorizations

Yan Zhu

School of Mathematics and System Science

Shandong University P. R. China

zhuyan@math.sdu.edu.cn

- **Some results on orthogonal factorizations**(*Outline*)

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1. Introduction

In 1992, Alspach, Heinrich and Liu posed the following open problems:

- Given a factorization $\mathscr{F}$ of G, does there exist a subgraph H of G with a given property to which $\mathscr{F}$ is orthogonal?
- Given a subgraph H of G, does there exist a factorization $\mathscr{F}$ of G with a given property orthogonal to H?

Basic Definitions

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Let g and f be two integer-valued functions defined on $V(G)$ such that $0 \leq g(x) \leq f(x) \leq d_G(x)$ for every vertex $x \in V(G)$.

- A (g, f)-factor of G is a spanning subgraph H of G satisfying $g(x) \leq d_H(x) \leq f(x)$ for all $x \in V(H)$. In particular, if G itself is a (g, f)-factor, then G is called a (g, f)-graph.
- A (g, f)-factorization $\mathscr{F} = \{F_1, F_2, \cdots, F_m\}$ of a graph G is a partition of $E(G)$ into edge-disjoint (g, f)-factors $F_1, F_2, \cdots, F_m$.

- Let a and b be two nonnegative integers. If for all $x \in V(G)$, $g(x) = a$ and $f(x) = b$, then a (g, f)-factor is called an $[a, b]$-factor.
- If for all $x \in V(G)$, $g(x) = f(x)$, then a (g, f)-factor is also called an f-factor.
- If for all $x \in V(G)$, $g(x) = f(x) = k$, where k is an integer, then a (g, f)-factor is also called a k-factor.
- Similarly, we can define $[0, k_j]_1^m$-factorization, $[a, b]$-factorization, f-factorization, k-factorization, $[a, b]$-graph, f-graph and k-graph etc.

- Let H be a subgraph of G with mr edges (called an mr-subgraph of G). A factorization $\mathscr{F} = \{F_1, F_2, \cdots, F_m\}$ of G is called r-orthogonal (or orthogonal, when $r = 1$) to H if $|E(H) \cap E(F_i)| = r, 1 \leq i \leq m$.

- Let H be an mr-subgraph of G. If for every partition $H_1, H_2, \cdots, H_m$ of $E(H)$ where $|E(H_i)| = r$, $1 \leq i \leq m$ and $H_i \cap H_j = \phi$ for $i \neq j$, there is a factorization $\mathscr{F} = \{F_1, F_2, \cdots, F_m\}$ of G satisfying $E(H_i) \subseteq E(F_i)$, $1 \leq i \leq m$, then $\mathscr{F}$ is called randomly (m, r)-orthogonal to H.

- Let S and T be two disjoint subsets of $V(G)$. The number of edges joining S and T is denoted by $e(S,T)$. Let C be a component of $G-(S\cup T)$ such that $g(x)=f(x)$ for all $x\in V(C)$. Then we say that C is odd or even according to $e(T,V(C))+\sum_{x\in V(C)} f(x)$ being odd or even. A component of $G-(S\cup T)$ which is neither odd nor even is called neutral. The number of odd component of $G-(S\cup T)$ is denoted by $h(S,T)$.
- We write $f(S)=\sum_{x\in S} f(x)$, $g(T)=\sum_{x\in T} g(x)$ and $d_G(T)=\sum_{x\in T} d_G(x)$.

- Petersen's Theorem(1891). *A graph G has a 2-factorization if and only if G is a $2d$-regular graph for some positive integer d.*

- In 1970, L. Lovász posed the sufficient and necessary condition for a graph to have (g, f)-factors.

