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Pseudo greedy algorithm and upper bound for hamiltonian chromatic number of paths

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访 问 主 页
标 题 页
◀◀ ▶▶
◀ ▶
第 *1* 页 共 *20* 页
返 回
全 屏 显 示
关 闭
退 出

Pseudo greedy algorithm and upper bound for hamiltonian chromatic number of paths

1. Basic concepts

2. Main results

3. The pseudo greedy algorithm

4. The rationality of the pseudo greedy algorithm and the Sketch of the proof

5. Some examples

6. References

1 Basic concepts

The concepts of Radio k-colorings:

Radio k-coloring can be regarded as a generalization of ordinary coloring of graphs, which were motivated by (FM Radio) Channel Assignments Problem (see [9, 10]) and introduced by Chartrand, Erwin, Harary and Zhang [1]. Usually, for avoiding interference, FM stations located within a certain proximity of one another must be assigned distinct channels, and the nearer two stations are to each other, the greater the differences in their assigned channels must be. For a set of given stations, the task is that how to allocate channels (positive integers) to stations, so that interference is prohibited, and the span of channels allocated is minimized. Then the concept of radio k-coloring presents a model for the above problem.

访 问 主 页
标 题 页
◀◀ ▶▶
◀ ▶
第 3 页 共 20 页
返 回
全 屏 显 示
关 闭
退 出

For a connected graph G of order n and diameter d and a integer k with $1 \leq k \leq d$, a radio k-coloring of G is a function

$$c : V(G) \rightarrow \mathbf{N},$$

such that

$$d(u,v) + |c(u) - c(v)| \geq k+1$$

for every pair u and v of distinct vertices of G, where $d(u,v)$ denotes the distance between u and v (the length of a shortest $u-v$ path) in G.

Clearly, radio 1-colorings and ordinary colorings are synonymous.

The *value* $\mathrm{rc}_k(c)$ of a radio k-coloring c of G is the maximum color assigned to a vertex of G; while the *radio k-chromatic number* $\mathrm{rc}_k(G)$ of G is $\min\{\mathrm{rc}_k(c)\}$ taken over all k-colorings of G.

访问主页
标题页
◀◀ ▶▶
◀ ▶

返回
全屏显示
关闭
退出

In particular, radio d-coloring is referred to as *radio labeling*, and the *radio d-chromatic number* is called the *radio number*.

It is easy to see that in a radio d-coloring of a connected graph, the colors assigned to adjacent vertices must differ by at least d, the colors assigned to two vertices whose distance is 2 must differ by at least $d-1$, and so on, up to the vertices whose distance is d, that is, *antipodal vertices*, whose colors are only required to be different.

Radio $(d-1)$-coloring is referred to as *radio antipodal coloring* or, more simply, as an *antipodal coloring*, and the *radio $(d-1)$-chromatic number* is called the *antipodal chromatic number*, denoted by $\mathrm{ac}(G)$.

Note that in a antipodal coloring of G, it is possible for two vertices u and v to be colored the same, but only if they are antipodal, that is, $d(u, v) =$diam $(G) = d$.

Basic concepts
Main results
The pseudo greedy . . .
The rationality of the . . .
Some examples
Reference

访问主页
标题页
◀◀ ▶▶
◀ ▶

返回
全屏显示
关闭
退出

Radio k-coloring and radio labeling of graphs were studied in [1, 2], radio antipodal coloring of paths were studied in [3, 4].

The concept of hamiltonian coloring:

The concept of hamiltonian coloring was inspired by antipodal coloring of path, and was introduced by Chartrand, Nebeský and Zhang [5].

For vertices u and v in a connected graph G, let $D(u, v)$ denote the length of a longest $u - v$ path in G. Thus for every connected graph G of order n and diameter d, both $d(u, v)$ and $D(u, v)$ are metric on $V(G)$. Note that if G is a path, then

$$d(u, v) + |c(u) - c(v)| \geq d \tag{1}$$

is equivalent to

$$D(u, v) + |c(u) - c(v)| \geq n - 1, \tag{2}$$

which suggests an extension of the coloring c that satisfies (2) for an arbitrary connected graph G.

