On f-colorings of simple graphs

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1. Introduction

Let G be a graph and let f be a function which assigns a positive integer $f(v)$ to each vertex $v \in V(G)$.

An *f-coloring* of G is an edge-coloring such that each vertex v has at most $f(v)$ edges colored with the same color. The minimum number of colors needed to f-color G is called an *f-chromatic index* of G, and denoted by $\chi'_f(G)$.

If $f(v) = 1$ for all $v \in V(G)$, the f-coloring problem is reduced to the proper edge-coloring problem.

Since the proper edge-coloring problem is NP-complete (Holyer, SIAM J. Comput., 1981), the f-coloring problem is also NP-complete in general.

Hakimi and Kariv (JGT, 1986) studied the f-coloring problem and obtained some upper bounds on $\chi'_f(G)$. Nakano et al. (IEEE Trans. Circuit and Syst., 1988) obtained another upper bound on $\chi'_f(G)$.

In the proper edge-coloring, one of the most celebrated results is that

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$$

for any simple graph G (Vizing, 1964).

Let

$$\Delta_f(G) = \max_{v \in V(G)} \{\lceil \frac{d(v)}{f(v)} \rceil\}.$$

Similarly, we have:

Lemma 1. *Let G be a simple graph. Then*

$$\Delta_f(G) \leq \chi'_f(G) \leq \max_{v \in V(G)} \{\lceil \frac{d(v)+1}{f(v)} \rceil\} \leq \Delta_f(G)+1.$$

We say that G is of C_f 1 if $\chi'_f(G) = \Delta_f(G)$; and that G is of C_f 2 if $\chi'_f(G) = \Delta_f(G) + 1$.

Zhang and Liu (2004-2005) generalized and studied the classification of simple graphs on f-colorings.

2. MAIN RESULTS

Note that any subgraph G' of G has $f_{G'}(v) = f_G(v)$ for all $v \in V(G')$.

Theorem 2. (Hakimi and Kariv, JGT, 1986) *Let G be a bipartite graph. Then $\chi'_f(G) = \Delta_f(G)$.*

Theorem 3. (Hakimi and Kariv, JGT, 1986) *Let G be a graph and $f(v)$ be even for all $v \in V(G)$. Then $\chi'_f(G) = \Delta_f(G)$.*

Let

$$V_0^*(G) = \{v : \Delta_f(G) = \frac{d(v)}{f(v)}, v \in V(G)\}.$$

Theorem 4. (Zhang and Liu, JAMC, 2005) *Let G be a simple graph. If $V_0^*(G) = \emptyset$, then G is of C_f 1.*

The f-$core$ of a graph G is the subgraph of G induced by the vertices of $V_0^*(G)$ and is denoted by G_{Δ_f}.

Theorem 5. (Zhang and Liu, AML, 2006) *Let G be a simple graph. Then G is of C_f 1 if G_{Δ_f} is a forest.*

The number $\frac{d(v)}{f(v)}$ is called the f-$ratio$ of vertex v of G, and denoted by $d_f(v)$.

Clearly, $d_f(v) = \Delta_f(G)$ if and only if $v \in V_0^*(G)$.

Let us call a graph G $\Delta_f(G)$-$peelable$, if all the vertices of G can be iteratively peeled away using the following operation: Peel off any vertex that has at most one remaining neighbor of f-ratio $\Delta_f(G)$.

Theorem 6. (Zhang and Liu) *Let G be a simple graph. If G is $\Delta_f(G)$-peelable, then G is of C_f 1.*

If the f-core of G is a forest, then G is $\Delta_f(G)$-peelable, since we can remove all the vertices of $V_0^*(G)$ by iteratively removal of the remaining vertices with degree one in G_{Δ_f}.

In fact, Theorem 6 gives a more general class of graphs than Theorem 5.

We have $\Delta_f(G) = 3$ and
$V_0^*(G) = \{v_1, v_3, v_4, v_5, v_7, v_8, v_9\}$.
G is $\Delta_f(G)$-peelable in the order $v_1, v_2, \ldots, v_{10}$, though the f-core of G is not a forest.

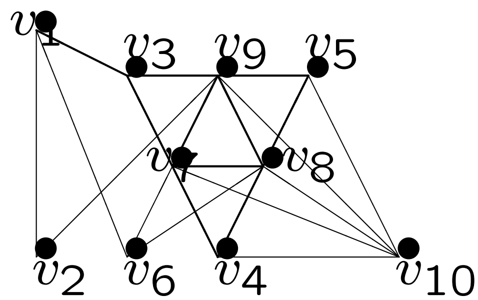

Fig. 1. A graph G with $f(v_i) = 1$ $(1 \le i \le 5)$ and $f(v_j) = 2$ $(6 \le j \le 10)$

Let

$$f^* = \min_{v \in V(G)} \{f(v)\}.$$

Theorem 7. (Zhang and Liu, JAMC, 2005) *Let G be a complete graph K_n. If k and n are odd integers, $f(v) = k$ and $k \mid (n-1)$ for all $v \in V$, then G is of C_f 2. Otherwise, G is of C_f 1.*

