A construction for large sets of disjoint Kirkman triple systems

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1 Introduction

A Steiner system $S(t, k, v)$ is a pair $(X, \mathcal{B})$, where X is a v-element set and $\mathcal{B}$ is a set of k-subset of X, called *blocks* with the property that every t-element subset of X is contained in a unique block. An $S(2, 3, v)$ is called a *Steiner triple system* and an $S(3, 4, v)$ is called a *Steiner quadruple system*. They are shortly denoted by STS(v) and SQS(v), respectively. An $S(t, k, v)$ is called i-resolvable, $0 < i < t$, if its block set can be partitioned into $S(i, k, v)$. Such a partition is called an i-resolution of the $S(t, k, v)$.

A 1-resolvable STS(v) is also called a *Kirkman triple system* of order v and shortly denoted by KTS(v). The KTS(v) is known to exist for any non-negative integer $v \equiv 3 \ (mod\ 6)$. (Ray-Chandhuri and Wilson, 1971)

Two KTS(v) $(X, \mathcal{A})$ and $(X, \mathcal{B})$ are called *disjoint* if $\mathcal{A} \cap \mathcal{B} = \emptyset$. A set of $v-2$ pairwise disjoint KTS(v) is called a *large set of disjoint Kirkman triple systems* of order v and briefly denoted by LKTS(v).

Example There is an LKTS(9).

Point set: $Z_7 \cup \{x, y\}$.

An initial KTS(9) with the following blocks.

$0\ x\ y$	$0\ 1\ 6$	$0\ 2\ 5$	$0\ 3\ 4$
$1\ 2\ 4$	$x\ 2\ 3$	$x\ 4\ 6$	$x\ 1\ 5$
$6\ 3\ 5$	$y\ 4\ 5$	$y\ 1\ 3$	$y\ 2\ 6$

Known results on LKTS

(1) $\exists$ LKTS(v) $\Rightarrow$ LKTS($3v$). (S. Zhang and L. Zhu, 2002)

(2) $\exists$ an LKTS($3^a 5^b m \prod_{i=1}^{r} (2 \cdot 13^{n_i} + 1) \prod_{j=1}^{p} (2 \cdot 7^{m_j} + 1)$) for $m \in M = \{1, 11, 17, 35, 43, 67, 91, 123\} \cup \{2^{2l+1}25^s + 1 : l, s \geq 0\}$, $a, n_i, m_j \geq 1$ $(1 \leq i \leq r, 1 \leq j \leq p)$, $b, r, p \geq 0$, $a+r+p \geq 2$ when $b \geq 1$ and $m \neq 1$.

2 A construction for LKTSs via 2-resolvable SQSs

A *group divisible t-design* (or t-GDD) of order v and block size k denoted by GDD(t, k, v) is a triple $(X, \mathcal{G}, \mathcal{B})$ such that

(1) X is a set of v elements (called *points*),

(2) $\mathcal{G} = \{G_1, G_2, \ldots\}$ is a set of non-empty subsets (called *groups*) of X which partition X,

(3) $\mathcal{B}$ is a family of k-subsets of X (called *blocks*) such that each block intersects any given group in at most one point,

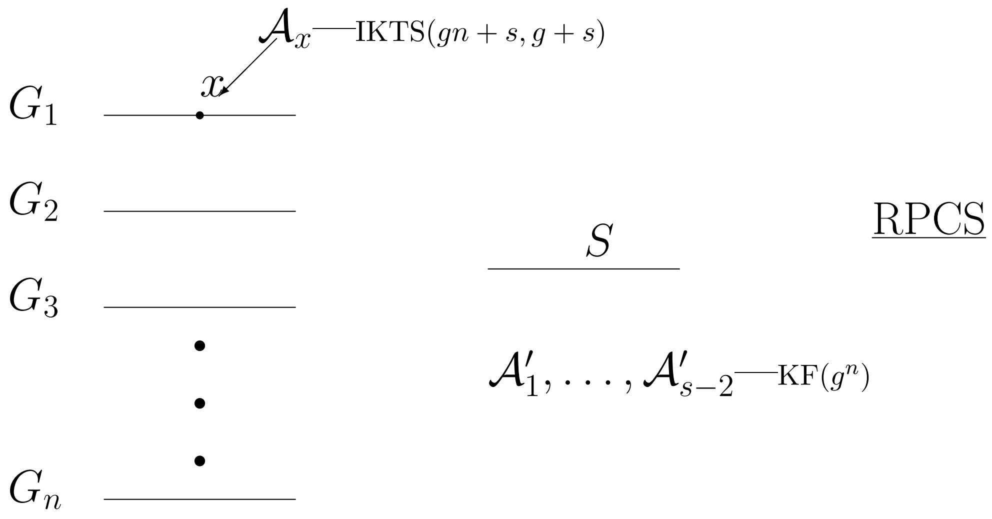
(4) each t-set of points from t distinct groups is contained in exactly one block.

