A unified approach to polynomial sequences with only real zeros

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(joint work with Prof. Yi Wang)

1 Unimodal and log-concave sequences

Def 1. Let $\{a_0, a_1, \ldots, a_n\}$ be a sequence of positive numbers. The sequence is said to be **unimodal** (UM) if there exists an index $0 \le m \le n$, called the *mode* of the sequence, such that

$$a_0 \le \cdots \le a_{m-1} \le a_m \ge a_{m+1} \ge \cdots \ge a_n.$$

It is **log-concave** (LC) if $a_{i-1}a_{i+1} \le a_i^2$ for all $0 < i < n$.

2 Newton Inequality

If $\sum_{k=0}^{n} a_k x^k \in \mathrm{RZ}$, then $a_i^2 \geq a_{k-1}a_{k+1}\left(1+\frac{1}{k}\right)\left(1+\frac{1}{n-k}\right)$.
Therefore $a_0, \ldots, a_n$ is LC and UM with at most two modes.

Ex 1. $\sum_{i=0}^{n} \binom{n}{i} x^i = (x+1)^n \in \mathrm{RZ}$. So the sequence $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ has a mode $n/2$ or two modes $(n \pm 1)/2$.

Notation: 1. Let RZ be the set of polynomials with only **R**eal **Z**eros.
2. Let PF consist of the polynomials in RZ with positive coefficients.

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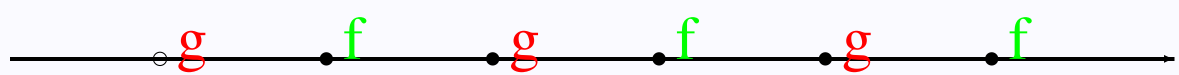
3 Interlacing of polynomials

Def 2. Suppose that $f(x) = \alpha \prod_{i=1}^{n}(x - r_i), \quad r_n \leq \cdots \leq r_1$ and

$$g(x) = \beta \prod_{j=1}^{m}(x - s_j), \quad s_m \leq \cdots \leq s_1.$$

We say that g **interlaces** f, denoted by $g \preceq f$, if the roots of g and f satisfy



Denote by $g \prec f$ if all roots of f and g are distinct.

Prop 1. $P \in \mathrm{RZ} \Longrightarrow P' \preceq_{\mathrm{int}} P$.

Ex 2. Let $P = (x+1)^3(x+5), \quad r_1 = r_2 = r_3 = -1, r_4 = -5.$
Then $P' = 4(x+1)^2(x+4), \quad s_1 = s_2 = -1, s_3 = -4.$

4 Problems

1. Let $F(x) = a(x)f(x) + b(x)g(x)$. Suppose that $f, g \in \mathrm{RZ}$ and $g \preceq f$. Under what conditions $F \in \mathrm{RZ}$ and $f \preceq F$?
2. Let $F(x) = a(x)f(x) + \sum_{j=1}^{k} b_j(x)g_j(x)$. Suppose that $f, g_j \in \mathrm{RZ}$ and $g_j \preceq f$ for all j. Under what conditions $F \in \mathrm{RZ}$ and $f \preceq F$?
3. Let $F(x) = a(x)f(x)+b(x)g(x)$ and $G(x) = c(x)f(x)+d(x)g(x)$. Suppose that $f, g \in \mathrm{RZ}$ and $g \preceq f$. Under what conditions $F, G \in \mathrm{RZ}$ and $G \preceq F$?

5 Theorem 1 and Applications

Thm 1. Let F, f, g be three real polynomials satisfying
(1). $F(x) = a(x)f(x) + b(x)g(x)$, where $a(x), b(x)$ are two real polynomials such that $\deg F = \deg f$ or $\deg f + 1$.
(2). $f, g \in \mathrm{RZ}$ and $g \preceq f$.
(3). F and g have leading coefficients of the same sign.
If $b(r) \le 0$ whenever $f(r) = 0$, then $F \in \mathrm{RZ}$ and $f \preceq F$. In particular, if $g \prec f$ and $b(r) < 0$ whenever $f(r) = 0$, then $f \prec F$.