访问主页
标题页
◀◀ ▶▶
◀ ▶

返回
全屏显示
关闭
退出

A hamiltonian coloring of a connected graph G is an assignment c of colors (positive integers) to the vertices of G such that

$$D(u,v) + |c(u) - c(v)| \geq n - 1$$

for every two distinct vertices u and v of G, where $D(u,v)$ is the length of a longest $u-v$ path in G and n is the order of G. The *value* $\mathrm{hc}(c)$ of a hamiltonian coloring c is the maximum color assigned to a vertex of G. The *hamiltonian chromatic number* $\mathrm{hc}(G)$ of G is $\min\{\mathrm{hc}(c)\}$ taken over all hamiltonian colorings c of G.

Since $D(u,v) = d(u,v)$ and $d =$diam $(G) = n - 1$ for paths P_n, therefore it holds that hc(P_n) =ac(P_n).

Basic concepts
Main results
The pseudo greedy . . .
The rationality of the . . .
Some examples
Reference

访 问 主 页
标 题 页
◀◀ ▶▶
◀ ▶
第 7 页 共 20 页
返 回
全 屏 显 示
关 闭
退 出

2 Main results

Hamiltonian coloring for some classes of graphs were studied in [5, 6, 7, 8].

For hc(P_n) =ac(P_n), in [3], Chartrand, Erwin and Zhang showed that

Theorem 1([3]). For every positive integer $n \geq 1$, hc(P_n) =ac(P_n) $\leq \binom{n-1}{2}+1$.

In [3], Chartrand, Erwin and Zhang have shown that the equality holds in Theorem 1 for $1 \leq n \leq 6$. At the same time they posed a conjecture as follows.

Conjecture 1 ([3]). For every positive integer n, $ac(P_n) = \binom{n-1}{2} + 1$.

In [4], Chartrand, Nebeský and Zhang disprove Conjecture 1 for all odd integers $n \geq 9$, they showed that

Theorem 2([4]). For all odd integers $n \geq 7$, $ac(P_n) \leq \binom{n-1}{2} - \frac{n-1}{2} + 4$.

As $\mathrm{hc}(P_n) = \mathrm{ac}(P_n)$, from Theorem 1 and Theorem 2, Chartrand, Nebeský and Zhang presented a upper bound for the hamiltonian chromatic number of paths P_n as follows.

Theorem 3([5]). For every positive integer n, $hc(P_n) = ac(P_n) \leq \binom{n-1}{2} + 1$. Furthermore, for all odd integers $n \geq 7$, $hc(P_n) = ac(P_n) \leq \binom{n-1}{2} - \frac{n-1}{2} + 4$.

In this paper we will introduce a pseudo greedy algorithm for hamiltonian coloring for P_n, and by this algorithm, we will show that

Theorem 4. For all even integers $n \geq 10$, $hc(P_n) = ac(P_n) \leq \binom{n-1}{2} - \frac{n}{2} - \lfloor \frac{10}{n} \rfloor + 6$.

Clearly, for all even integers $n \geq 10$, it holds that $-\frac{n}{2} - \lfloor \frac{10}{n} \rfloor + 6 \leq 0$, so Theorem 4 provides an improved upper bound for hamiltonian (antipodal) of P_n, and disproves Conjecture 1 for all even integers $n \geq 10$.

