On $f$-colorings of simple graphs

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1. Introduction

Let $G$ be a graph and let $f$ be a function which assigns a positive integer $f(v)$ to each vertex $v \in V(G)$.

An $f$-coloring of $G$ is an edge-coloring such that each vertex $v$ has at most $f(v)$ edges colored with the same color. The minimum number of colors needed to $f$-color $G$ is called an $f$-chromatic index of $G$, and denoted by $\chi'_f(G)$. 
If $f(v) = 1$ for all $v \in V(G)$, the $f$-coloring problem is reduced to the proper edge-coloring problem.

Since the proper edge-coloring problem is NP-complete (Holyer, SIAM J. Comput., 1981), the $f$-coloring problem is also NP-complete in general.
Hakimi and Kariv (JGT, 1986) studied the $f$-coloring problem and obtained some upper bounds on \( \chi_f'(G) \). Nakano et al. (IEEE Trans. Circuit and Syst., 1988) obtained another upper bound on \( \chi_f'(G) \).
In the proper edge-coloring, one of the most celebrated results is that

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$$

for any simple graph $G$ (Vizing, 1964).
Let
\[ \Delta_f(G) = \max_{v \in V(G)} \left\lceil \frac{d(v)}{f(v)} \right\rceil. \]

Similarly, we have:

**Lemma 1.** Let \( G \) be a simple graph. Then
\[ \Delta_f(G) \leq \chi'_f(G) \leq \max_{v \in V(G)} \left\lceil \frac{d(v) + 1}{f(v)} \right\rceil \leq \Delta_f(G) + 1. \]
We say that $G$ is of $C_f 1$ if $\chi'_f(G) = \Delta_f(G)$; and that $G$ is of $C_f 2$ if $\chi'_f(G) = \Delta_f(G) + 1$.

2. MAIN RESULTS

Note that any subgraph $G'$ of $G$ has $f_{G'}(v) = f_G(v)$ for all $v \in V(G')$.

**Theorem 2.** (Hakimi and Kariv, JGT, 1986)
Let $G$ be a bipartite graph. Then $\chi_f'(G) = \Delta_f(G)$.

**Theorem 3.** (Hakimi and Kariv, JGT, 1986)
Let $G$ be a graph and $f(v)$ be even for all $v \in V(G)$. Then $\chi_f'(G) = \Delta_f(G)$.
Let

\[ V_0^*(G) = \{ v : \Delta_f(G) = \frac{d(v)}{f(v)}, v \in V(G) \}. \]

**Theorem 4.** (Zhang and Liu, JAMC, 2005)

Let \( G \) be a simple graph. If \( V_0^*(G) = \emptyset \), then \( G \) is of \( C_f \) 1.
The *f-core* of a graph *G* is the subgraph of *G* induced by the vertices of \( V_0^*(G) \) and is denoted by \( G_{\Delta f} \).

**Theorem 5.** (Zhang and Liu, AML, 2006) Let *G* be a simple graph. Then *G* is of \( C_f 1 \) if \( G_{\Delta f} \) is a forest.
The number $\frac{d(v)}{f(v)}$ is called the $f$-ratio of vertex $v$ of $G$, and denoted by $d_f(v)$.

Clearly, $d_f(v) = \Delta_f(G)$ if and only if $v \in V_0^*(G)$. 
Let us call a graph $G \Delta_f(G)$-peelable, if all the vertices of $G$ can be iteratively peeled away using the following operation: Peel off any vertex that has at most one remaining neighbor of $f$-ratio $\Delta_f(G)$.

**Theorem 6.** (Zhang and Liu) Let $G$ be a simple graph. If $G$ is $\Delta_f(G)$-peelable, then $G$ is of $C_f$ 1.
If the $f$-core of $G$ is a forest, then $G$ is $\Delta_f(G)$-peelable, since we can remove all the vertices of $V_0^*(G)$ by iteratively removal of the remaining vertices with degree one in $G_{\Delta_f}$.

In fact, Theorem 6 gives a more general class of graphs than Theorem 5.
We have $\Delta_f(G) = 3$ and $V_0^*(G) = \{v_1, v_3, v_4, v_5, v_7, v_8, v_9\}$. 
$G$ is $\Delta_f(G)$-peelable in the order $v_1, v_2, \ldots, v_{10}$, though the $f$-core of $G$ is not a forest.

Fig. 1. A graph $G$ with $f(v_i) = 1$ ($1 \leq i \leq 5$) and $f(v_j) = 2$ ($6 \leq j \leq 10$)
Let

\[ f^* = \min_{v \in V(G)} \{f(v)\}. \]

**Theorem 7.** (Zhang and Liu, JAMC, 2005) Let \( G \) be a complete graph \( K_n \). If \( k \) and \( n \) are odd integers, \( f(v) = k \) and \( k \mid (n - 1) \) for all \( v \in V \), then \( G \) is of \( C_f 2 \). Otherwise, \( G \) is of \( C_f 1 \).
Theorem 8. (Zhang, Wang and Liu) Let $G$ be a regular graph of degree $d(G) = \Delta$. Then $G$ is of $C_f 1$ if $f^* \nmid \Delta$ or $f^*$ is even.

