The (f, g)-Difference Operator, (f, g)-Expansion Formula, and Their Applications

马欣荣

苏州大学数学系

报告的主要内容是

A. Lagrange inversion formula and three known results

B. Discrete analogue of the Lagrange-inversion formula

C. The (f, g)-difference operator

D. The (f, g)-expansion formula and Ismail's argument

E. The $(1 - xy, x - y)$-expansion formula

F. The (f, g)-expansion form of Gessel and Stanton's q-analogue

G. Applications of the (f, g)-expansion formula to basic hypergeometric series

A. The Lagrange inversion formula and three known results

Lagrange inversion formula (cf.[46, §7.32])

Theorem 1. *Assume that* $F(x)$ *and* $\phi(x)$ *are analytic around* $x = 0, \phi(0) \neq 0$. *Then*

$$F(x) = \sum_{n=0}^{\infty} a_n \left(\frac{x}{\phi(x)} \right)^n, \tag{1}$$

and the coefficients a_n *are given by*

$$a_n = n! \frac{d^{n-1}}{dx^{n-1}} \left[\phi^n(x) \frac{dF(x)}{dx} \right]_{x=0}.$$

where $\frac{d}{dx}$ denotes the usual derivative operator.

In the past years, various q-analogues (as generalizations) of the Lagrange inversion formula, as an active field of research with an increasing number of applications to q-series and the Rogers-Ramanujan identities, have been studied by numerous authors (cf.[1, 8, 18, 25, 26, 42, 43]).

For a good survey about results and open problems on this topic, we would like to refer the reader to Stanton's paper [43].

Carlitz's analogue [9, (1973)]

Theorem 2. *For any formal series $F(x)$, it holds that*

$$F(x) = \sum_{k=0}^{\infty} \frac{x^k}{(q,x;q)_k} \left[\mathcal{D}_{q,x}^k\{F(x)(x;q)_{k-1}\}\right]_{x=0}, \tag{2}$$

where $\mathcal{D}_{q,x}$ denotes the q-difference operator.

Gessel and Stanton's analogue [18, (1983)]

Theorem 3. *For any formal power series $F(x)$,*

$$F(x) = \sum_{n \geq k \geq 0}^{\infty} a_k \frac{(Ap^k q^k; p)_{n-k}}{(q;q)_{n-k}} q^{-nk} x^n \tag{3}$$

if and only if

$$a_n = \sum_{k=0}^{n} (-1)^{n-k} q^{\binom{n-k+1}{2}+nk} \frac{(1 - Ap^k q^k)(Aq^n p^{n-1}; p^{-1})_{n-k-1}}{(q;q)_{n-k}} F(q^k). \tag{4}$$

Their result is based on the deep insight that the essential character of the Lagrange inversion formula is equivalent to finding the inverse of an infinite lower triangular matrix $F = (B_{n,k})$ subject to $B_{n,k} = 0$ unless $n \geq k$, $B_{n,n} \neq 0$, that is, another unique matrix $G = (B^{-1}_{n,k})$ satisfying

$$\sum\nolimits_{n \geq i \geq k} B^{-1}_{n,i} B_{i,k} = \delta_{n,k},$$

where δ denotes the usual Kronecker delta.

Liu 's expansion formula [30, (2002)]

Theorem 4. *Given any formal series $F(x)$, it holds that*

$$F(x) = \sum_{k=0}^{\infty} \frac{(1 - aq^{2k})(aq/x;q)_k x^k}{(q;q)_k (x;q)_k} \left[\mathcal{D}^k_{q,x}\{F(x)(x;q)_{k-1}\}\right]_{x=aq}. \quad (5)$$

B. Discrete analogue of the Lagrange-inversion formula

基于对以上结论的观察, 我们可以引入以下概念

Definition 1. *Let $f(x,y)$ and $g(x,y)$ be two arbitrary nonzero functions over $\mathbb{C}$ in variables x,y. Suppose that for all four numbers a,b,c,x,*

$$f(x,a)g(b,c)+f(x,b)g(c,a)+f(x,c)g(a,b)=0, \tag{6}$$

then $f(x,y)$ is called orthogonal to $g(x,y)$. In particular, $f(x,y)$ is called self-orthogonal if $f(x,y)$ is orthogonal to itself.

