Colorings of generalized Mycielskians of graphs

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1. Introduction
   - Graph Constructions
   - Definitions of colorings
2. $\chi_c(\mu_m(G))$ for $G = K_n$ and C_n
3. $\chi_f(\mu_m(G))$ and $\chi_k(\mu_m(G))$

Mycielskians of graphs

- (Mycielski, 1955) Let G be a graph with vertex set V and edge set E. The Mycielskian of G, $\mu(G)$, has vertex set $V \cup V' \cup \{u\}$ (where $V' = \{x' | x \in V\}$) and edge set $E \cup \{xy' | xy \in E\} \cup \{y'u | y' \in V'\}$.

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- $\mu^n(K_2)$ has chromatic number $n + 2$ and clique number 2.

Generalized Mycielskians of graphs (also called cones over graphs)

- Let G be a graph with vertex set $V^0 = \{v_1^0, v_2^0, \ldots, v_n^0\}$ and edge set E^0. Given an integer $p \geq 1$ the *p-Mycielskian* of G, denoted by $\mu_p(G)$, has vertex set

$$V^0 \cup V^1 \cup V^2 \cup \cdots \cup V^p \cup \{u\},$$

where $V^i = \{v_j^i : v_j^0 \in V^0\}$ is the i-th distinct copy of V^0 for $i = 1, 2, \ldots p$, and edge set

$$E^0 \cup (\cup_{i=0}^{p-1}\{v_j^i v_{j'}^{i+1} : v_j^0 v_{j'}^0 \in E^0\}) \cup \{v_j^p u : v_j^p \in V^p\}.$$

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- $\mu_1(G)$ is exactly the Mycielskian of G.

$K_{p/q}$ and $KG(n, k)$

- Let $K_{p/q}$ be the graph with vertex set $\{0, 1, \ldots, p-1\}$ in which ij is an edge if and only if $q \leq |i - j| \leq p - q$.

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- Given two positive integers $n \geq 2k$, the Kneser graph $KG(n,k)$ has vertices all the k-subsets of $\{1, 2, \ldots, n\}$ and two vertices u and v are adjacent if u and v do not intersect.

Circular chromatic number

- Given two integers k, d such that $k \geq 2d \geq 1$, a (k, d)-*coloring* of a graph G is a coloring c of the vertices of G with colors $0, 1, 2, \ldots, k-1$ such that for any two adjacent vertices x and y of G, we have $d \leq |c(x) - c(y)| \leq k - d$.

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- $\chi_c(G) = \inf\{p/q : G \text{ is homomorphic to } K_{p/q}\}$.

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- Clearly $\chi_1(G)$ is the ordinary chromatic number of G.

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- Let $m(G)$ denote the least integer k such that $\chi_f(G) = \chi_k(G)/k$.
- $m(G)$ may not be the denominator of the minimal fractional representation of $\chi_f(G)$, though it is always a multiple of this denominator.

$\chi_c(\mu_m(K_n))$

Theorem 1 (Lam, Lin, Song, Gu, 2003, JCTB)

For any odd integer $n \geq 3$ and any integer $m \geq 0$, $\chi_c(\mu_m(K_n)) = \chi(\mu_m(K_n)) = n + 1$.

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- What is the limit of $\chi_c(\mu_m(G))$ as m approaches infinity for any G?

Corollaries

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Corollary

For any integer k and $\varepsilon > 0$, there exist arbitrarily large $(2k+1)$-edge-critical graphs G with connectivity $2k$ and maximum degree at most $4k-2$ such that $\chi_c(G) < 2k + \varepsilon$.

$\chi_c(\mu_m(C_n))$

(Stiebitz, 1985; Tardif, 2001, JGT)

For any integer $m \geq 0$ and any $k \geq 1$, $\chi(\mu_m(C_{2k+1})) = 4$.

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- For any integer $m \geq 0$ and any $k \geq 1$, $\chi_c(\mu_m(C_{2k+1})) = 4$.
- If G is a nonempty bipartite graph then for any integer $m \geq 0$, $\chi_c(\mu_m(G)) = \chi_c(C_{2m+3}) = 2 + 1/(m+1)$.

$\chi_f(\mu_m(G))$

Theorem 4 (Larsen, Propp, and Ullman, 1995, JGT)

For any graph G,

$$\chi_f(\mu(G)) = \chi_f(G) + \frac{1}{\chi_f(G)}.$$

$\chi_f(\mu_m(G))$

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Theorem 5 (Tardif, 2001, JGT)

For any graph G and any nonnegative integer p,

$$\chi_f(\mu_p(G)) = \chi_f(G) + \frac{1}{\sum_{k=0}^{p}(\chi_f(G) - 1)^k}.$$

$\chi_k(\mu(G))$

Theorem 6 (Lin, 2005, submitted)

For any graph G and any integer $k \geq 1$,

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$\chi_2(\mu(KG(n, 2))) = n + 2$ for $n \geq 5$.

Theorem 8 (Zhu etc., 2006)

$\chi_k(\mu(KG(n, k))) = n + k$ for n properly large.