  Lovász's Theorem(1970). *Let G be a graph, g and f be two integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. Then G has a (g, f)-factor if and only if for all disjoint subsets S and T of $V(G)$,*

$$\delta(S,T) = d_G(T) - e(S,T) - g(T) - h(S,T) + f(S) \geq 0.$$

- Liu's Theorem(1995). *Let G be a graph and let g and f be two integer-valued functions defined on $V(G)$ such that $0 \leq g(x) < f(x)$ for all $x \in V(G)$. Then for any edge e of G there is a (g, f)-factor containing e if and only if for all disjoint subsets S and T of $V(G)$,*

$$\delta(S,T) = d_G(T) - e(S,T) - g(T) + f(S) \geq \varepsilon(S,T).$$

We define $\varepsilon(S,T)$ as follows:

$\varepsilon(S,T) = 2$ if $e = uv$ where $u, v \in S$;

$\varepsilon(S,T) = 1$ if there is a neutral component C of $G - (S \cup T)$ such that $e \in E(V(C), S)$;

$\varepsilon(S,T) = 0$, otherwise.

2. Subgraph orthogonal to a given factorization

Alspach posed the following conjectures on orthogonal factorization problems at a combinatorics meeting.

Conjecture 1 (1988). *Let G be a $2d$-regular graph, $\mathscr{F} = \{F_1, F_2, \cdots, F_d\}$ be a 2-factorization of G, then there exists a d-matching M of G to which $\mathscr{F}$ is orthogonal.*

Conjecture 2 (1988). *Let G be a $2d$-regular graph, $\mathscr{F} = \{F_1, F_2, \cdots, F_d\}$ be a 2-factorization of G, then there exists a edge partition $\mathscr{M} = \{M_1, M_2, \cdots, M_m\}$ of G to which $\mathscr{F}$ is orthogonal, where each M_i is a d-matching.*

2. Subgraph orthogonal to a given factorization

- ⋆ Liu, Alspach and Heinrich, Kouider and Sotteau, and Stong have partly confirmed Conjecture 1 successively in [9, 15, 23] respectively.
- ⋆ Liu, Alspach and Heinrich discussed the k-factorization of G and partly proved Conjecture 2.
- ⋆ Feng and Liu discussed the existence of matching or $[a, b]$-subgraphs to which a given $[a, b]$-factorization of G is orthogonal.

- **Theorem 2.1** (Stong). *Let G be a $2d$-regular graph with $|V(G)| \geq 3d-2$, $\mathscr{F} = \{F_1, F_2, \cdots, F_d\}$ be a 2-factorization of G, then there exists a d-matching M of G to which $\mathscr{F}$ is orthogonal.*

- **Theorem 2.2** (Liu, Alspach and Heinrich). *Let G be a kd-regular graph with $V(G) \geq 12d-11$, $\mathscr{F} = \{F_1, F_2, \cdots, F_d\}$ be a k-factorization of G, then there exists a edge partition $\mathscr{M} = \{M_1, M_2, \cdots, M_m\}$ of G to which $\mathscr{F}$ is orthogonal, where each M_i is a d-matching.*

Let $k = 2$. It follows from Theorem 2.2 that Conjecture 2 holds when $V(G) \geq 12d-11$.

2. *Subgraph orthogonal to a given factorization*

- **Theorem 2.3** (Liu, Alspach and Heinrich). *Let G be a kd-regular graph with $V(G) \geq 4d-3$, $\mathscr{F} = \{F_1, F_2, \cdots, F_d\}$ be a k-factorization of G, then for every $e \in E(G)$, there exists a d-matching M of G such that $e \in M$, and $\mathscr{F}$ is orthogonal to M.*

- **Theorem 2.4** (Liu, Alspach and Heinrich). *Let G be a kd-regular graph, where $k \geq 2$ is an integer. Let $\mathscr{F} = \{F_1, F_2, \cdots, F_d\}$ be a k-factorization of G, then for every $e \in E(G)$, there exists a $[1, k]$-subgraph H of G satisfying $e \in H$ and $\mathscr{F}$ is orthogonal to H.*

- **Theorem 2.5** (Feng and Liu). *Let a and b be two integers satisfying $\sqrt{2b} \le a \le b$. Let G be a graph and e be an edge of G, $\mathscr{F} = \{F_1, F_2, \cdots, F_d\}$ be an $[a, b]$-factorization of G. Then there exists a $[1, a]$-subgraph H with one edge in each of the $[a, b]$-factors $F_1, F_2, \cdots, F_m$ and with $e \in H$.*

Taking $a = b = k$ in Theorem 2.5 we get $k \ge 2$, and therefore obtain Theorem 2.4.