Write $f(x,y)\perp g(x,y)$, or in short, $f\perp g$ if $f(x,y)$ is orthogonal to $g(x,y)$. As it stands, $f\perp g$ does not mean $g\perp f$. This definition allows us to rewrite

$$\mathit{Ker}\mathcal{L}_3=\{f(x,y)|f\perp f\},\ \mathit{Ker}\mathcal{L}_3^{(g)}=\{f(x,y)|f\perp g\}.$$

Now, we use Ω to denote an open subset of the complex plane, and $\mathcal{H}(\Omega)$ the space of analytic functions over Ω.

Definition 2. *With the assumption as above. Let $F(x) \in \mathcal{H}(\Omega)$. The following series*

$$\sum_{k=0}^{\infty} G(k) f(x_k, b_k) \frac{\prod_{i=0}^{k-1} g(b_i, x)}{\prod_{i=1}^{k} f(x_i, x)}, \tag{7}$$

where the coefficients $G(n)$ are given by

$$\sum_{k=0}^{n} F(b_k) \frac{\prod_{i=1}^{n-1} f(x_i, b_k)}{\prod_{i=0, i\neq k}^{n} g(b_i, b_k)},$$

is called the (f, g)-series generated by $F(x)$ with respect to two *sequences $\{b_i\}$ and $\{x_i\}$ over Ω. We denote it by*

$$F(x) \sim \sum_{k=0}^{\infty} G(k) f(x_k, b_k) \frac{\prod_{i=0}^{k-1} g(b_i, x)}{\prod_{i=1}^{k} f(x_i, x)} \tag{8}$$

Very similar to the situation in the theory of Fourier series, we come up against two questions:

The convergence problem Does the series (7) converge at some point $x \in \Omega$?

The representation problem If (7) does converge at $x \in \Omega$, is its sum $F(x)$? more precisely,

$$F(x) = \sum_{k=0}^{\infty} G(k) f(x_k, b_k) \frac{\prod_{i=0}^{k-1} g(b_i, x)}{\prod_{i=1}^{k} f(x_i, x)}. \tag{9}$$

这两个问题都是围绕(f,g)-inversion formula的新问题。为了研究这个反演公式在基本超几何理论内的作用，我们把精力 放在表示问题上。为此引入

Definition 3. *Let $F(x) \in \mathcal{H}(\Omega)$ and $f(x,y) \in Ker\mathcal{L}_3^{(g)}$. If there exist three sequences $\{x_i\}, \{b_i\} \subseteq \Omega$, and $\{\chi(i)\}$, such that for any $x \in \Omega$,*

$$F(x) = \sum_{k=0}^{\infty} \chi(k) f(x_k, b_k) \frac{\prod_{i=0}^{k-1} g(b_i, x)}{\prod_{i=1}^{k} f(x_i, x)}. \tag{10}$$

then it is called the (f,g)-expansion formula of $F(x)$ with respect to two sequences $\{b_i\}$ and $\{x_i\}$ over Ω.

Whereas many nonterminating summation and transformation formulas from the theory of basic hypergeometric series, just like Examples 2 and 3, fit into such a framework, apparently there is no general theorem (aside from those formulas mentioned in Section 1) known on the existence of such phenomena.

C. The (f, g)-difference operator

In what follows, we use $\mathbb{C}(x)$ to denote the linear space of all functions over $\mathbb{C}$ of a variable x.

Definition 4. *Let $f(x, y) \in Ker\mathcal{L}_3^{(g)}$, $\{x_i\}$ and $\{b_i\}$ be arbitrary sequences such that none of the denominators in (11) vanish. Then the mapping*

$$\mathbb{D}^{(n)}_{(f,g)} \begin{bmatrix} b_0, b_1, \ldots, b_n \\ x_1, \ldots, x_{n-1} \end{bmatrix} \{\bullet\} : \mathbb{C}(x) \longrightarrow \mathbb{C},$$

such that

$$\mathbb{D}^{(n)}_{(f,g)} \begin{bmatrix} b_0, b_1, \ldots, b_n \\ x_1, \ldots, x_{n-1} \end{bmatrix} \{F(x)\} = \sum_{k=0}^{n} F(b_k) \frac{\prod_{i=1}^{n-1} f(x_i, b_k)}{\prod_{i=0, i \neq k}^{n} g(b_i, b_k)} \qquad (11)$$

is said to be the nth order (f, g)-difference operator with respect to

$n+1$ *pairwise distinct nodes* $b_0, b_1, b_2, \cdots, b_n$ *and* $n-1$ *parameters* $x_1, x_2, \cdots, x_{n-1}$.

