Brooks-type Theorem for the Pair-list Colouring

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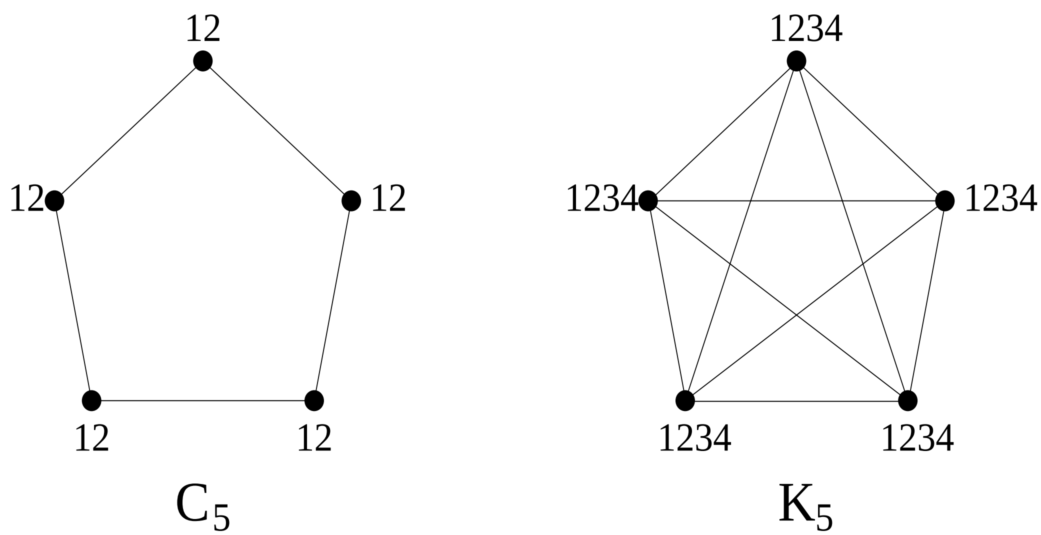
(Joint with Tomas Feder and Pavol Hell)

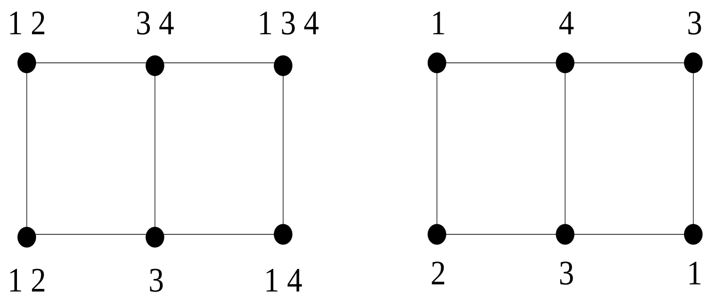
Theorem [Brooks]

- $\chi(G) \leq \Delta(G) + 1$, and
- $\chi(G) \leq \Delta(G)$ except when $G = C_{2k+1}$ or K_n.



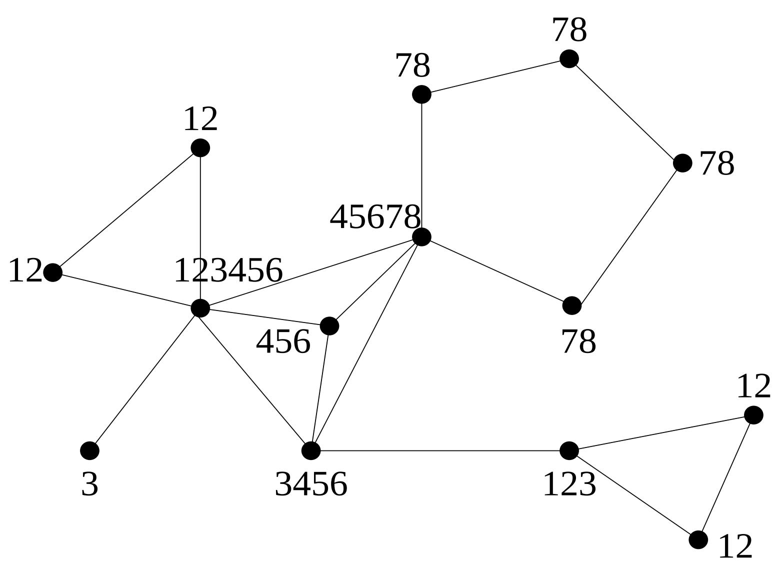
- 3-colouring NP-c
- 3-colouring $(\Delta \leq 3) \in P$
- 3-colouring $(\Delta \leq 4)$ NP-c

