On f-Edge Cover-Coloring of Graphs

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Introduction

♠ $G(V, E)$: a graph allows multiple edges but no loops and has a finite vertex set V and a finite and nonempty edge set E.

♠ Degree and the minimum degree
$d(v)$: the number of edges of G incident with vertex v.
$\delta = \min\{d(v) : v \in V\}$

♠ $E(u, v)$: the set of edges with end vertices u and v.

♠ the multiplicity :
$\mu(u, v) = |E(u, v)|$
$\mu(v) = \max\{\mu(v, u) : u \in V\}$
$\mu(G) = \max\{\mu(v) : v \in V(G)\}$.

♡ The reader is referred to [1] for the undefined terms.

Introduction

- ♠ C is a k-edge-coloring of G: if $C : E \to \{1, 2, \ldots, k\}$.
- ♠ $C_v^{-1}(i)$: the number of edges of G incident with vertex v that receive color i by the coloring C.
- ♠ Assume that a positive integer $f(v)$ with $1 \leq f(v) \leq d(v)$ is associated with each vertex $v \in V$.
  C is an f-edge cover-coloring of G: if for each vertex $v \in V$, $C_v^{-1}(i) \geq f(v)$ for $i = 1, 2, \ldots, k$.
- ♠ the f-edge cover chromatic index of G denoted by $\chi'_{fc}(G)$: the maximum positive integer k for which an f-edge cover k-edge coloring of G exists.
- ♡ When $f(v) = 1$ for all $v \in V$, the f-edge cover-coloring is called edge-cover coloring. Accordingly the f-edge cover chromatic index is called the edge cover chromatic index denoted by $\chi'_c(G)$.
- ♡ Let $\delta_f = \min\{\lfloor d(v)/f(v) \rfloor : v \in V\}$. It is trivial that $\chi'_{fc}(G) \leq \delta_f$.

Introduction—-the related result

- ♡ Gupta proved that $\delta(G) - \mu(G \leq \chi_c^{'}(G) \leq \delta(G)$ for any graph[2].
- ♡ Lianying Miao and Guizhen Liu, Edge covered coloring and fractional edge covered coloring [3].
- ♡ Hilton gave some special results on edge cover coloring of multigraphs [4].
- ♡ In [5] the generalization of edge coloring is discussed. We will use some results in [5] for our proofs.
- ♡ Hilton and de Werra studied the equitable edge colorings of simple graphs [6].

Introduction—-our main result

- ♡ We present two interesting special cases for which $\chi'_{fc}(G) = \delta_f$.
- ♡ We show that $\min_{v\in V}\{\lfloor d(v)/f(v)\rfloor\} \geq \chi'_{fc}(G) \geq \min_{v\in V}\{\lfloor (d(v)-\mu(v))/f(v)\rfloor\}$, which reduces to Gupta's theorem when $f(v) = 1$ for all vertex $v \in V$.

The analysis of two special cases—-Lemma 2.1

Let $E(i) = \{e \in E : C(e) = i\}$. Let $G(u; i, j)$ be the components of the edge induced subgraph of G on $E(i) \cup E(j)$ which contains u.

Lemma

Lemma 2.1 *Let $G(V, E)$ be a connected graph. Then G has a 2-edge-coloring C such that:*

(a) If G is Eulerian and $|E|$ is odd, then we can make the 2-edge-coloring C such that for an arbitrary $u \in V$, $|C_u^{-1}(1) - C_u^{-1}(2)| = 2$ and $C_v^{-1}(1) - C_v^{-1}(2) = 0$ for all $v \in V \setminus \{u\}$.

(b) If G is Eulerian and $|E|$ is even, then $C_v^{-1}(1) - C_v^{-1}(2) = 0$ for all $v \in V$.

(c) If G is not Eulerian, then $|C_v^{-1}(1) - C_v^{-1}(2)| \leq 1$ for all $v \in V$.