The following example displays that $\mathbb{D}^{(n)}_{(f,g)}$ is a generalization of the divided difference and the q-difference operator in numerical analysis and special function.

Example 1. *Let* $f(x,y) = 1, g(x,y) = x - y, b_i = x_{i+1}, i = 0, 1, 2, \cdots$. *Then*

$$\mathbb{D}^{(n)}_{(g)}[x_1, x_2, \cdots, x_{n+1}]\{F(x)\} = (-1)^n F[x_1, x_2, \cdots, x_{n+1}], \qquad (12)$$

where the classical divided difference $F[x_1, x_2, \cdots, x_{n+1}]$ *is recursively* *defined by*

$$
\begin{aligned}
&F[x_1] = F(x_1);\\
&F[x_1, x_2] = \frac{F(x_1) - F(x_2)}{x_1 - x_2};\\
&F[x_1, x_2, x_3] = \frac{F[x_1, x_2] - F[x_2, x_3]}{x_1 - x_3};\\
&\cdots;\\
&F[x_1, x_2, \cdots, x_n] = \frac{F[x_1, x_2, \cdots, x_{n-1}] - F[x_2, x_3, \cdots, x_n]}{x_1 - x_n}.
\end{aligned}
$$

See [3, p.123] for more details.

As might be expected from this definition by specializing

$b_i = x + h(i-1)$, $\mathbb{D}_{(g)}^{(n)}$ can be expressed in a simpler form

$$\mathbb{D}_{(g)}^{(n)}[x, x+h, x+2h, \cdots, x+nh]\{F(x)\} = \frac{1}{n!h^n}\nabla_h^n F(x),$$

where ∇_h is the usual backward difference operator defined as $\nabla_h\{F(x)\} = F(x) - F(x+h)$. In the meantime, assume that $F(x)$ is n-times differentiable at x. Then it is easily found that

$$\lim_{h\to 0} \mathbb{D}_{(g)}^{(n)}[x, x+h, x+2h, \cdots, x+nh]\{F(x)\} = \frac{1}{n!}\frac{d^n}{dx^n}F(x). \qquad (13)$$

Another interesting case is that with the specification $b_i = xq^i, i = 0, 1, 2, \cdots$, we have

$$\mathbb{D}_{(g)}^{(1)}[x, xq]\{F(x)\} = \frac{1}{(q-1)}\mathcal{D}_{q,x}F(x),$$

where $\mathcal{D}_{q,x}$ denotes the q-difference operator $(\text{cf.}[24])$ *appeared previously in (2)*

$$\mathcal{D}_{q,x}F(x) = \frac{F(x) - F(qx)}{x}. \tag{14}$$

(f, g)-差分算子也具有良好的性质:

Theorem 5. [递推关系] *Let $f(x, y) \in Ker\mathcal{L}_3^{(g)}$, and $\mathbb{D}_{(f,g)}^{(n)}$ be defined as above. Then for any integer $n \geq 0$,*

$$\begin{aligned}
&\mathbb{D}_{(f,g)}^{(n+1)} \begin{bmatrix} b_0, b_1, \ldots, b_{n+1} \\ x_1, \ldots, x_n \end{bmatrix} \{F(x)\} \\
= &\quad \frac{f(x_n, b_0)}{g(b_{n+1}, b_0)} \mathbb{D}_{(f,g)}^{(n)} \begin{bmatrix} b_0, b_1, \ldots, b_n \\ x_1, \ldots, x_{n-1} \end{bmatrix} \{F(x)\} \\
+ &\quad \frac{f(x_n, b_{n+1})}{g(b_0, b_{n+1})} \mathbb{D}_{(f,g)}^{(n)} \begin{bmatrix} b_1, b_2, \ldots, b_{n+1} \\ x_1, \ldots, x_{n-1} \end{bmatrix} \{F(x)\}.
\end{aligned} \tag{15}$$