- **Theorem 2.6** (Feng and Liu). *Let a and b be two integers such that $a \le b$. Let G be a graph with $|V(G)| \ge \frac{40a+40b-18}{15a+10b-6}d$, where $d \ge 1$ is an integer. Then if $\mathscr{F} = \{F_1, F_2, \cdots, F_d\}$ is an $[a, b]$-factorization of G, there exists a d-matching of G to which $\mathscr{F}$ is orthogonal.*

3. Factorization orthogonal to a given subgraph

3.1. Factorization orthogonal to a given subgraph

This is a basic result given by Liu ([18]) on orthogonal factorization problems.

- **Theorem 3.1** (Liu). *Let G be an $(mg+m-1, mf-m+1)$-graph. Let H be a star of G with m edges, then there exists a (g, f)-factorization of G orthogonal to H.*

- **Theorem 3.2** (Li and Liu). *Let G be an $(mg+m-1, mf-m+1)$-graph, where $m \geq 1$ is an integer. Let H be a subgraph of G with m edges, then there exists a (g, f)-factorization of G orthogonal to H.*

- **Theorem 3.3** (Liu and Long). *Let G be an $(mg+m-1, mf-m+1)$-graph, g and f be two integer-valued functions defined on $V(G)$ such that $2k-1 \leq g(x) \leq f(x)$. If H is an mk-subgraphs of G, then G have (g, f)-factorizations randomly k-orthogonal to H.*

- **Theorem 3.4** (Zhao, Liu and Yan). *Let G be an $(mg+m-1, mf-m+1)$-graph, g and f be two integer-valued functions defined on $V(G)$ such that $2k-2 \leq g(x) \leq f(x)$. If H is an mk-subgraphs of G, then G have (g,f)-factorizations randomly k-orthogonal to H if $m \leq k$.*

- **Theorem 3.5** (Zhao, Liu and Yan). *Let G be an $(mg+m-1, mf-m+1)$-graph, g and f be two integer-valued functions defined on $V(G)$ such that $2k-2 \leq g(x) \leq f(x)-3$. If H is an mk-subgraphs of G, then G have (g,f)-factorizations randomly k-orthogonal to H.*

- **Theorem 3.6** (Liu and Zhu). *Let G be a bipartite $(mg+m-1, mf-m+1)$-graph, g and f be two integer-valued functions defined on $V(G)$ such that $k \leq g(x) \leq f(x)$. If H is an mk-subgraphs of G, then G have (g,f)-factorizations randomly k-orthogonal to H.*

- **Theorem 3.7** (Liu and Zhu). *Let G be a bipartite $(mg+m-1, mf-m+1)$-graph, g and f be two integer-valued functions defined on $V(G)$ such that $k-1 \leq g(x) \leq f(x)$. If H is an mk-subgraphs of G, then G has a (g,f)-factorization k-orthogonal to H.*

3. Factorization orthogonal to a given subgraph

- **Theorem 3.8** (Liu). *Let d be a positive integer and let G be a $2d$-regular graph. Let H be a d-star such that a set of any two edges of H is not a cut-set of any 4-regular subgraph of G. Then there is a 2-factorization of G orthogonal to H.*

3. Factorization orthogonal to a given subgraph

- **Theorem 3.9** (Liao and Xie). *Let G be an $(mg+kr, mf-kr)$-graph, where m, k and r are positive integers with $k < m$, and H be any subgraph of G with kr edges. If $g \geq r-1$ then G contains a subgraph R such that R has a (g, f)-factorization which is r-orthogonal to H.*