Applications of Theorem 1

1. Stirling polynomials: $S_0(x) = 1$ and

$$S_n(x) = xS_{n-1}(x) + xS'_{n-1}(x).$$

2. Eulerian polynomials: $A_0(x) = 1$ and

$$A_n(x) = nxA_{n-1}(x) + x(1-x)A'_{n-1}(x).$$

3. Orthogonal polynomials: $p_{-1}(x) = 0, p_0(x) = 1$ and

$$p_n(x) = (a_n x + b_n)p_{n-1}(x) - c_n p_{n-2}(x), \quad a_n, c_n > 0.$$

4. Narayana polynomials: $N_1(x) = x, N_2(x) = x + x^2$ and

$$(n+1)N_n(x) = (2n-1)(1+x)N_{n-1}(x) - (n-2)(1-x)^2 N_{n-2}(x).$$

6 Theorem 2 and Applications

Thm 2. Let $F, f, g_1, \ldots, g_k$ be real polynomials satisfying
(1). $F(x) = a(x)f(x) + b_1(x)g_1(x) + \cdots + b_k(x)g_k(x)$, where a, b_j are real polynomials such that $degF = \deg f$ or $\deg f + 1$.
(2). $f, g_j \in \mathrm{RZ}$ and $g_j \preceq f$ for each j.
(3). F and g_j have leading coefficients of the same sign.
If $b_j(r) \le 0$ for each j and each zero r of f, then $F \in \mathrm{RZ}$ and $f \preceq F$. In particular, if for each zero r of f, there is an index j such that $g_j \prec f$ and $b_j(r) < 0$, then $f \prec F$.

Applications of Thm 2

1. Brenti derangement polynomials: $d_0(x) = 1, d_1(x) = 0$ and

$$d_n(x) = (n-1)xd_{n-1}(x) + x(1-x)d'_{n-1}(x) + (n-1)xd_{n-2}(x).$$

2. Matching polynomials:

$$M(G,x) = xM(G-\{v\},x) - \sum_{u\sim v} M(G-\{v,u\},x).$$

3. Heilmann-Lieb partition functions: $W \geq 0$ and

$$Q(G,x) = xQ(G-\{v\},x) - \sum_{u\sim v} W(u,v)Q(G-\{v,u\},x).$$

7 Theorem 3 and Applications

Thm 3. Let f, g, F, G be four standard real polynomials satisfying
(1). $F(x) = a(x)f(x) + b(x)g(x)$ and $G(x) = c(x)f(x) + d(x)g(x)$ where $a(x)$, $b(x)$, $c(x)$, $d(x)$ are real polynomials such that $\deg F = \deg G$ or $\deg G + 1$.
(2). $f, g \in \mathrm{RZ}$ and $g \preceq f$.
(3). $\Delta(x) = a(x)d(x) - b(x)c(x) \geq 0$ whenever $G(x) = 0$.
Suppose either that $c(x)$ is a positive constant and $\deg G \leq \deg g + 1$ or that $d(x)$ is a positive constant and $\deg G \leq \deg f$. Then $F, G \in \mathrm{RZ}$ and $G \preceq F$. In particular, if $g \prec f$ and $\Delta(x) > 0$ whenever $G(x) = 0$, then $G \prec F$.

Applications of Thm 3: folklore results

Coro 1. Let $a, b, c, d \geq 0$. Suppose that $f, g \in \mathrm{RZ}$ are standard and $g \preceq f$. Then the following statements hold.

1. If $ad \geq bc$, then $cf+dg \preceq af+bg$. In particular, $g \preceq af+bg \preceq f$.
2. If $af - bg$ is standard, then $cf + dg \preceq af - bg$. In particular, $f, g \preceq af - bg$.
3. If $-af+bg$ is standard, then $-af+bg \preceq cf+dg$, and in particular, $-af + bg \preceq f, g$.