Theorem 6. [Leibinz formula] *Let $f(x,y) \in Ker\mathcal{L}_3^{(g)}$, and $\mathbb{D}_{(f,g)}^{(n)}$ be defined as above. Then*

$$\mathbb{D}_{(f,g)}^{(n)}\begin{bmatrix} b_0, b_1, \ldots, b_n \\ x_1, \ldots, x_{n-1} \end{bmatrix}\{F(x)H(x)\}$$
$$= \sum_{k=0}^{n} f(x_k, b_k)\, \mathbb{D}_{(f,g)}^{(k)}\begin{bmatrix} b_0, b_1, \ldots, b_k \\ x_1, \ldots, x_{k-1} \end{bmatrix}\{H(x)\} \tag{16}$$
$$\mathbb{D}_{(f,g)}^{(n-k)}\begin{bmatrix} b_k, b_{k+1}, \ldots, b_n \\ x_{k+1}, \ldots, x_{n-1} \end{bmatrix}\{F(x)\}.$$

D.The (f,g)-expansion formula and Ismail's argument

下面的结论既说明$F(x)$的(f,g)-展开式若存在则必是唯一的,同时说明(f,g)-差分算子的作用.

Theorem 7. *The coefficients $\chi(n)$ in (10) are uniquely determined by $\{G(i)\}$, i.e.,*

$$\chi(n) = \mathbb{D}^{(n)}_{(f,g)} \begin{bmatrix} b_0, b_1, \ldots, b_n \\ x_1, \ldots, x_{n-1} \end{bmatrix} \{F(x)\}. \tag{17}$$

It is worth pointing out that even if the (f,g)-series generated by $F(x)$ does not converge to itself, it always agrees with $F(x)$ at infinite points

$b_n, n = 0, 1, \cdots$, i.e.,

$$F(b_n) = \sum_{k=0}^{\infty} G(k) f(x_k, b_k) \frac{\prod_{i=0}^{k-1} g(b_i, x)}{\prod_{i=1}^{k} f(x_i, x)} |_{x=b_n}. \tag{18}$$

什么是"Ismail 原理"？

Generally speaking, the "Ismail's argument" can be summarized briefly as follows: if one wants to prove two analytic functions $F(x) = G(x)$, all that is necessary is to show that they agree infinitely often near a point that is an interior point of the set of analyticity.

Hence, in order to guarantee that they are equal over Ω, it is sufficient to require that the series and $F(x)$ be analytic around $x = b$ while b can be chosen as an accumulation point of $\{b_i\}$ in the interior of Ω. Combining this idea with the original "Ismail's argument" due to Ismail, we get all that described in the present paper.

Theorem 8. [Generalized Ismail's argument] *Let $F(x) \in \mathcal{H}(\Omega)$, $f(x, y) \in$ Ker$\mathcal{L}_3^{(g)}$. Let $S_n(x)$ be the sequence of partial sums of the (f, g)-series generated by $F(x)$, say*

$$S_n(x) = \sum_{k=0}^{n} G(k) f(x_k, b_k) \frac{\prod_{i=0}^{k-1} g(b_i, x)}{\prod_{i=1}^{k} f(x_i, x)} \tag{19}$$

where the coefficients

$$G(k) = \mathbb{D}_{(f,g)}^{(k)} \begin{bmatrix} b_0, b_1, \ldots, b_k \\ x_1, \ldots, x_{k-1} \end{bmatrix} \{F(x)\}. \tag{20}$$

Assume further that

(i) $\lim_{n \mapsto \infty} b_n = b \in \Omega$;

(ii) $S_n(x)$ *converges uniformly to* $S(x)$ *in a neighborhood of* $b \in \Omega$;

(iii) *for any integer* $i \geq 0, g(b_i, x)/f(x_{i+1}, x)$ *is analytic at* $x = b$.

Then there exists a subset $\Omega_1 \subseteq \Omega$ *containing* b *such that for* $x \in \Omega_1$,

$$F(x) = S(x) = \sum_{k=0}^{\infty} G(k) f(x_k, b_k) \frac{\prod_{i=0}^{k-1} g(b_i, x)}{\prod_{i=1}^{k} f(x_i, x)}. \tag{21}$$

Our argument is based on the (f, g)-inversion formula.