- **Corollary**. *Let G be an $(mg+k, mf-k)$-graph, where m, k are positive integers with $k < m$, and H be any subgraph of G with k edges. Then G contains a subgraph R such that R has a (g, f)-factorization which is orthogonal to H.*

3. Factorization orthogonal to a given subgraph

- **Theorem 3.10** (Feng). *Let G be a $[0, k_1 + k_2 + \cdots + k_m - m + 1]$-graph, where $m \geq 1$ is an integer, $k_1, k_2, \cdots, k_m$ are positive integers. Let H be an arbitrary subgraph of G with m edges, then there exists a $[0, k_j]_1^m$-factorization of G orthogonal to H.*

- **Theorem 3.11** (Liu). *Let G be a $(0, mf - m + 1)$-graph. Let H be a subgraph of G with m edges, then there exists a $(0, f)$-factorization of G orthogonal to H.*

- **Theorem 3.12** (Feng). *Let k be a positive integer, G be a $(0, mf - m + 1)$-graph, where f is an integer-valued function defined on $V(G)$ satisfying for each $x \in V(G)$, $f(x) \geq k + 2$. Let H be a star of G with m edges, then there exists a (g, f)-factorization of G orthogonal to H.*

- **Theorem 3.13** (Liu and Zhu). *Let G be a bipartite (mg, mf)-graph. Then for any m-star H in G, there is a (g, f)-factorizations of G orthogonal to H.*

3.2. Factorization orthogonal to k subgraphs

Later, some people began researching the existence of factorizations orthogonal to k vertex-disjoint or edge-disjoint subgraphs.

- **Theorem 3.14** (Lam, Liu, Li and Shiu). *Let G be an $(mg+m-1, mf-m+1)$-graph, where g and f are two integer-valued functions defined on $V(G)$ such that $k \leq g(x) \leq f(x)$. Let $H_1, H_2, \cdots, H_k$ be mutually vertex-disjoint m-subgraphs of G. Then, G has a (g, f)-factorization orthogonal to every H_i, $1 \leq i \leq k$.*

- **Theorem 3.15** (Liu and Dong). *Let G be a bipartite $(mg+m-1, mf-m+1)$-graph, where g and f are two integer-valued functions defined on $V(G)$ such that $\frac{k}{2} \leq g(x) \leq f(x)$. Let $H_1, H_2, \cdots, H_k$ be k vertex-disjoint m-subgraphs of G. Then, G has a (g, f)-factorization orthogonal to every H_i, $1 \leq i \leq k$.*

- **Theorem 3.16** (Yan and Liu). *Let G be an $(mg+k, mf-k)$-graph, where $1 \leq k < m$, g and f are two integer-valued functions defined on $V(G)$ such that $r \leq g(x) \leq f(x)$. Let $H_1, H_2, \cdots, H_r$ be vertex-disjoint k-subgraphs of G. Then G has k edge disjoint (g, f)-factors which are orthogonal to every H_i, $1 \leq i \leq r$.*

- **Theorem 3.17** (Liu and Dong). *Let G be a bipartite $(mg+k, mf-k)$-graph, where $1 \leq k < m$, g and f are two integer-valued functions defined on $V(G)$ such that $\frac{r}{2} \leq g(x) < f(x)$. Let $H_1, H_2, \cdots, H_r$ be r vertex-disjoint k-subgraphs of G. Then there exists a subgraph R which has a (g, f)-factorization orthogonal to every H_i, $1 \leq i \leq r$.*

3. Factorization orthogonal to k vertex-disjoint subgraphs

- **Theorem 3.18** (Feng). *Let k be a positive integer, G be a $(0, mf - m + 1)$-graph, where f is an integer-valued function defined on $V(G)$ such that $f(x) \geq k + 2$. Let $H_1, H_2, \cdots, H_k$ be vertex-disjoint m-subgraphs of G. Then there exists a $(0, f)$-factorization of G orthogonal to every H_i, $1 \leq i \leq k$.*