List colouring:



Theorem [Erdös, Rubin & Taylor]

When $|L(v)| \geq deg(v)$, G has an L-colouring, except G is a *Gallai tree* with special lists:



- L-colouring with $|L(v)| \geq deg(v) \in P$

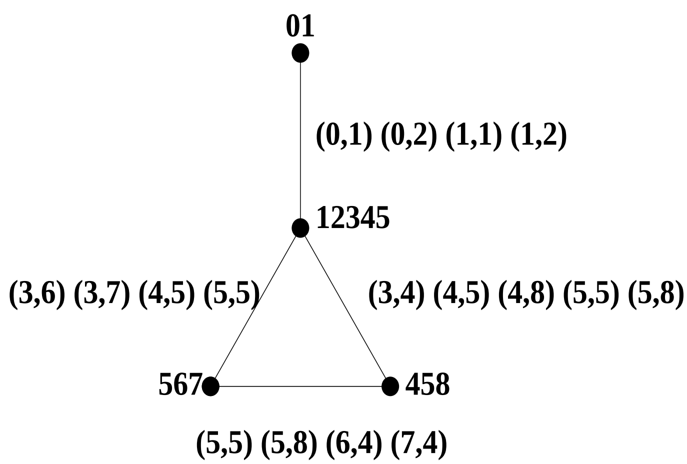
Pair-lists: G with $L(v) \subseteq \mathbf{N} \quad \forall v \in V$, and

$$T(u,v) \subseteq L(u) \times L(v) \quad \forall uv \in E.$$

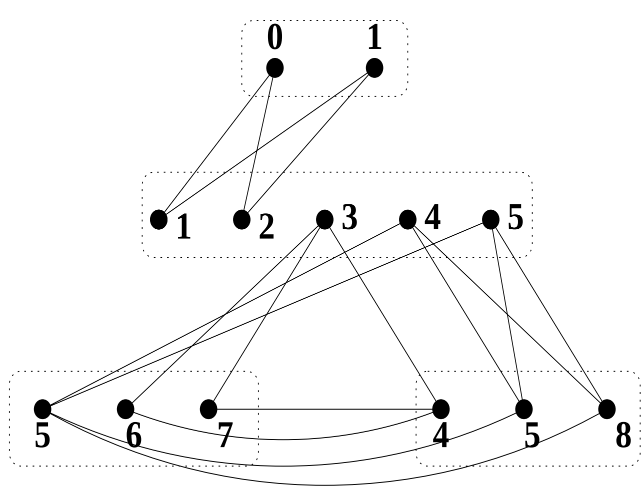
Pair-list colouring: $f : G \to \mathbf{N}$ s.t.