Lemma 1. *Let* $F = (B_{n,k})_{n,k \in \mathbb{Z}}$ *and* $G = (B^{-1}_{n,k})_{n,k \in \mathbb{Z}}$ *be two matrices with entries given by*

$$B_{n,k} = \frac{\prod_{i=k}^{n-1} f(x_i, b_k)}{\prod_{i=k+1}^{n} g(b_i, b_k)} \quad \text{and} \tag{22}$$

$$B^{-1}_{n,k} = \frac{f(x_k, b_k)}{f(x_n, b_n)} \frac{\prod_{i=k+1}^{n} f(x_i, b_n)}{\prod_{i=k}^{n-1} g(b_i, b_n)}, \quad \text{respectively,} \tag{23}$$

where $\mathbb{Z}$ denotes the set of integers, $\{x_i\}$ and $\{b_i\}$ are arbitrary sequences such that none of the denominators in the right hand sides of (22) and (23) vanish. Then $F = (B_{n,k})_{n,k\in\mathbb{Z}}$ and $G = (B^{-1}_{n,k})_{n,k\in\mathbb{Z}}$ is a matrix inversion if and only $f(x,y) \in$ Ker$\mathcal{L}_3^{(g)}$.

这个反演公式统一了目前主要的已知结论: Gould-Hsu inversion(1973年); Krattethaler's formula(1996年); Warnaar's formula(2002年).

Remark 1. *Note that Conditions* (i) *and* (ii) *are also necessary. Otherwise, the conclusion may be false. For instance, let $F(x) = \sin(\pi x)$ and parameter sequence $b_k = k, k = 1, 2, \cdots$. Assume that*

$$\sin(\pi x) = \sum_{k=0}^{\infty} G(k)x(x-1)(x-2)\cdots(x-k+1).$$

By the $(1, x-y)$*-inversion, it would follow that*

$$G(n) = \mathbb{D}^{(n)}_{(x-y)}[0, 1, \ldots, n]\{\sin(\pi\, x)\} \equiv 0$$

for $n \geq 1$*, i.e.,* $\sin(\pi x) = 1$*, which is obviously contrary to the known fact.*

Remark 2. *Note that Conditions* (i) *and* (ii) *are also necessary. Otherwise, the conclusion may be false. For instance, let* $F(x) = \sin(\pi x)$ *and parameter sequence* $b_k = k, k = 1, 2, \cdots$. *Assume that*

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for $n \geq 1$, *i.e.*, $\sin(\pi x) = 1$, *which is obviously contrary to the known fact.*

Remark 3. *广义的Ismail原理也可以视为Lagrange反演公式的离散模拟(discrete analogue),即在(1)中*

$$\left(\frac{x}{\phi(x)}\right)^n \longmapsto f(x_n, b_n)\frac{\prod_{i=0}^{n-1} g(b_i, x)}{\prod_{i=1}^{n} f(x_i, x)}.$$

Five important cases of Theorem 8 are worthy of note, which are obtained by specializing

$$\begin{aligned} f(x,y) &= 1, & g(x,y) &= x-y;\\ &= x-y, & &= x-y;\\ &= 1-xy, & &= x-y;\\ &= (1-axy)(1-b\tfrac{x}{y}), & &= (x-y)(1-\tfrac{b}{axy});\\ &= y\theta(xy)\theta(\tfrac{x}{y}), & &= y\theta(xy)\theta(\tfrac{x}{y})) \end{aligned}$$

respectively. We now summarize these results without proof.

Corollary 1. *With the same assumption as Theorem 8. Then the* *following hold*

$$F(x)=\sum_{k=0}^{\infty}\left\{\sum_{j=0}^{k}\frac{F(b_j)}{\prod_{i=0,i\neq j}^{k}(b_i-b_j)}\right\}\prod_{i=0}^{k-1}(b_i-x) \tag{24}$$

$$=\sum_{k=0}^{\infty}(x_k-b_k)\left\{\sum_{j=0}^{k}F(b_j)\frac{\prod_{i=1}^{k-1}(x_i-b_j)}{\prod_{i=0,i\neq j}^{k}(b_i-b_j)}\right\}\frac{\prod_{i=0}^{k-1}(b_i-x)}{\prod_{i=1}^{k}(x_i-x)} \tag{25}$$

$$=\sum_{k=0}^{\infty}(1-x_kb_k)\left\{\sum_{j=0}^{k}F(b_j)\frac{\prod_{i=1}^{k-1}(1-x_ib_j)}{\prod_{i=0,i\neq j}^{k}(b_i-b_j)}\right\}\frac{\prod_{i=0}^{k-1}(b_i-x)}{\prod_{i=1}^{k}(1-xx_i)} \tag{26}$$