Theorem 9. (Zhang, Wang and Liu) Let $n \geq 1$. Let $G$ be a regular graph of order $2n + 1$ and degree $d(G) = \Delta$. Then $G$ is of $C_f 2$ if $f(v) = f^*$ is odd for all $v \in V$ and $f^* \mid \Delta$. 
Theorem 10. (Zhang, Wang and Liu) Let $G$ be a regular graph with order $n$ and degree $d(G) = \Delta$, and let $G \neq K_n$. Let $w \in V$ be the only vertex such that $f(w) > f^*$. Then $G$ is of $C_f 1$ if and only if $G \setminus w$ is of $C_f 1$.

Theorem 11. (Zhang, Wang and Liu) Let $n \geq 1$. Let $G$ be a regular graph with order $2n$ and degree $d(G) = \Delta$, where $G \neq K_{2n}$. Let $f(v) = f^*$ for all $v \in V$. $G$ is of $C_f 1$ if and only if $G \setminus w$ is of $C_f 1$, where $w \in V$. 
Problem 1. *Find the necessary or sufficient conditions for a simple graph to be of $C_f$ 1 or $C_f$ 2.*

Problem 2. *Find the classes of graphs which are of $C_f$ 1 for arbitrary positive integer function $f$.***
Problem 3. Discuss the properties of critical graphs on $f$-colorings.

Problem 4. Find the sufficient conditions for a regular graph of degree $d(G) = \Delta$ to be of $C_f$ $1$ or $C_f$ $2$ when $f^* \mid \Delta$ and $f^*$ is odd.
3. Application to equitable edge-colorings of simple graphs

Given an edge-coloring of $G$ with $k$ colors in $C$, for $v \in V(G)$, let $c_i(v)$ denote the set of edges incident with $v$ colored with $c_i$ ($1 \leq i \leq k$). Call an edge-coloring of $G$ with $k$ colors in $C$ equitable if

$$||c_i(v)| - |c_j(v)|| \leq 1 \ (\forall \ 1 \leq i < j \leq k)$$

for every $v \in V(G)$. 
Define $V_k(G) = \{v \in V(G) : k \mid d(v)\}$. Call the subgraph of $G$ induced by $V_k(G)$ the $k$-core of $G$.

**Conjecture 1.** (Hilton, 2005) Let $G$ be a simple graph and let $k \geq 2$. If the $k$-core of $G$ is a forest, then $G$ has an equitable edge-coloring with $k$ colors.
By virtue of some methods in $f$-colorings of graphs, we obtain a new sufficient condition for equitable edge-colorings of simple graphs, which not only verifies Conjecture 1, but also substantially extends it to a more general class of graphs.
For a graph $G$, let $C$ denote the set of colors available to color the edges of $G$.

Define $m(v, \alpha) = f(v) - d(v, \alpha)$ and $M(v) = \{\alpha : m(v, \alpha) \geq 1, \alpha \in C\}$. 
**Theorem 12.** (Zhang and Liu, 2006) Let $G$ be a simple graph and let $k \geq 2$. Let $f(v) = \left\lceil \frac{d(v)}{k} \right\rceil$ for each $v \in V(G)$. Let $C = \{c_1, c_2, \ldots, c_k\}$. If the edges of $G$ can be $f$-colored with $k$ colors of $C$ in the order $e_1, e_2, \ldots, e_{\varepsilon(G)}$ in such a way that, for every $j$ ($1 \leq j \leq \varepsilon(G)$), when $f$-coloring the $j$th edge $e_j = w_jv_j$, there are $M(v) \neq \emptyset$ for all $v \in N_G(w_j)$ or for all $v \in N_G(v_j)$, then $G$ has an equitable edge-coloring with $k$ colors.
It is rather difficult to decide whether or not a graph has the properties described in Theorem 12. Fortunately, we find a much easier sufficient condition for a graph to have the properties described in Theorem 12.

**Theorem 13.** (Zhang and Liu, 2006) Let $G$ be a simple graph and let $k \geq 2$. Let $f(v) = \lceil \frac{d(v)}{k} \rceil$ for each $v \in V(G)$. If $G$ is $\Delta_f(G)$-peelable, then $G$ has an equitable edge-coloring with $k$ colors.
Let \( f(v) = \left\lceil \frac{d(v)}{k} \right\rceil \) for each \( v \in V(G) \).

It is easy to see that when \( V_k(G) \neq \emptyset \), the \( k \)-core of \( G \) is exactly the \( f \)-core of \( G \). Since a simple graph \( G \) whose \( k \)-core is a forest is \( \Delta_f(G) \)-peelable, Conjecture 1 is true.
Theorem 14. (Zhang and Liu, 2006) Let $G$ be a simple graph and let $k \geq 2$. If the $k$-core of $G$ is a forest, then $G$ has an equitable edge-coloring with $k$ colors.

As mentioned in Section 2, Theorem 13 is strictly stronger than Theorem 14.
Problem 5. Let $G$ be a graph and let $k \geq 2$. Let $f(v) = \lceil \frac{d(v)}{k} \rceil$ for each $v \in V(G)$. Suppose that $V_{k}(G) \neq \emptyset$. $G$ has an equitable edge-coloring with $k$ colors if and only if $G$ is of $C_f$ 1?
Thank you!
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