$$f(v) \in L(v) \quad \forall v \in V$$

$$f(u)f(v) \notin T(u,v) \quad \forall uv \in E$$



G



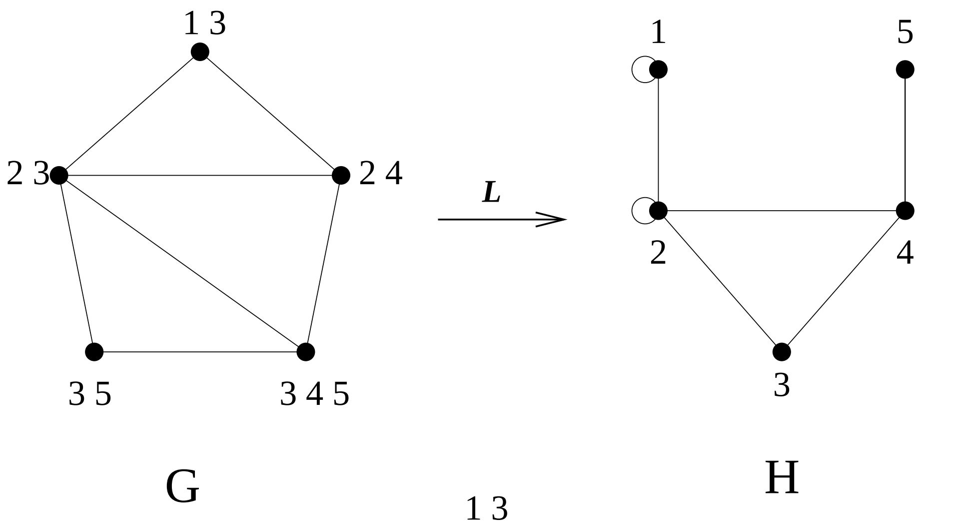
GxT

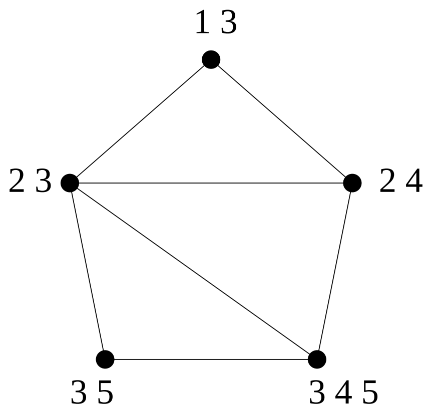
<u>List homomorphism</u>: $f : G \xrightarrow{L} H$

$$f(v) \in L(v) \qquad \forall v \in V(G)$$

$$f(u)f(v) \in E(H) \qquad \forall uv \in E(G)$$

[Feder, Hell and Huang]





(3,3) (3,5) (5,5) (5,3)

The list homomorphism problem: Fix H

Instance: G with lists L

Question: $G \xrightarrow{L} H$?

When $L = V(H)$, *the H-colouring problem*

- $\in P$ if H contains a loop or is bipartite
- NP-c otherwise [Hell and Nešetřil]

Some related problems

- Minimum cost homomorphism problems
- List partition problems
- Retract problems
- Extension problems
- Constrait-satisfaction problems

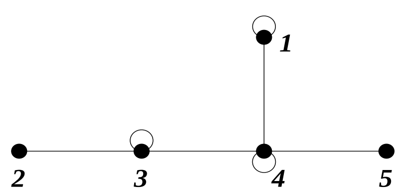
Theorem [Feder, Hell, & Huang] L-hom$H$ is

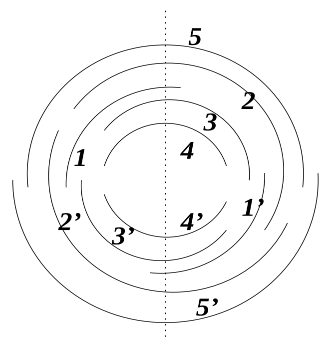
- $\in P$, if H is a bi-arc graph
- NP-c, otherwise

(Same when restricted to input G of $\Delta \leq 3$ and $L(v) \leq 3$.)

Bi-arc H: $\exists \mathcal{F} = \{(N_x, S_x) : \; x \in V(H)\}$ s.t.

1. $N_x \cap S_y = \emptyset$ and $S_x \cap N_y = \emptyset \quad \forall xy \in E(H)$;
2. $N_x \cap S_y \neq \emptyset$ and $S_x \cap N_y \neq \emptyset \quad \forall xy \notin E(H)$.





- Reflexive H is bi-arc $\iff$ H is an interval graph.
- Irreflexive H is bi-arc $\iff$ H is bipartite and $\overline{H}$ is a circular arc graph.