$$=\sum_{k=0}^{\infty}G_1(k)(1-ax_kb_k)(1-bx_k/b_k)\frac{\prod_{i=0}^{k-1}(b_i-x)(1-\frac{b}{ab_ix})}{\prod_{i=1}^{k}(1-ax_ix)(1-b\frac{x_i}{x})} \tag{27}$$

$$=\sum_{k=0}^{\infty}G_2(k)b_k\theta(x_kb_k)\theta\left(\frac{x_k}{b_k}\right)\frac{\prod_{i=0}^{k-1}\theta(b_ix)\theta(\frac{b_i}{x})}{\prod_{i=1}^{k}\theta(x_ix)\theta(\frac{x_i}{x})}, \tag{28}$$

where the coefficients

$$
\begin{aligned}
G_1(k) &= \mathbb{D}^{(k)}_{\left((1-axy)(1-b\frac{x}{y}),(x-y)(1-\frac{b}{axy})\right)} \begin{bmatrix} b_0, b_1, \ldots, b_k \\ x_1, \ldots, x_{k-1} \end{bmatrix} \{F(x)\}, \\
G_2(k) &= \mathbb{D}^{(k)}_{\left(y\theta(xy)\theta(\frac{x}{y}),y\theta(xy)\theta(\frac{x}{y})\right)} \begin{bmatrix} b_0, b_1, \ldots, b_k \\ x_1, \ldots, x_{k-1} \end{bmatrix} \{F(x)\}, \qquad \textit{and}
\end{aligned}
$$

$\theta(x) = (x;q)_\infty(\frac{q}{x};q)_\infty$, $|q| < 1$.

We remark that the theta function $\theta(x)$ satisfying Jacobi's *triple product identity*

$$\theta(x)(q;q)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k}{2}} x^k, x \neq 0$$

has been frequently used in the study of the elliptic hypergeometric series.

E. The $(1-xy, x-y)$-expansion theorem

在广义的Ismail原理的基础上，我们可以建立

Theorem 9. [The $(1-xy, x-y)$-expansion theorem] *Let* $F(x) \in \mathcal{H}(\Omega)$*,* $\{b_i\}, \{x_i\} \subseteq \Omega$ *such that* $\{b_i\}$ *are pairwise distinct and* $\{x_i\}$ *is bounded,* $\lim_{i\mapsto\infty} b_i = b \neq b_0$ *and* $\inf\{|1/x_i - b| : i \geq 0\} > 0$*. Suppose further that* $\limsup_{k\mapsto\infty} |\lambda_k| < \infty$ *where*

$$\lambda_k = \frac{\mathbb{D}^{(k)}_{(1-xy,x-y)}\begin{bmatrix} b_1, b_2, \ldots, b_{k+1} \\ x_1, \ldots, x_{k-1} \end{bmatrix}\{F(x)\}}{\mathbb{D}^{(k)}_{(1-xy,x-y)}\begin{bmatrix} b_0, b_1, \ldots, b_k \\ x_1, \ldots, x_{k-1} \end{bmatrix}\{F(x)\}}. \tag{29}$$

Then there exists an open set Ω_1 containing b such that for $x \in \Omega_1$,

$$F(x) = \sum_{k=0}^{\infty}(1 - x_k b_k)\left\{\mathbb{D}^{(k)}_{(1-xy,x-y)}\begin{bmatrix} b_0, b_1, \ldots, b_k \\ x_1, \ldots, x_{k-1}\end{bmatrix}\{F(x)\}\right\}\frac{\prod_{i=0}^{k-1}(b_i - x)}{\prod_{i=1}^{k}(1 - x x_i)}. \tag{30}$$

Remark 4. *It is also worthy of note that Fu and Lascoux, in their paper [13], established a similar expansion formula in the setting of formal power series by virtue of a divided difference operator acting on multivariate function. For a discussion of this divided difference operator and related matters, the reader may consult [29].*

E1.The $(1-xy, x-y)$-expansion form of Gessel and Stanton's q-analogue

一个被人们忽视(包括Gessel and Stanton本人)的事实:定理 9其实是蕴含了 Gessel and Stanton的结果的(f,g)-展开公式形式. 这一公式 统一了刘治国的展开公式 (因此Carlitz's q-模拟.)