访问主页
标题页
◀◀ ▶▶
◀ ▶

返回
全屏显示
关闭
退出

3 The pseudo greedy algorithm

For P_n, let $n = 2k$ is even and $n \geq 10$. Write

$$V(P_n) = \{v_1, v_2, \ldots, v_n\}(\text{see Figure1}).$$

For $i = 1, 2, \ldots, n$, let $p(i)$ denote the subscript of the vertex $v_{p(i)}$ that we shall color it at the ith time, and $h(i)$ denote the color that we shall assign to the vertex $v_{p(i)}$ at the ith time. Let $c(v_{p(i)}) = h(i), i = 1, 2, \ldots, n$. At first, we use the color $h(1) = 1$ to color a vertex $v_{p(1)}$, where $v_{p(1)}$ is one of *central vertices* of P_n, say $v_{p(1)} = v_{\frac{n}{2}}$.

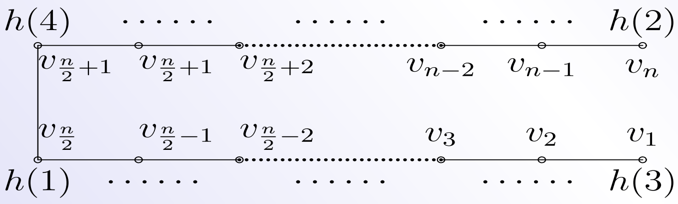

Figure 1: The pseudo greedy algorithm for hamiltonian coloring for $P_n(n = 2k \geq 10)$

Suppose that for $1 \leq i \leq n-1$ the vertices in $\{v_{p(1)}, v_{p(2)}, \ldots, v_{p(i)}\} \subset V(P_n)$ have been colored with $c(v_{p(l)}) = h(l)$ for all $1 \leq l \leq i$, then we choose a color $h(i+1) \in \mathbf{N}$ as small as possible to color one vertex $v_{p(i+1)} \in V(P_n) \backslash \{v_{p(1)}, v_{p(2)}, \ldots, v_{p(i)}\}$, such that

$$d(v_{p(i+1)}, v_{p(l)}) + |c(v_{p(i+1)}) - c(v_{p(l)})| = |p(i+1) - p(l)| + (h(i+1) - h(l)) \geq n-1$$

for all $1 \leq l \leq i$. And if there are two value for $p(i+1)$ can be chosen, then we take $p(i+1)$ satisfying the condition that $v_{p(i+1)}$ close to central vertices of P_n as near as possible. Finally, we obtain that $\mathrm{hc}(c) = h(n)$ and hence $\mathrm{hc}(P_n) \leq \mathrm{hc}(c)$.

Based on the above idea, we provide the pseudo greedy algorithm for hamiltonian coloring for P_n as follows.

```
Initialization: Put $i := 0$; $A := \Phi$;
Step 1: Let $p(1) = \frac{n}{2}$; $h(1) = 1$.
Step 2: Let $i := i + 1$; $A := A \cup \{p(i)\}$; $t = h(i) + 1$; if $i > n$, then go to Step 9.
Step 3: If $p(i) \leq \frac{n}{2}$, then go to Step 4 else go to Step 8.
Step 4: For $j = p(i) + 1$ to $n$ (with step length $+1$) and $j \notin A$ do Step 5.
Step 5: If $\{|j - p(i)| + (t - h(i)) \geq n - 1$ and $|j - p(i-1)| + (t - h(i-1)) \geq n - 1\}$, then go to Step 6 else go to Step 7.
Step 6: Let $p(i+1) := j$; $h(i+1) := t$; go to Step 2.
Step 7: Let $t := t + 1$; go to Step 3.
Step 8: For $j = p(i) - 1$ to 1 (with step length $-1$) and $j \notin A$ do Step 5.
Step 9. For $i = 1$ to $n$ output $p(i)$ and $h(i)$.
Step 10: Stop.
```

4 The rationality of the pseudo greedy algorithm and the Sketch of the proof

To show the rationality of the pseudo greedy algorithm for hamiltonian coloring for P_n, we only need to prove that $c : c(v_{p(i)}) = h(i)$ $(i = 1, 2, \ldots, n)$ is a hamiltonian coloring for P_n. Since that for any $1 \leq i < j \leq n$, $d(v_{p(j)}, v_{p(i)}) + |c(v_{p(j)}) - c(v_{p(i)})| = |p(j) - p(i)| + (h(j) - h(i))$, it suffices to prove the following proposition.