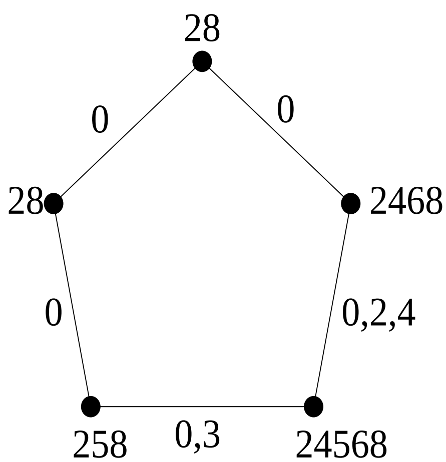
T-lists: G with $L(v) \subseteq \mathbf{N} \quad \forall v \in V$, and

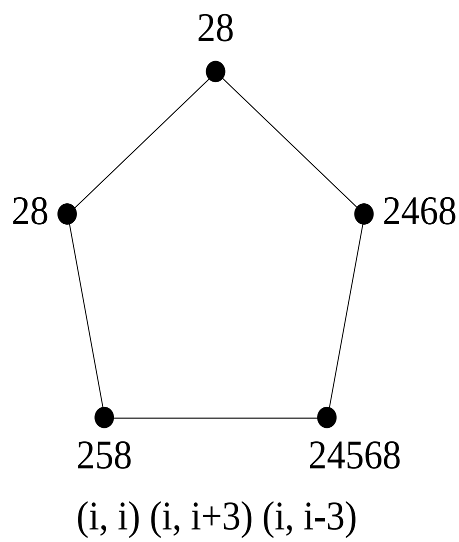
$$t(u,v) \subseteq \mathbf{N} \quad \forall uv \in E.$$

List T-colouring: $f : G \to \mathbf{N}$ s.t.

$$f(v) \in L(v) \quad \forall v \in V$$

$$|f(u) - f(v)| \notin t(u,v) \quad \forall uv \in E$$





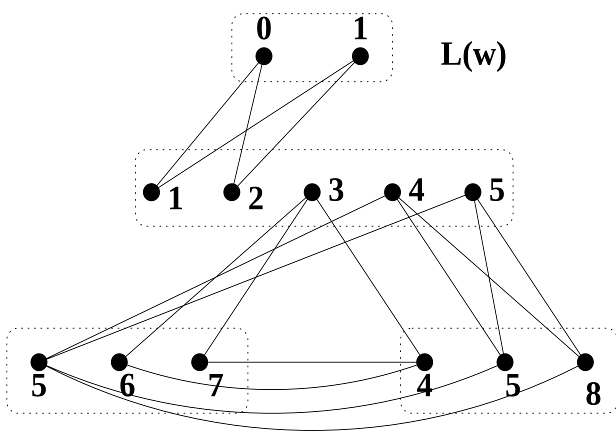
Balanced T-lists: $|L(v)| = \sum_{x \in N(v)} |t(vx)| \quad \forall v \in V$

[Fiala, Král, and Škrekovski]

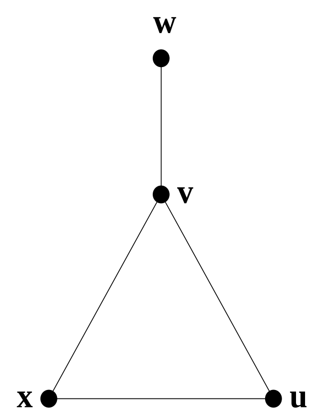
Pair-balanced vertices: $v \in V(G)$ s.t.

(P_1) For each $u \in N(v)$, any assignment f of $N(v) - u$ can be extended to v,
i.e., $\exists a \in L(v)$ with $(a, f(x)) \in L(v) \times L(x) - T(v, x)\ \forall x \in N(v) - u$

(P_2) If an assignment g of $N(v)$ can not be extended to v, then for any $a \in L(v)$, there is a unique $x \in N(v)$ with $(a, g(x)) \in T(v, x)$.