Theorem 8. (Zhang, Wang and Liu) *Let* G *be a regular graph of degree* $d(G) = \Delta$*. Then* G *is of* C_f 1 *if* $f^* \nmid \Delta$ *or* f^* *is even.*

Theorem 9. (Zhang, Wang and Liu) *Let* $n \geq 1$*. Let* G *be a regular graph of order* $2n+1$ *and degree* $d(G) = \Delta$*. Then* G *is of* C_f 2 *if* $f(v) = f^*$ *is odd for all* $v \in V$ *and* $f^* \mid \Delta$*.*

Theorem 10. (Zhang, Wang and Liu) *Let G be a regular graph with order n and degree $d(G) = \Delta$, and let $G \neq K_n$. Let $w \in V$ be the only vertex such that $f(w) > f^*$. Then G is of C_f 1 if and only if $G \setminus w$ is of C_f 1.*

Theorem 11. (Zhang, Wang and Liu) *Let $n \geq 1$. Let G be a regular graph with order $2n$ and degree $d(G) = \Delta$, where $G \neq K_{2n}$. Let $f(v) = f^*$ for all $v \in V$. G is of C_f 1 if and only if $G \setminus w$ is of C_f 1, where $w \in V$.*

Problem 1. *Find the necessary or sufficient conditions for a simple graph to be of* C_f *1 or* C_f *2.*

Problem 2. *Find the classes of graphs which are of* C_f *1 for arbitrary positive integer function* f.

Problem 3. *Discuss the properties of critical graphs on f-colorings.*

Problem 4. *Find the sufficient conditions for a regular graph of degree $d(G) = \Delta$ to be of* C_f 1 *or* C_f 2 *when $f^* \mid \Delta$ and f^* is odd.*

3. Application to equitable edge-colorings of simple graphs

Given an edge-coloring of G with k colors in C, for $v \in V(G)$, let $c_i(v)$ denote the set of edges incident with v colored with c_i ($1 \le i \le k$). Call an edge-coloring of G with k colors in C *equitable* if

$$||c_i(v)| - |c_j(v)|| \le 1 \ (\forall\ 1 \le i < j \le k)$$

for every $v \in V(G)$.

Define $V_k(G) = \{v \in V(G) : k \mid d(v)\}$. Call the subgraph of G induced by $V_k(G)$ the k-*core* of G.

Conjecture 1. (Hilton, 2005) *Let G be a simple graph and let $k \geq 2$. If the k-core of G is a forest, then G has an equitable edge-coloring with k colors.*

By virtue of some methods in f-colorings of graphs, we obtain a new sufficient condition for equitable edge-colorings of simple graphs, which not only verifies Conjecture 1, but also substantially extends it to a more general class of graphs.

For a graph G, let C denote the set of colors available to color the edges of G.

Define $m(v, \alpha) = f(v) - d(v, \alpha)$
and $M(v) = \{\alpha : m(v, \alpha) \geq 1, \alpha \in C\}$.

Theorem 12. (Zhang and Liu, 2006) *Let G be a simple graph and let $k \geq 2$. Let $f(v) = \lceil \frac{d(v)}{k} \rceil$ for each $v \in V(G)$. Let $C = \{c_1, c_2, \ldots, c_k\}$. If the edges of G can be f-colored with k colors of C in the order $e_1, e_2, \ldots, e_{\varepsilon(G)}$ in such a way that, for every j ($1 \leq j \leq \varepsilon(G)$), when f-coloring the jth edge $e_j = w_j v_j$, there are $M(v) \neq \emptyset$ for all $v \in N_G(w_j)$ or for all $v \in N_G(v_j)$, then G has an equitable edge-coloring with k colors.*

It is rather difficult to decide whether or not a graph has the properties described in Theorem 12. Fortunately, we find a much easier sufficient condition for a graph to have the properties described in Theorem 12.

Theorem 13. (Zhang and Liu, 2006) *Let G be a simple graph and let $k \geq 2$. Let $f(v) = \lceil \frac{d(v)}{k} \rceil$ for each $v \in V(G)$. If G is $\Delta_f(G)$-peelable, then G has an equitable edge-coloring with k colors.*

Let $f(v) = \lceil \frac{d(v)}{k} \rceil$ for each $v \in V(G)$.

It is easy to see that when $V_k(G) \neq \emptyset$, the k-core of G is exactly the f-core of G. Since a simple graph G whose k-core is a forest is $\Delta_f(G)$-peelable, Conjecture 1 is true.

Theorem 14. (Zhang and Liu, 2006) *Let G be a simple graph and let $k \geq 2$. If the k-core of G is a forest, then G has an equitable edge-coloring with k colors.*

As mentioned in Section 2, Theorem 13 is strictly stronger than Theorem 14.

Problem 5. *Let G be a graph and let $k \geq 2$. Let $f(v) = \lceil \frac{d(v)}{k} \rceil$ for each $v \in V(G)$. Suppose that $V_k(G) \neq \emptyset$. G has an equitable edge-coloring with k colors if and only if G is of C_f 1?*

Thank you !

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