The *type* of a GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. We denote the type by $1^{u_1}2^{u_2}\cdots$, where there are precisely u_i occurrences of i, $i \geq 1$.

A group divisible design, $(X, \mathcal{G}, \mathcal{B})$, is *resolvable* (briefly by RGDD) if there exists a partition $\Gamma = \{P_1, P_2, \ldots, P_r\}$ of $\mathcal{B}$ such that each part P_i is itself a partition of X. The parts P_i are called *parallel classes*, and the partition Γ is called a *resolution*.

A *generalized frame* $F(t, k, v\{m\})$ is a GDD(t, k, vm) $(X, \mathcal{G}, \mathcal{B})$ of type m^v such that the block set $\mathcal{B}$ can be partitioned into subsets $\mathcal{B}_r$, $r \in R$, each $\mathcal{B}_r$ being the block set of a GDD$(t-1, k, (v-1)m)$ of type m^{v-1} missing some group $G \in \mathcal{G}$.

An $F(2, 3, v\{m\})$ is called a *Kirkman frame*, briefly a $KF(m^v)$, and each element (i.e. a GDD$(1, 3, (v-1)m)$) of the $F(2, 3, v\{m\})$ is called a *partial parallel class* (or *holey parallel class*).



G_i—groups $\quad |G_i| = g \quad S$—Stem $\quad |S| = s$

$X = (\cup G_i) \cup S$—Point set $\quad (g^n : s)$—type

A *resolvable partitionable candelabra system* with block size three and type $(g^n : s)$ (denoted by $RPCS(g^n : s)$) is a quadruple $(X, S, \mathcal{G}, \mathcal{A})$, where

(i) X is an $s + ng$-elements set;

(ii) S is an s-subset of X;

(iii) $\Gamma = \{G_1, G_2, \ldots, G_n\}$ is a set of g-subsets of $X \backslash S$, which partition $X \backslash S$;

(iv) $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{gn+s-2}\}$ is a set of subsets of $X^{(3)} \setminus (\cup_{G\in\mathcal{G}}(G \cup S)^{(3)})$, which partition $X^{(3)} \setminus (\cup_{G\in\mathcal{G}}(G \cup S)^{(3)})$, with the properties: (1) for each group G, there are exactly g $\mathcal{A}_i$'s $(1 \leq i \leq gn)$ such that $\mathcal{A}_i$ is the block set of an IKTS$(gn + s, g + s)$ with the hole $G \cup S$; (2) for $gn+1 \leq i \leq gn+s-2$, $(X \backslash S, \mathcal{G}, \mathcal{A}_i)$ is a $KF(g^n)$.

Theorem 1 (Filling in holes) Suppose there exists an $RPCS$ $(g^n : s)$ with $s \geq 2$. If there is an LKTS$(g+s, s)$ containing a subdesign LKTS(s), then there is an LKTS$(ng+s)$ containing a subdesign LKTS$(g+s)$.

Theorem 2 There is an RPCS$(6^k : 3)$ for $k \in \{4, 7, 13\}$. (Lei 2002, Ji and Lei 2004)

Theorem 3 Suppose a 1-fan S$(3, (K_1, K_2), v)$ exists. If there exist $OLKF(g^k)$ for all $k \in K_2$ and $RPCS(g^k : s)$ for all $k \in K_1$, then there exists an $RPCS(g^v : s)$. (Lei 2002)

Let $g \geq 3$ and $(X, \mathcal{G}, \mathcal{A})$ be a GDD$(3, 3, g(n+1)-1)$ of type $g^n(g-1)^1$, where G_0 is the group of size $g-1$. Such a GDD is shortly denoted by RPGDD$(g^n(g-1)^1)$ if the block set $\mathcal{A}$ can be partitioned into $\mathcal{A}_x$ ($x \in G$, $G \in \mathcal{G}$ and $G \neq G_0$) and $\mathcal{A}_1, \ldots, \mathcal{A}_{g-3}$ with the following two properties: (i) each $\mathcal{A}_x$ is the block set of an RGDD$(2, 3, gn)$ of type g^n with the group set $(\mathcal{G} \setminus \{G_0, G\}) \cup \{G_0 \cup \{x\}\}$, (2) each $(X \setminus G_0, \mathcal{G} \setminus \{G_0\}, \mathcal{A}_i)$ is an RGDD$(2, 3, gn)$ of type g^n.

Example There is an RPGDD$(3^3 2^1)$.