GxT



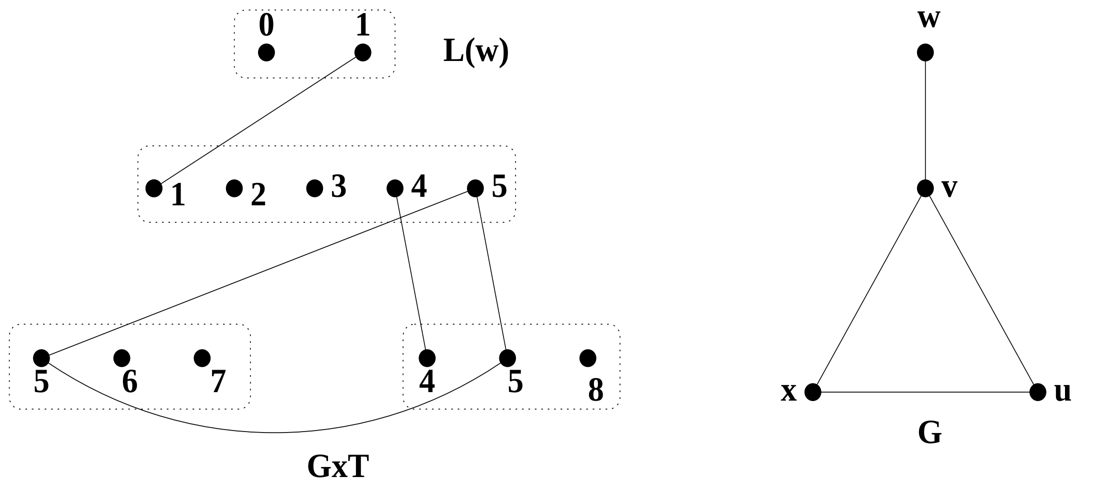
G

T-degree: $deg_T(v) = \sum_{x \in N(v)} \max\{deg(c) : c \in L(x)\}$
(E.g., $deg_T(w) = 2$ and $deg_T(u) = 4$.)

Fact: $|L(v)| \geq deg_T(v) \Rightarrow v$ is pair-balanced.

Pair-balanced instance: every vertex is pair-balanced

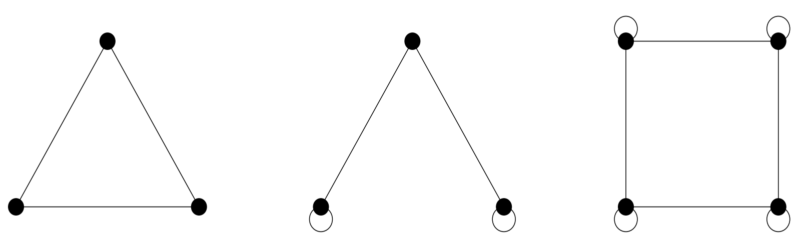
When $T(u,v) = \{(i,i) \in L(u) \times L(v)\}$, ...



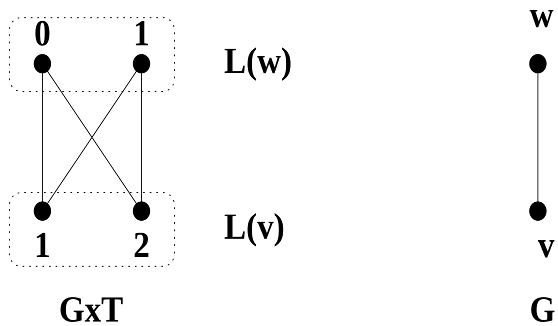
$$deg_G(v) \geq deg_T(v).$$

So if $|L(v)| \geq deg_G(v)$, then v is a pair-balanced vertex.

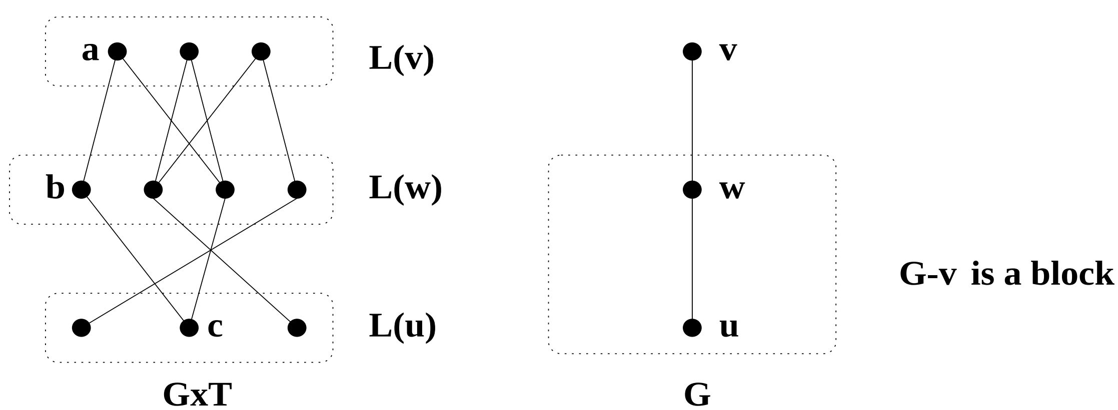
Also true in the case of L-hom$H$ when H is *nearly complete* (i.e., each vertex of H is not adjacent to precisely one vertex including itself). E.g.