(d) In all cases, we have

$$| \, |E(1)| - |E(2)| \, | = \begin{cases} 0, \text{if } |E| \text{ is even} \\ 1, \text{if } |E| \text{ is odd} \end{cases}$$

The analysis of two special cases—-Lemma 2.2

Lemma

Lemma 2.2 *Let k be an integer which is larger than 1. Then there exists a k-edge-coloring C of G such that*
(a) $|C_v^{-1}(i) - C_v^{-1}(j) \leq 2$ for all $v \in V$, $i, j \in \{1, 2, \ldots, k\}$, and furthermore if for some $u \in V$, $i, j \in \{1, 2, \ldots, k\}$, $|C_u^{-1}(i) - C_u^{-1}(j)| = 2$, then $G(u; i, j)$ is Eulerian with an odd number of edges, and
(b)$|\ |E(i)| - |E(j)|\ | \leq 1$ for all i and $j \in \{1, 2, \ldots, k\}$.

♡The above two lemmas are proved in [5] which are important for our proofs.

The analysis of two special cases—-Theorem 2.3

Applying Lemmas 2.1 and 2.2, we can evaluate edge cover chromatic index for bipartite graphs.

Theorem

Theorem 2.3 *Let $G(V,E)$ be a bipartite graph and $\delta_f = \min\{\lfloor d(v)/f(v) \rfloor : v \in V\}$. Then $\chi'_{fc}(G) = \delta_f$. Furthermore if $\delta_f = k \geq 2$, there exists an f-edge cover-coloring C of G for which $||E(i)| - |E(j)|| \leq 1$ and $|C_v^{-1}(i) - C_v^{-1}(j)| \leq 1$ for all $v \in V$, $i, j \in \{1, 2, \ldots, k\}$.*

♡**Idea.** We first give one kind of coloring with the condition subscribed in Lemma 2.2, then prove the coloring is required.

The analysis of two special cases—-Lemma 2.4

Let $G(V, E)$ be a graph. By an orientation of G, we mean a digraph $\overrightarrow{G}$ obtained from G by assigning a direction to each edge of G. By $d^+(v)$ and $d^-(v)$, we mean the indegree and outdegree of vertex v in $\overrightarrow{G}$, respectively. Note that $d^+(v) + d^-(v) = d(v)$ for every $v \in V$.

Lemma

Lemma 2.4 *There exists an orientation $\overrightarrow{G}$ of G such that*

$$\lfloor \frac{d(v)}{2} \rfloor \leq d^+(v) \leq \lceil \frac{d(v)}{2} \rceil \text{ for each } v \in V.$$

The analysis of two special cases—-Theorem 2.5 and Corollary 2.6

Using the above lemma, we give the following theorem.

Theorem

Theorem 2.5 *Let $G(V, E)$ be a graph and $\chi'_{fc}(G)$, $d(v)$ and $f(v)$ be defined as above. Suppose that $f(v) > 1$ for all $v \in V$ and let*

$$\delta'_f = \min_{v\in V}\{\min(\lfloor\lfloor\frac{d(v)}{2}\rfloor/\lfloor\frac{f(v)}{2}\rfloor\rfloor, \lfloor\lceil\frac{d(v)}{2}\rceil/\lceil\frac{f(v)}{2}\rceil\rfloor)\}.$$

Then $\chi'_{fc}(G) \geq \delta'_f$.

Corollary

Corollary 2.6 *Let $G(V, E)$ be a graph. If $f(v) > 1$ for all $v \in V$, then $\chi'_{fc}(G) \geq \frac{1}{2}\delta_f$.*

The analysis of two special cases—-Corollary 2.7

Corollary

Corollary 2.7 *If $f(v)$ is positive and even for all $v \in V(G)$, then* $\chi'_{fc}(G) = \delta_f = \min\{\lfloor d(v)/f(v) \rfloor \ |v \in V\}$.