Similar results hold when $g \prec f$.

Applications of Thm 3: A simple proof of Wang-Yeh thm

Wang-Yeh Thm. Let f and g be two standard RZ polynomials and $g \preceq f$. Suppose that $ad \le bc$. Then

$$F(x) = (ax+b)f(x) + (cx+d)g(x) \in \mathrm{RZ}.$$

Proof. Let $G(x) = af(x) + cg(x)$. Then $F, G \in \mathrm{RZ}$ and $G \preceq F$. □

Applications of Thm 3: Nice Matrix

Let $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ be a polynomial matrix, where $a, b, c, d \in \mathbb{R}^+[x]$.

Def 3. We say that the polynomial matrix M is **nice** if

(1). $\deg a \leq 1, \deg b \leq 2, \deg d \leq 1$ and c is a positive constant.

(2). $\det(M) \geq 0$ for $x \leq 0$.

Prop 2. M is nice $\Longrightarrow b \in \mathrm{RZ}$ and $d \preceq b$.

Proof. $d(r) = 0, ad \geq bc \Longrightarrow b(r) \leq 0.$ □

Genus Polynomials

Ex 3. Given a finite graph G and a nonnegative integer k, let $\gamma(G, k)$ denote the number of distinct embeddings of the graph G into an oriented surface of genus k. Define the genus polynomial

$$GP(G, x) = \sum_{k \geq 0} \gamma(G, k) x^k.$$

Stahl Conjecture. $GP(G, x) \in \mathrm{RZ}.$

Stahl considered the H-linear family of graphs obtained by consistently amalgamating additional copies of a graph H. For such a family $\{G_n\}$, there is a square matrix M and vector v with entries in $\mathbb{Z}[x]$ such that the genus polynomial of G_n is the first entry of $M^n v$.

Stahl Example

1. $M_1 = \begin{pmatrix} 4 & 2 \\ 6x & 0 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ x \end{pmatrix}.$ □
2. $M_2 = \begin{pmatrix} 0 & 4 \\ 2x & 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$ □
3. $M_3 = 4\begin{pmatrix} 2+3x & 1 \\ 4x & 2x \end{pmatrix}, \quad v_3 = 2\begin{pmatrix} 1+x \\ 2x \end{pmatrix}.$ □
4. $M_4 = 6\begin{pmatrix} 3x & 3 \\ 2x & 1+3x \end{pmatrix}, \quad v_4 = 2\begin{pmatrix} 2 \\ 1+x \end{pmatrix}.$
5. $M_5 = \begin{pmatrix} 192x & 96+288x \\ 72+192x^2 & 24+288x \end{pmatrix}, \quad v_5 = \begin{pmatrix} 18+18x \\ 6+30x \end{pmatrix}.$
6. $M_6 = \begin{pmatrix} 8+68x & 4+16x \\ 32x+48x^2 & 16x \end{pmatrix}, \quad v_6 = \begin{pmatrix} 2+14x \\ 8x+8x^2 \end{pmatrix}.$
7. $M_7 = 4\begin{pmatrix} 2+65x+54x^2 & 1+22x \\ 16x+104x^2 & 8x+16x^2 \end{pmatrix}, \quad v_7 = \begin{pmatrix} 2+58x+36x^2 \\ 16+80x \end{pmatrix}.$

Stahl Question

Let $M(x) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ where $a, b, c, d \in \mathbb{R}[x]$.

1. Under what conditions can it be guaranteed that if $(f(x), g(x))$ is a pair of polynomials whose zeros interlace, then so do the zeros of the two components of the vector $(f(x), g(x))M(x)$ interlace?
2. Under what conditions can it be guaranteed that the zeros of each of the entries of $M^k(x)$ are all real for $k = 1, 2, \ldots$.

Remark 1. Let $M^k = \begin{pmatrix} a_k & c_k \\ b_k & d_k \end{pmatrix}$. Then $(a_{k+1}, c_{k+1}) = (a_k, c_k)M$.

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Thank you for your attention!