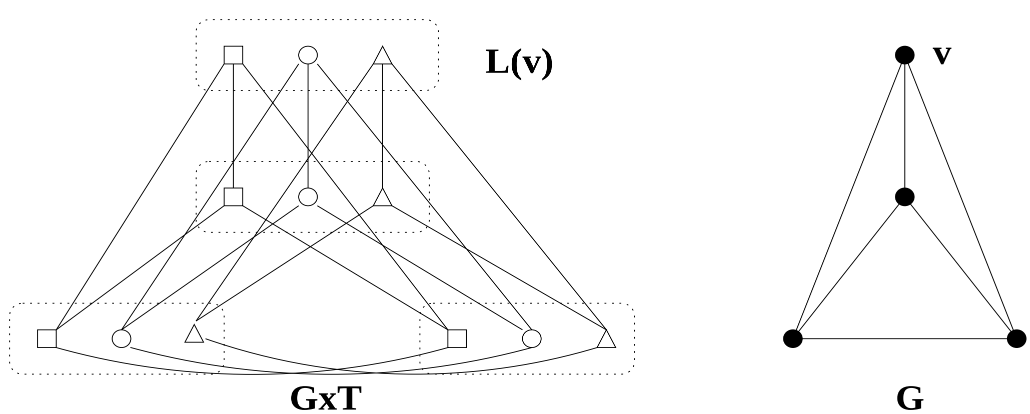


When $G = K_2$,

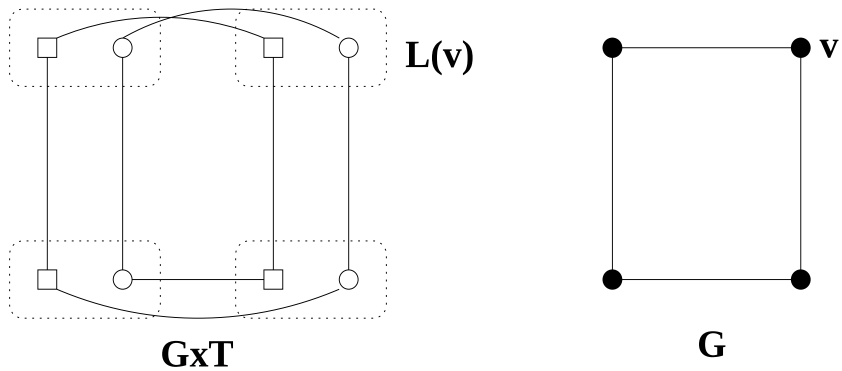


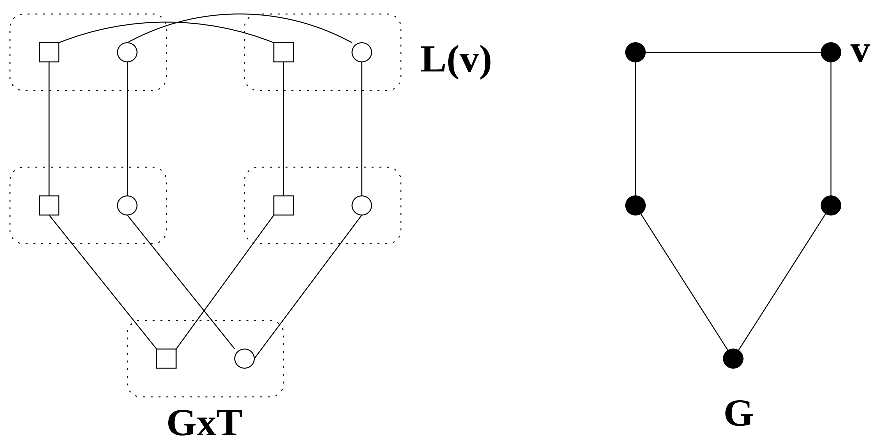
When G is a block of minimum degree at least 3, G must be a clique.