Point set: $X = Z_9 \cup \{x, y\}$

Group set: $\mathcal{G} = \{G_i = \{i, i+3, i+6\} : 0 \leq i \leq 2\} \cup \{\{x, y\}\}$.

An initial RTD$(2, 3, 3)$ with group set $\{G_1, G_2, \{0, x, y\}\}$.

$$\begin{array}{ccc} 0\,1\,2 & 7\,8\,x & 4\,5\,y \\ 0\,4\,8 & 1\,5\,x & 2\,7\,y \\ 0\,5\,7 & 2\,4\,x & 1\,8\,y \end{array}$$

Construction $\exists$ 2-resolvable SQS$(2g+2) \Rightarrow \exists\ RPCS(6^g : 3)$

Proof: Let $(X \cup \{\infty_1, \infty_2\}, \mathcal{B})$ be a 2-resolvable SQS$(2g+2)$.

For $B \in \mathcal{B}$ with $B \cap \{\infty_1, \infty_2\} = \emptyset$, construct an $F(3,3,4\{3\})$ on $B \times Z_3$ with groups $\{x\} \times Z_3$, $x \in B$ such that each element TD$(2,3,3)$ $\mathcal{A}_B(x,h)$ is an RTD.

For $B \in \mathcal{B}$ with $B \cap \{\infty_1, \infty_2\} = \{\infty_1\}$, construct an RPGDD$(3^3 2^1)$ on $((B \setminus \{\infty_1\}) \times Z_3) \cup \{(y,0),(y,1)\}$ with groups $\{x\} \times Z_3$, $x \in B \setminus \{\infty_1\}$, and $\{(y,0),(y,1)\}$. It has 9 RTD$(2,3,3)$ $\mathcal{C}_B(x,h)$, $x \in B \setminus \{\infty_1\}$ and $h \in Z_3$.

For $B \in \mathcal{B}$ with $B \cap \{\infty_1, \infty_2\} = \{\infty_2\}$, construct a $CS(3,4,10)$ of type $(3^3 : 1)$ on $((B \setminus \{\infty_2\}) \times Z_3) \cup \{(y,2)\}$ with groups $\{x\} \times Z_3$, $x \in B \setminus \{\infty_2\}$, and a stem $\{(y,2)\}$. Denote its block set by $\mathcal{D}_B$. For $x \in B \setminus \{\infty_2\}$ and $h \in Z_3$, let $\mathcal{D}_B(x,h) = \{D \setminus \{(x,h)\} : D \in \mathcal{D}_B, (x,h) \in D\}$, and let $\mathcal{D}_B(y,2) = \{D \setminus \{(y,2)\} : D \in \mathcal{D}_B, (y,2) \in D\}$.

For $x \in X$ and $h \in Z_3$, let

$$\mathcal{F}(x,h) = (\cup_{x\in B, B\in\mathcal{B}, B\cap\{\infty_1,\infty_2\}=\emptyset} \mathcal{A}_B(x,h))$$
$$\cup(\cup_{x\in B, B\in\mathcal{B}, B\cap\{\infty_1,\infty_2\}=\{\infty_1\}} \mathcal{C}_B(x,h))$$
$$\cup(\cup_{x\in B, B\in\mathcal{B}, B\cap\{\infty_1,\infty_2\}=\{\infty_2\}} \mathcal{D}_B(x,h)),$$

and let

$$\mathcal{F}(y,2) = \cup_{B\in\mathcal{B}, B\cap\{\infty_1,\infty_2\}=\{\infty_2\}} \mathcal{D}_B(y,2).$$

Then all $\mathcal{F}(x,h)$, $\mathcal{F}(y,2)$ form an $RPCS(6^g : 3)$ on $X' = (X \cup \{y\}) \times Z_3$ with group set $\mathcal{G}' = \{G' = G \times Z_3 : G \in \mathcal{G}\}$ and a stem $S = \{y\} \times Z_3$, where $\mathcal{G} = \{B \setminus \{\infty_1,\infty_2\} : \{\infty_1,\infty_2\} \subset B, B \in \mathcal{B}\}$.

Main Theorem $\exists$ 2-resolvable SQS(v) $\Rightarrow$ LKTS($3v-3$).

4 Main result

(1) $\exists$ 2-resolvable SQS(4^n). (Baker 1976)

(2) $\exists$ 2-resolvable SQS($2 \cdot p^n + 2$), $p \in \{7, 31, 127\}$ and n being positive integer. (Teirlinck, 1994)

Theorem $\exists$ LKTS(v) for $v = 3 \cdot 4^n - 3$, or $v = 6 \cdot p^n + 3$ with $p \in \{7, 31, 127\}$, for any positive integer n.

Research problem

(i) Find more 2-resolvable SQS(v).

(ii) Find more LKTS(v).

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