$\chi_k(\mu_p(K_n))$

Theorem 9 (Lin, 2005, submitted)

Let $p \geq 0$, $n \geq 2$ and $k \geq 1$ be integers. Then

$$\chi_k(\mu_p(K_n)) = \begin{cases} nk + \lceil \frac{k}{n+1} \rceil, & \text{if } n = 2; \\ nk + \lceil \frac{(n-2)k}{(n-1)^{p+1}-1} \rceil, & \text{if } n \geq 3. \end{cases}$$

$\chi_k(\mu_p(G))$

Theorem 10 (Lin, 2005, submitted)

Suppose G is a graph and p a nonnegative integer. If $\chi_f(G) = a/b$ and $\chi_b(G) = a$, then $\chi_{bt}(\mu_p(G)) = at + b^{p+1}$, where $t = \sum_{i=0}^{p}(a-b)^i b^{p-i}$.

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Corollary

For any two positive integers a and b with $a > 2b$ and $\gcd(a,b) = 1$. Let $\lambda = (a/b - 1)^{1/a}$ and $C = 1/(a-2b)$. Then $\mu_p(K_{a/b})$ has $n = (p+1)a + 1$ vertices and $m(\mu_p(K_{a/b})) = C\lambda^{n-1} - 1/C$.

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$\lambda = 1.2599, 1.3161, 1.3195, 1.3077, 1.2917, 1.2754, 1.2599$ for $b = 1$ and $a = 3, 4, 5, 6, 7, 8, 9$. When $b = 1$ and $a = 5$, the growth rate 1.3195 is the largest, which is a little smaller than the growth rate 1.346193 obtained by Fisher (1995, JGT).

$\chi_k(\mu_m(G))$

Theorem 11 (Pan and Zhu, 2006, manuscript)

For any finite graph G, for any positive integers m, k, s, if $m \geq 2\lceil k/s \rceil$, then $\chi_k(\mu_m(G)) \leq \chi_k(G) + s$.

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Corollary

For any graph G and any integer $k \geq 1$, if $m \geq 2k$, then $\chi_k(\mu_m(G)) \leq \chi_k(G) + 1$.

Graphs with $\chi_k(\mu_m(G)) = \chi_k(G)$

Theorem 12 (Pan and Zhu, 2006, manuscript)

If G is a graph with $\chi_k(G) > k\chi_c(G)$ and $\chi_k(G)$ is even, then for sufficiently large m, $\chi_k(\mu_m(G)) = \chi_k(G)$.

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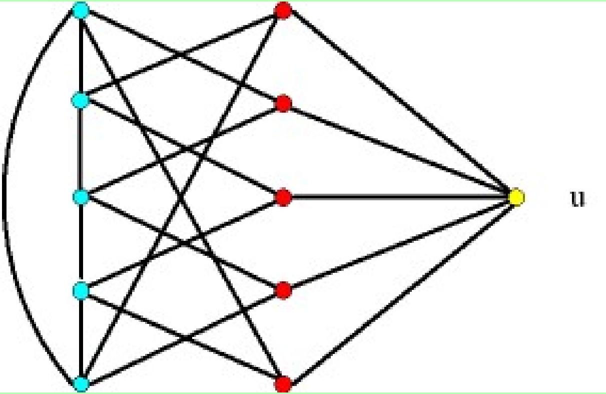
Theorem 13 (Pan and Zhu, 2006, manuscript)

Suppose $\chi_c(G) = p/q$ and p is odd. Then for any $k \geq q$, if $\chi_k(G) > k\chi_c(G)$, then for sufficiently large m, $\chi_k(\mu_m(G)) = \chi_k(G)$.

Thanks!

The Mycielskian of 5-cycle



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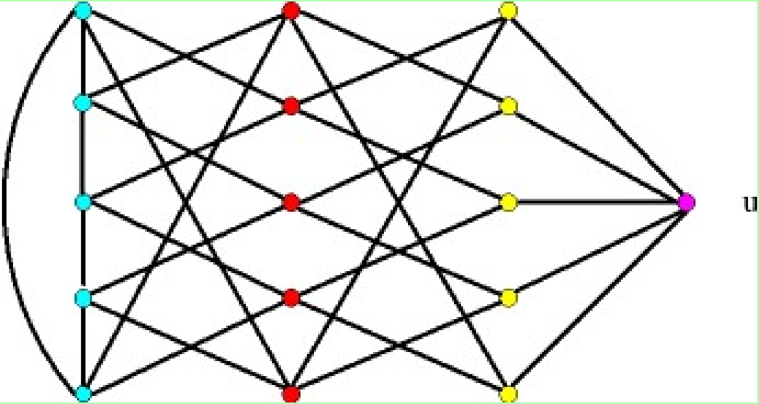
The 2-Mycielskian of 5-cycle



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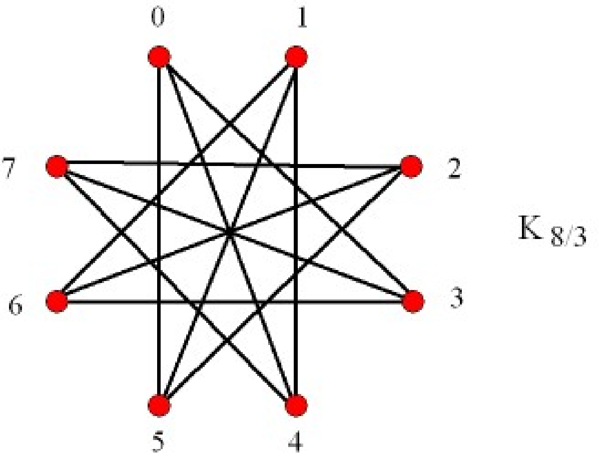
$K_{8/3}$



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