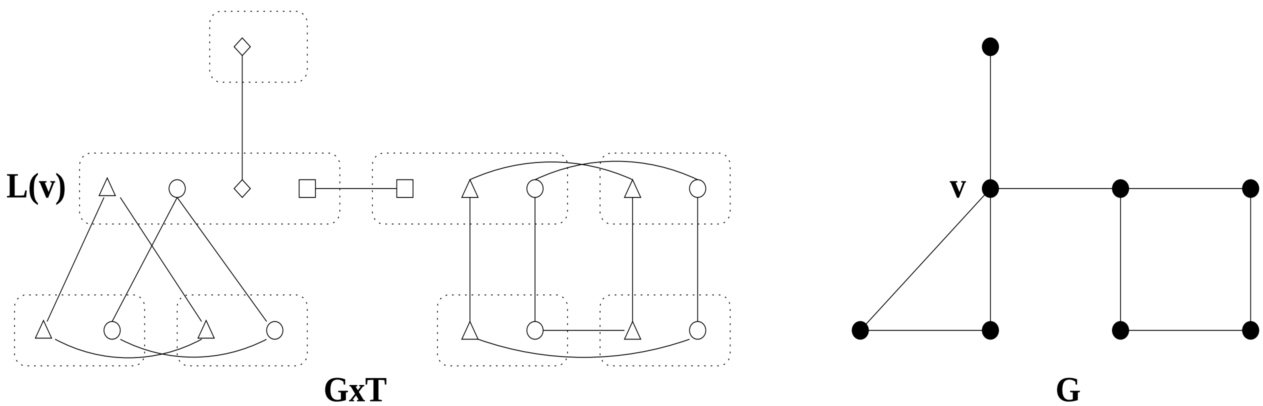
When G is a block and contains a vertex of degree two, G must be a cycle.

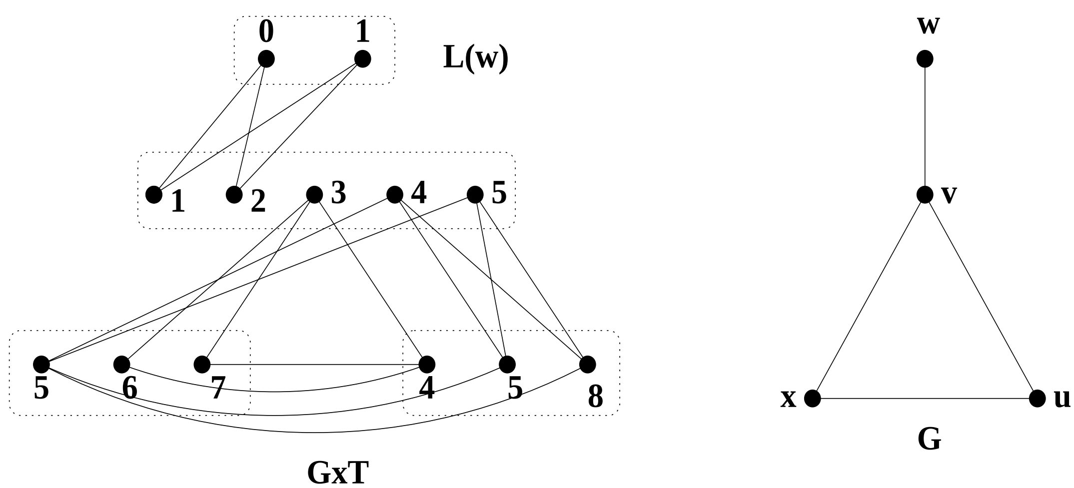




Theorem [Feder, Hell and Huang]

A pair-balanced instance has a solution, except G is a *Gallai configuration* with special pair-list:





Corollary When H is nearly complete, G with lists $|L(v)| \geq deg_G(v)$ $(\forall v \in V)$ has a L-hom, except G is a Gallai-tree with special lists.