Theorem 10. [Gessel and Stanton's q-analogue (3)/(4))] *Let $|p|, |q| < 1$ and $F(x) = \sum_{n=0}^{\infty} a_n x^n$ be arbitrary power series with the nonzero radius R of convergence and be also convergent at $x = R$. Assume further R is not of the form $q^n (n \geq 0)$, $\lim_{n\mapsto\infty} a_{n+1}/a_n = c_0$, and $0 < m|c_0|^n \leq |a_n| \leq M|c_0|^n$. Then*

$$F(x) = \sum_{k=0}^{\infty} G_3(k)(1 - Ap^k q^k) \frac{\prod_{i=0}^{k-1}(q^i - x)}{(q;q)_k (Apx;p)_k}$$

where the coefficients

$$G_3(n) = \sum_{k=0}^{n} (-1)^{n-k} q^{\binom{k+1}{2} - nk} \begin{bmatrix} n \\ k \end{bmatrix}_q (Apq^k; p)_{n-1} F(q^k); \quad (31)$$

Remark 5. *We obtain it only by specializing $b_i = q^i, x_i = Ap^i$ in (30). It can be regarded as initial study on the representation problem of $F(x)$ in terms of (f, g)-series.*

Remark 6. *Liu's q-expansion formula (5) follows from the q-analogue of Gessel and Stanton by substituting $p \mapsto q, A \mapsto a, x \mapsto x/aq$, and Identity (2) is the special case $a \mapsto 0$ of (5).*

G. Two illustrative examples

Example 2. *For $|q| < 1$ and a variable $z : |z| < 1$,*

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty} \tag{32}$$

is equivalent to its finite form

$$\sum_{k=0}^{n} (-1)^k q^{\binom{k+1}{2}-nk} \begin{bmatrix} n \\ k \end{bmatrix}_q (z;q)_k = z^n \Longleftrightarrow \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q c^k = (c;q)_n. \tag{33}$$

Proof. Now, we replace the parameter a by a new variable x and take z

as a parameter, and then reformulate (32) as

$$\sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k}(x;q)_k = \frac{(zx;q)_\infty}{(z;q)_\infty},$$

which turns out to be equivalent to, under the substitution $x \mapsto 1/x, z \mapsto zx$,

$$\sum_{k=0}^{\infty} \frac{(-1)^k z^k}{(q;q)_k} \prod_{i=0}^{k-1} (q^i - x) = \frac{(z;q)_\infty}{(zx;q)_\infty}. \tag{34}$$

Letting $x = q^n$ yields a terminating summation formula

$$\sum_{k=0}^{n} \frac{(-1)^k z^k}{(q;q)_k} \prod_{i=0}^{k-1} (q^i - q^n) = (z;q)_n. \tag{35}$$

Now, applying the $(1, x-y)$-inversion given in Lemma 1 to (35) to arrive at

$$\frac{(-1)^n z^n}{(q;q)_n} = \sum_{k=0}^{n} \frac{(z;q)_k}{\prod_{i=0, i\neq k}^{n} (q^i - q^k)},$$

which reduces after simplification to the desired result. Conversely, in the light of the $(1, x-y)$-inversion, (33) is equivalent to (35), the latter states that (34) is valid for $x = q^n, n = 0, 1, 2, \cdots$. Then by the generalized "Ismail's argument", (34), i.e., the q-binomial theorem holds.

∎

Another example of the $(1-xy, x-y)$-expansion formula, also a good case to illustrate the "Ismail's argument", is the following generalized Lebesgue identity due to Carlitz.

Example 3. *For three indeterminates q, x, a with $|q| < 1$ and $|x| < 1$,*

$$\sum_{k=0}^{\infty}(-1)^k q^{\binom{k}{2}} \frac{(x;q)_k a^k}{(q,bx;q)_k} = \frac{(a,x;q)_\infty}{(bx;q)_\infty} \sum_{k=0}^{\infty} \frac{(b;q)_k}{(q,a;q)_k} x^k. \tag{36}$$

Proof. At first, apply the substitution $b \mapsto b/y, x \mapsto xy, a \mapsto y$ to reformulate equivalently the identity in question as

$$F(y) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{(q;q)_k} \frac{\prod_{i=0}^{k-1} g(b_i,y)}{\prod_{i=1}^{k} f(x_i,y)}, \tag{37}$$

where $f(x,y) = 1 - xy, g(x,y) = x - y, b_i = bq^i, x_i = q^{i-1}$, and $F(y)$ is defined by

$$F(y) = \sum_{i=0}^{\infty} \frac{(-1)^i q^{\binom{i}{2}} (bxq^i;q)_\infty}{(q;q)_i} \frac{y^i}{(y,xyq^i;q)_\infty}. \tag{38}$$