Proposition 1. For any $1 \leq i < j \leq n$, $|p(j) - p(i)| + (h(j) - h(i)) \geq n - 1$.

By Proposition 1, we know that $c : c(v_{p(i)}) = h(i)$ $(i = 1, 2, \ldots, n)$ is a hamiltonian coloring for P_n. Then hc(P_n)=ac$(P_n) \leq c(P_n) = h(n)$. Thus, we only need to show that $h(n) = C_{n-1}^2 - \frac{n}{2} - \lfloor \frac{10}{n} \rfloor + 6$ holds for all even integers $n \geq 10$.

Basic concepts
Main results
The pseudo greedy . . .
The rationality of the . . .
Some examples
Reference

访问主页
标题页
◀◀ ▶▶
◀ ▶
第 13 页 共 20 页
返回
全屏显示
关闭
退出

From the results output by computer, we find that

$$\begin{aligned} h(i+1)-h(i)+|p(i+1)-p(i)| &= n-1, \ \ i\in\{1,2,\ldots,n-1\}\backslash\{2,6\}, \\ h(i+1)-h(i)+|p(i+1)-p(i)| &= n, \ \ i\in\{2,6\}. \end{aligned}$$

Then we have

$$h(n) \ = \ h(1)+2n+(n-3)(n-1)-\sum_{i=1}^{n-1}(|p(i+1)-p(i)|)$$

Firstly, we let $n \geq 12$ and write $n = 2k = 10+2(4r+s)$, where $r = 0,1,2,\ldots$, $s=1,2,3,4$. Then, by some detailed analysis, we can find that

$$\sum_{i=1}^{n-1}(|p(i+1)-p(i)|)=\sigma_1+\sigma_2+\sigma_3+\sigma_4,$$

where

$$\sigma_1=\sum_{i=1}^{9}(|p(i+1)-p(i)|)=12k-12,$$

访问主页
标题页
◀◀ ▶▶
◀ ▶

返回
全屏显示
关闭
退出

$$\sigma_2 = \sum_{i=0}^{2r-1}[\sum_{j=1}^{3}(|p(10+3i+j)-p(10+3i+j-1)|)] = 2r(4k-2r-9),$$
$$\sigma_3 = \sum_{i=0}^{2s-1}(|p(10+6r+i+1)-p(10+6r+i)|) = 2s(k-1)-1,$$
$$\sigma_4 = \sum_{i=0}^{2r-1}(|p(10+6r+2s+i+1)-p(10+6r+2s+i)|) = 2r(k-s-2r).$$

Therefore,

$$\begin{aligned} h(n) &= n^2-2n+4-(\sigma_1+\sigma_2+\sigma_3+\sigma_4) \\ &= n^2-2n+4-[(12k-12)+2r(4k-2r-9) \\ &\quad +(2s(k-1)-1)+2r(k-s-2r)] \\ &= \frac{n_2}{2}-2n+7 \\ &= C_{n-1}^2-\frac{n}{2}-\lfloor\frac{10}{n}\rfloor+6. \end{aligned}$$

Secondly, if $n = 2k = 10$, in the above proof we take $r = 0$ and $s = 0$. Then by (1) and (2) hc(P_{10})=ac$(P_{10}) \leq c(P_{10}) = h(10) = h(1) + 2\times 10 + 7(10-1) - \sigma_1|_{k=5} = 1 + 83 - (12\times 5 - 12) = 36 = C_{10-1}^2 - \frac{10}{2} - \lfloor\frac{10}{10}\rfloor + 6$. Thus, for all even integers $n \geq 10$, hc(P_n) =ac$(P_n) \leq C_{n-1}^2 - \frac{n}{2} - \lfloor\frac{10}{n}\rfloor + 6$.