♡Suppose that we could find an orientation $\overrightarrow{G}$ of G such that for some positive integer k and each $v \in V$, $k\lfloor f(v)/2 \rfloor \leq \min\{d^-(v), d^+(v)\} \leq d(v) - k\lceil f(v)/2 \rceil$. Then , following the same reasoning as in Theorem 2.5, one can show that $\chi'_{fc}(G) \geq k$.

A bound of f-edge cover chromatic index of graphs—-Theorem 3.1

Theorem

Theorem 3.1 *Let $G(V,E)$ be a graph. Let $f(v)$, $d(v)$, and $\mu(v)$ be defined as in Section 1. Then*

$$\min_{v\in V}\{\lfloor d(v)/f(v)\rfloor\} \geq \chi'_{fc}(G) \geq \min_{v\in V}\{\lfloor (d(v)-\mu(v))/f(v)\rfloor\}.$$

♡**Idea.**Assume that G has an edge-coloring C with k colors $1,2,\ldots,k$. For each $v\in V$ and $1\leq i\leq k$, let

- ♡ $k=\min_{v\in V}\{\lfloor (d(v)-\mu(v))/f(v)\rfloor\}$.
- ♡ $\sigma_i(v)=\max(0,f(v)-C_v^{-1}(i))$
- ♡ $\epsilon(v)=\max\limits_{1\leq i\leq k}\sigma_i(v)$
- ♡ $\sigma(v)=\sum\limits_{i=1}^{k}\sigma_i(v)$.

A bound of f-edge cover chromatic index of graphs—-exchange chain

In order to 'improve' the coloring of an edge-colored graph, we shall use the concept of an exchange chain.

An (α, β)-*exchange chain* K of G is a sequence $(v_0, e_1, v_1, e_2, \ldots, v_{r-1}, e_r, v_r)$ of vertices and edges of G in which

(i) for $1 \leq i \leq r$, the vertices v_{i-1} and v_i are distinct and are both incident with the edge e_i,

(ii) the edges are all distinct and are colored α and β alternately,

(iii) e_1 is colored α and $C_{v_0}^{-1}(\alpha) > C_{v_0}^{-1}(\beta)$; similarly, let γ denote the color of e_r and $\bar{\gamma}$ denote the other color of $\{\alpha, \beta\}$, then $C_{v_r}^{-1}(\gamma) > C_{v_r}^{-1}(\bar{\gamma})$.

♡It is clear, however, that an interchange of colors in an exchange chain never makes the coloring worse.

some variations of f-edge cover-coloring problem

Let $G(V, E)$ be a graph and $f(v)$ be defined as before. An f-edge cover equitable coloring is an edge coloring $C : E \to \{1, 2, \ldots, k\}$ such that every color appears at each vertex v at least $f(v)$ times, and $|C_v^{-1}(i) - C_v^{-1}(j)| \leq 1$ for every $i, j \in \{1, 2, \ldots, k\}$. The maximum number k for which an f-edge cover equitable coloring exists, denoted by $q'_{fc}(G)$, is called equitable f-edge cover chromatic index of G. Obviously, $q'_{fc}(G) \leq \chi'_{fc}(G)$.

From Theorem 2.3, we have the following corollary.

Corollary

Let $G(V, E)$ be a bipartite graph, we have $q'_{fc}(G) = \chi'_{fc}(G)$.

some variations of f-edge cover-coloring problem

Finally we present some problems for future research as follows.

- ♣ For some special class of graph G, define $q'_{fc}(G)$ and $\chi'_{fc}(G)$.
- ♣ Characterizes graph G such that $q'_{fc}(G) = \chi'_{fc}(G)$.
- ♣ Characterizes graph G such that $q'_{fc}(G) = \delta_f$ or $\chi'_{fc}(G) = \delta_f$.
- ♣ Does $\delta_f \geq q'_{fc}(G) \geq \min_{v \in V}\{\lfloor (d(v) - \mu(v))/f(v) \rfloor\}$ stand true for any graph G?

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