- **Corollary** (Feng). *Let G be a $(0, mf - m + 1)$-graph, where f is an integer-valued function defined on $V(G)$ such that $f(x) \geq k + 2$. Then for every km-matching M of G, there exists a $(0, f)$-factorization $\mathscr{F} = \{F_1, F_2, \cdots, F_m\}$ of G such that each F_i shares exactly k edges in common with M.*

3. Factorization orthogonal to k edge-disjoint subgraphs

- **Theorem 3.19** (Feng). *Let G be an $(mg+m-1, mf-m+1)$-graph, where g and f be integer-valued functions defined on $V(G)$ such that $\frac{9}{4}k \leq g(x) \leq f(x)$. Let $H_1, H_2, \cdots, H_k$ be edge-disjoint m-subgraphs of G. Then there exists a (g, f)-factorization orthogonal to every H_i, $1 \leq i \leq k$.*

- **Theorem 3.20** (Feng). *Let G be an $(mg+m-1, mf-m+1)$-graph, where g and f be integer-valued functions defined on $V(G)$ such that $r \leq g(x) \leq f(x)$. Let H be an star of G with mr-edges. Then for every partition $H_1, H_2, \cdots, H_m$ of H such that $|E(H_i)| = r$, $1 \leq i \leq m$, $H_i \cap H_j = \phi$ when $i \neq j$, there exists a (g, f)-factorization $\mathscr{F} = \{F_1, F_2, \cdots, F_m\}$ of G satisfying $E(H_i) \subseteq E(F_i)$, $1 \leq i \leq m$.*

- **Theorem 3.21** (Yuan). *Let G be an $(mg+(m-1)k, mf-(m-1)k$-graph, where $k-1 \leq g(x) \leq f(x)$. Let H be an mk-subgraph of G. Then for every partition $H_1, H_2, \cdots, H_m$ of H such that $|E(H_i)| = k$, $1 \leq i \leq m$ and $H_i \cap H_j = \phi$ for $i \neq j$. Then G has a (g, f)-factorization $\mathscr{F} = \{F_1, F_2, \cdots, F_m\}$, where $E(H_i) \subseteq E(F_i)$, $1 \leq i \leq m$.*

- **Theorem 3.22** (Zhao, Liu and Yan). *Let G be an $(mg+m-1, mf-m+1)$-graph, where $2k-2 \leq g(x) \leq f(x)$. Let H be an mk-subgraph of G. Then for every partition $H_1, H_2, \cdots, H_m$ of H such that $|E(H_i)| = k$, $1 \leq i \leq m$, H_i are matchings, $H_i \cap H_j = \phi$ for $i \neq j$. Then if $m \leq 2k-1$, G has a (g, f)-factorization $\mathscr{F} = \{F_1, F_2, \cdots, F_m\}$, where $E(H_i) \subseteq E(F_i)$, $1 \leq i \leq m$.*

4. Some new problems

Problem 4.1. *To prove that Conjecture* 1 *and* 2 *are true for every* $2d$*-regular graph with* $|V(G)| \geq 2d+1$.

Problem 4.2. *In Theorem* 3.9*, for any subgraph with* kr *edges in an* $(mg+kr, mf-kr)$*-graph* G*, is it possible to weaken the sufficient condition for the existence of a* (g,f)*-factorization* r*-orthogonal to the subgraph?*

Problem 4.3. *Let G be a $(0, mf - 1)$-graph. Let H be a subgraph of G with m edges. Does there exists a $(0, f)$-factorization of G orthogonal to H?*

Problem 4.4. *In Theorem 3.19, for any k subgraphs with m edges in an $(mg + m - 1, mf - m + 1)$-graph G, is it possible to weaken the sufficient condition for the existence of a (g, f)-factorization orthogonal to the k subgraphs to $2k \leq g(x) \leq f(x)$?*

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Thank you!