5 Some examples

Example 1. A hamiltonian (antipodal) coloring c for P_{10} with $hc(c) = ac(c) = C^2_{10-1} - \frac{10}{2} - \lfloor\frac{10}{10}\rfloor + 6 = 36$ (see Figure 2).

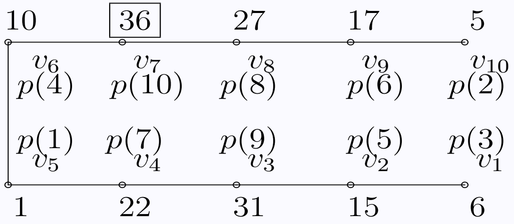

Figure 2: A hamiltonian (antipodal) coloring for P_{10}

Example 2. A hamiltonian (antipodal) coloring c for P_{28} with $hc(c) = ac(c) = C^2_{28-1} - \frac{28}{2} + 6 = 343$ (see Figure 3).

Here $n = 2k = 10 + 2(4r + s) = 28$, then $k = 14$, $r = 2$ and $s = 1$.

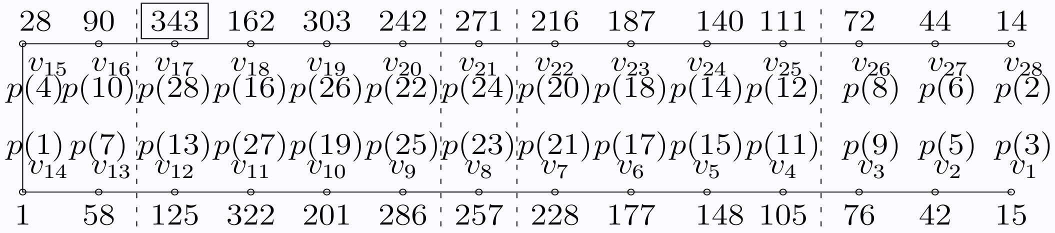

Figure 3: A hamiltonian (antipodal) coloring for P_{28}

Example 3. A hamiltonian (antipodal) coloring c for P_{32} with $hc(c) = ac(c) = C^2_{32-1} - \frac{32}{2} + 6 = 455$ (see Figure 4).

Here $n = 2k = 10 + 2(4r + s) = 32$, then $k = 16$, $r = 2$ and $s = 3$.

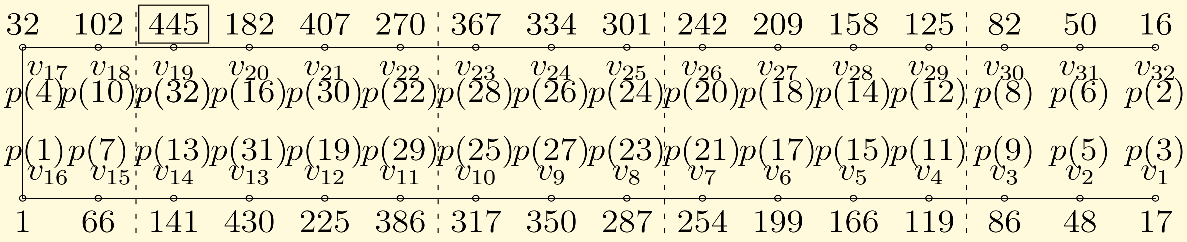

Figure 4: A hamiltonian (antipodal) coloring for P_{32}

访 问 主 页

标 题 页

◀◀ ▶▶

◀ ▶

返 回

全 屏 显 示

关 闭

退 出

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Basic concepts
Main results
The pseudo greedy . . .
The rationality of the . . .
Some examples
Reference

访问主页
标题页
◀◀ ▶▶
◀ ▶

返回
全屏显示
关闭
退出

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