By the $(1-xy, x-y)$-expansion formula, it suffices to show in a direct way that

$$\frac{(bx)^n}{1-bq^{2n-1}} = \sum_{k=0}^{n}(-1)^k q^{\binom{k+1}{2}-nk} \begin{bmatrix} n \\ k \end{bmatrix}_q (bq^k;q)_{n-1} F(bq^k). \quad (39)$$

For this, we calculate by using Heine's transformation formula to obtain

$$\begin{aligned} F(bq^k) &= \frac{(bx;q)_k}{(bq^k;q)_\infty} \sum_{i=0}^{\infty} \frac{(-1)^i q^{\binom{i}{2}} (bq^k)^i}{(q;q)_i} \frac{(bxq^k;q)_i}{(bx;q)_i} \\ &= \sum_{i=0}^{\infty} \frac{(q^{-k};q)_i (bxq^k)^i}{(q, bq^k;q)_i} = \sum_{i=0}^{k} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} k \\ i \end{bmatrix}_q \frac{(bx)^i}{(bq^k;q)_i}. \end{aligned}$$

Inserting this into the r.h.s. of (39) to simplify the resulting identity, we have

$$\text{RHS of (39)} = \sum_{n\geq k\geq i\geq 0}(-1)^{k+i}q^{\binom{k+1}{2}-nk+\binom{i}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}k\\i\end{bmatrix}_q\frac{(bq^k;q)_{n-1}(bx)^i}{(bq^k;q)_i}$$

$$= \sum_{i=0}^{n}(bx)^i q^{-Ni}\begin{bmatrix}n\\i\end{bmatrix}_q\sum_{K=0}^{N}(-1)^K q^{\binom{K+1}{2}-NK}\begin{bmatrix}N\\K\end{bmatrix}_q\frac{(b;q)_{N+K+2i-1}}{(b;q)_{K+2i}},$$

where $N=n-i, K=k-i$. Evidently, if $N\geq 1$, then

$$\frac{(b;q)_{N+K+2i-1}}{(b;q)_{K+2i}}=(bq^{K+2i};q)_{N-1}.$$

Since for fixed i, $(bq^{2i}x;q)_{N-1}$ is a polynomial of degree no more than $N-1$ in x, thus by a proposition of the difference operator, we find that the inner sum turns out to be

$$\sum_{K=0}^{N}(-1)^K q^{\binom{K+1}{2}-NK}\begin{bmatrix}N\\K\end{bmatrix}_q \frac{(b;q)_{N+K+2i-1}}{(b;q)_{K+2i}}=0.$$

Thus, the only nonzero term corresponding to $N=0$ of the r.h.s. of (39) is $(bx)^n/(1-bq^{2n-1})$. It gives the complete proof of (39). ∎

结束语

We hope that the generalized "Ismail's argument" or the representation of analytic functions in terms of (f,g)-series in this article is a new general approach to the basic hypergeometric series. 下面三个问题有待于进一步研究

1. 求$F(x)$, 使得$F(x) \in \mathcal{H}(\Omega)$ 的n阶 (f,g)-差分

$$\mathbb{D}^{(n)}_{(f,g)} \begin{bmatrix} b_0, b_1, \ldots, b_n \\ x_1, \ldots, x_{n-1} \end{bmatrix} \{F(x)\}$$

   是封闭的？

2. The (f,g)-expansion formula of $F(x)$ is in fact a rational approximation to $F(x)$ if $f(x,y)$ and $g(x,y)$ are polynomials of two

variables x and y. Thus, it is necessary to study application of this expansion formula as well as the n-th order (f, g)-difference operator in the theory of (numerical) approximation.

3. The same problem remains open to the expansion formula (28),i.e.,

$$F(x) = \sum_{k=0}^{\infty} G_2(k) b_k \theta(x_k b_k) \theta\left(\frac{x_k}{b_k}\right) \frac{\prod_{i=0}^{k-1} \theta(b_i x)\theta(\frac{b_i}{x})}{\prod_{i=1}^{k} \theta(x_i x)\theta(\frac{x_i}{x})}, \tag{40}$$

in the theory of elliptic hypergeometric series [45].

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